A Lightface Analysis of the Differentiability Rank

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History

Mazurkiewicz, 1936

$\{f : f \text{ is differentiable}\}$ is $\mathbf{\Pi^1_1}$-complete.

Kechris and Woodin, 1986

$$\{f : f \text{ is differentiable}\} = \bigcup_{\alpha<\omega_1} \{f : |f|_{KW} < \alpha\},$$

where each constituent of the union is Borel.

Effectivizations

Mazurkiewicz, 1936

$\{f : f \text{ is differentiable}\}$ is $\mathbf{\Pi^1_1}$-complete.

Effective version:
$\{e : f_e \text{ is differentiable}\}$ is Π^1_1-complete

Kechris and Woodin, 1986

$$\{f : f \text{ is differentiable}\} = \bigcup_{\alpha<\omega_1} \{f : |f|_{KW} < \alpha\},$$

where each constituent of the union is Borel.

Effective version:

$$\{e : f_e \text{ is differentiable}\} = \bigcup_{\alpha<\omega_1^{CK}} \{e : |f_e|_{KW} < \alpha\},$$

where each constituent of the union is HYP.

Our Goal

Theorem (W)

(a) *The set* $\{e : |f_e|_{KW} < \alpha + 1\}$ *is* $\Pi_{2\alpha+1}$*-complete for any constructive ordinal* $\alpha > 0$.

(b) *The set* $\{e : |f_e|_{KW} < \lambda\}$ *is* Σ_λ*-complete for* λ *a constructive limit ordinal.*

Remark: This result is expressed in the notation of Ash and Knight (2000). Here $(\emptyset^{(\omega)})'$ is a Σ_ω-complete set.

The Problem

How can we build differentiable functions which by their ranks encode the answers to arbitrary $\Pi_{2\alpha}$ questions?

The Differentiability Rank

Definition

Fix $f \in C[0,1], \varepsilon > 0$. For a closed set $P \subseteq [0,1]$, define

$$P'_{f,\varepsilon} = \{x \in P : \text{ for every open } U \ni x, \text{there are } p,q,r,s \in U \text{ such that} [p,q] \cap [r,s] \cap P \neq \emptyset, \text{ and } \left| \frac{f(p)-f(q)}{p-q} - \frac{f(r)-f(s)}{r-s} \right| > \varepsilon\}$$

Iterate this procedure through all the ordinals.

Definition

$P^0_{f,\varepsilon} = [0,1]$ $\quad$ $P^{\alpha+1}_{f,\varepsilon} = (P^\alpha_{f,\varepsilon})'_{f,\varepsilon}$ $\quad$ $P^\lambda_{f,\varepsilon} = \cap_{\alpha<\lambda} P^\alpha_{f,\varepsilon}$

The Differentiability Rank

Theorem (Kechris and Woodin, 1986)

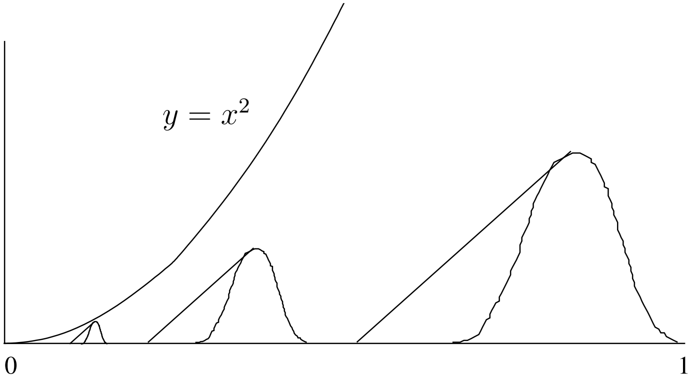
A function f is differentiable if and only if there is an $\alpha < \omega_1$ such that for all ε, $P^{\alpha}_{f,\varepsilon} = \emptyset$.

Definition (Kechris and Woodin, 1986)

For $f \in C[0,1]$, the **differentiability rank** of f, denoted $|f|_{KW}$, is the least α such that for all ε, $P^{\alpha}_{f,\varepsilon} = \emptyset$.

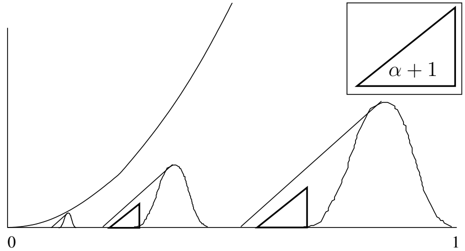
Examples

1. $|f|_{KW} = 1$ if and only if f is continuously differentiable
2. $x^2 \sin(\frac{1}{x})$ has rank 2
3. Here is an idealized rank 2 function:

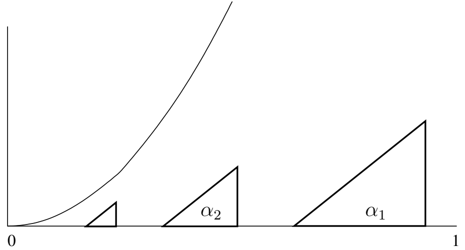


Examples

4. Building a function with higher rank:



5. A rank $\lambda + 1$ function, where λ is the limit of $\alpha_1, \alpha_2, \ldots$



Σ_α-completeness

Spector showed that $|a|_\mathcal{O} = |b|_\mathcal{O} \implies H_a \equiv_T H_b$. Thus $H_{2^a} \equiv_1 H_{2^b}$.

Definition (following Ash and Knight, 2000)

A set X is Σ_α if $X \leq_1 H_{2^a}$ for any a such that $|a|_\mathcal{O} = \alpha$. X is Σ_α-complete if $X \equiv_1 H_{2^a}$ for such a.

For example, X is Σ_ω-complete if and only if $X \equiv_1 (\emptyset^{(\omega)})'$.

Naive Upper Bound

We are proving this:

Theorem (W)

For any constructive ordinal $\alpha > 0$, *the set* $\{e : |f_e|_{KW} < \alpha + 1\}$ *is* $\Pi_{2\alpha+1}$*-complete.*

From the preceding definitions,

$$|f_e|_{KW} < \alpha + 1 \iff \forall \varepsilon P^{\alpha}_{f,\varepsilon} = \emptyset.$$

The statement $P^{\alpha}_{f,\varepsilon} = 0$ is naively $\Sigma_{2\alpha}$.

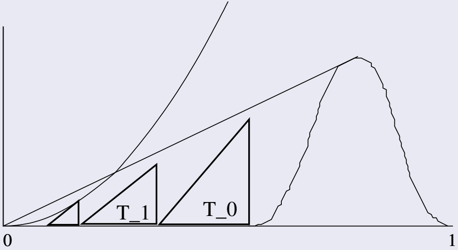
Core of the theorem

$\{e : P^{\alpha}_{f_e,\varepsilon} = \emptyset\}$ is $\Sigma_{2\alpha}$-complete.

Building Functions From Trees

Definition

If $T \subseteq \mathbb{N}^{<\mathbb{N}}$ is well-founded, define f_T by $f_T \equiv 0$ if $T = \emptyset$ and



otherwise.

Proposition

For any well-founded T, f_T is everywhere differentiable and uniformly computable from T.

A Rank on Trees

Now we define a rank on well-founded trees which agrees with the rank of the functions they generate.

Definition

For $T \subseteq \mathbb{N}^{<\mathbb{N}}$ a well-founded tree, the **limsup rank** of T, denoted $|T|_{ls}$, is defined as

$$|T|_{ls} = \max(\sup_n |T_n|_{ls}, [\limsup_n |T_n|_{ls}] + 1),$$

if $T \neq \emptyset$, and $|T|_{ls} = 0$ if $T = \emptyset$.

Proposition

For all well-founded T, $|T|_{ls} = |f_T|_{KW}$.

Examples

1. $|T|_{ls} = 3$
2. $|T|_{ls} = \omega + 1$

Forget Everything But Trees

To show that $P^{\alpha}_{f,\varepsilon} = \emptyset$ is $\Sigma_{2\alpha}$-complete, it suffices to do the following:

Combinatorial Task

Uniformly in a given $\Sigma_{2\alpha}$ question, produce T whose rank encodes the answer:

- If $\Sigma_{2\alpha}$, then $|T|_{ls} \leq \alpha$
- If $\Pi_{2\alpha}$, then $|T|_{ls} = \alpha + 1$

A Strategy For Finite α

"Let the children encode the evidence and witnesses."

Example: Σ_2/Π_2 Case

Given a statement $P = \forall x \exists y R(x,y)$, we want to build T so that

$$|T|_{ls} = \begin{cases} 2 & \text{if } P \\ \leq 1 & \text{if } \neg P \end{cases}.$$

This idea works if R is nice

Let $T = \{\emptyset\} \cup \{\langle x, y\rangle : R(x,y)\}$

This is how nice R has to be

If R satisfies the following, then T is as required:

1. (Unique witnesses) $R(x,y_1) \wedge R(x,y_2) \implies y_1 = y_2$
2. (Stable evidence) $\exists y R(x,y) \implies \forall z < x \exists y R(z,y)$.

Proof. Suppose P holds. Then infinitely many $\langle x, y_x\rangle \in T$, so $|T|_{ls} = 2$. Suppose $\neg P$ holds, in particular $\neg\exists y R(x_0, y)$. Then by stable evidence, $\langle z, y\rangle \notin T$ for all $z \geq x_0$. And by unique witnesses, T has at most x_0-many children of the form $\langle z, y\rangle$ for $z < x_0$. So T is finite. □

A Construction for Finite α

"Let the children encode the evidence and witnesses."

Lemma

From any Π_{2n+2} statement $\forall x \exists y R(x,y)$ one may uniformly produce a Π_{2n} formula $\tilde{R}$ such that

1. $\forall x \exists y R(x,y) \iff \forall x \exists y \tilde{R}(x,y)$
2. *$\tilde{R}$ has unique witnesses*
3. *$\tilde{R}$ has stable evidence*

Construction

Given a Π_{2n} statement $P \equiv \forall x \exists y R(x,y)$, define

$$T(P) = \{\emptyset\} \cup \{\langle x, y \rangle ^\frown \sigma : \sigma \in T(\tilde{R}(x,y))\}.$$

Then $|T|_{ls} = \begin{cases} n+1 & \text{if } P \\ \leq n & \text{if } \neg P \end{cases}.$

Proof: By induction on T.

Recap

Recall:

Combinatorial Task

Uniformly in a given $\Sigma_{2\alpha}$ question, produce T whose rank encodes the answer:

- If $\Sigma_{2\alpha}$, then $|T|_{ls} \leq \alpha$
- If $\Pi_{2\alpha}$, then $|T|_{ls} = \alpha + 1$

We have sketched how to do this for the case $\alpha < \omega$.

A Strategy for Infinite α

"Let the children evaluate multiple questions"

Example: Σ_ω/Π_ω case

Given a Π_ω statement P_ω we want to build T so that $|T|_{ls} = \begin{cases} \omega+1 & \text{if } P_\omega \\ <\omega & \text{if } \neg P_\omega \end{cases}$.

Uniformly we can decompose P_ω as $P_\omega \equiv \bigwedge_{i=1}^{\infty} P_i$, where each P_i is Π_{2i}.

This will work once we make $P \mapsto T(P)$ better

Let $T = \{\emptyset\} \cup \{n^\frown\sigma : \sigma \in T(\bigwedge_{i=1}^{n} P_i)\}$

Unfortunately, this T has rank $\omega+1$ regardless of what P is.

The Core

In order to make the preceding construction work, we need

Stronger Combinatorial Task

Uniformly in a finite sequence of statements $P_1, \dots, P_k$, where each P_i is $\Pi_{2\alpha_i}$, produce a tree $T(P_1, \dots, P_k)$ such that

$$|T|_{ls} = \begin{cases} \max_i \alpha_i + 1 & \text{if all statements hold} \\ \leq \alpha_i & \text{for each } i \text{ such that } P_i \text{ fails} \end{cases}$$

Assuming the stronger combinatorial task when the α_i are finite, we can encode $P_\omega \equiv \bigwedge_{i=1}^{\infty} P_i$ from the previous slide:

$$T = \{\emptyset\} \cup \{n^\frown\sigma : \sigma \in T(P_1, \dots, P_n)\}$$

One may check that $|T|_{ls} = \begin{cases} \omega + 1 & \text{if } P_\omega \\ (\text{the least } n \text{ such that } \neg P_n) + 1 & \text{if } \neg P_\omega \end{cases}$.

The Core

We have “reduced” the entire problem to this:

Stronger Combinatorial Task

Uniformly in a finite sequence of statements $P_1, \ldots, P_k$, where each P_i is $\Pi_{2\alpha_i}$, produce a tree $T(P_1, \ldots, P_k)$ such that

$$|T|_{ls} = \begin{cases} \max_i \alpha_i + 1 & \text{if all statements hold} \\ \leq \alpha_i & \text{for each } i \text{ such that } P_i \text{ fails} \end{cases}$$

We sketch the proof for the special case when $\alpha_i < \omega$ for all i.

Construction

Given $P_1, \ldots, P_k$, with complexity $\Pi_{\alpha_1}, \ldots, \Pi_{\alpha_k}$, construct $T(P_1, \ldots, P_k)$ by recursion as follows:

1. Renumber all the formulas so that $\alpha_1 \geq \cdots \geq \alpha_k$
2. Rewrite all the formulas in the form $P_i \equiv \forall x \exists y R_i(x, y)$, where R_i has unique witnesses and stable evidence. Also ensure that $R_i(x, y) \implies x < y$.
3. Put $\emptyset$ in T
4. For each $n = \langle m_0, \ldots, m_k \rangle$, define T_n (the nth subtree):
    1. $n \notin T$ unless $m_0 < m_1 < \cdots < m_k$
    2. If for any i, $\alpha_i = 1$ and $R_i(m_{i-1}, m_i)$ fails, $n \notin T$
    3. Otherwise, define T_n recursively as the tree obtained from the following statements:
        - $R_i(m_{i-1}, m_i)$ for each i with $\alpha_i > 1$
        - $\forall x \exists y R_i(x, y)$ for each i with $\alpha_i < \alpha_1$.

Verification

Case 1. Suppose each statement holds.
For each natural number m_0, define $\langle\overline{m}\rangle$ recursively by letting m_i be the unique y such that $R_i(m_{i-1}, m_i)$ holds.
Then $T_{\langle\overline{m}\rangle}$ was built from formulas:

- $R_i(m_{i-1}, m_i)$, which hold
- $\forall x \exists y R_i(x, y)$, which hold

Out of the above formulas, the most complex is $\Pi_{2(\alpha_1-1)}$. Therefore, by induction, $|T_{\langle\overline{m}\rangle}|_{ls} = (\alpha_1 - 1) + 1 = \alpha_1$. There are infinitely many such subtrees. So $|T|_{ls} = \alpha_1 + 1$.

Verification

Case 2. Let r be largest such that $\forall x \exists y R_r(x, y)$ fails.
Claims:

1. For each n, $|T_n|_{ls} \leq \alpha_r$.
2. For each choice of $m_0, \ldots, m_{r-1}$, there is at most one choice of $m_r, \ldots, m_k$ which makes $|T_{\langle \overline{m} \rangle}|_{ls} = \alpha_r$.
3. Let z be such that $\neg \exists y R_r(z, y)$. THere are only finitely many ways to put $m_0 < m_1 < \cdots < m_{r-1} < z$.
4. If $m_{r-1} \geq z$, then $|T_{\langle \overline{m} \rangle}|_{ls} \leq \alpha_r - 1$, because $R_r(m_{r-1}, m_r)$ does not hold.

Therefore, $\sup_n |T_n|_{ls} \leq \alpha_r$ (Claim 1) and $\limsup_n |T_n|_{ls} \leq \alpha_r - 1$ (Claims 2-4).
Thus $|T|_{ls} \leq \alpha_r$. □