Oberwolfach randomness and computing K-trivial sets

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Background

Random sets and c.e. sets

What are the possible interactions, in the Turing degrees, between c.e. sets and random sets?

- An incomplete c.e. set cannot compute a random set (follows from Arslanov)
- Chaitin's Ω computes every c.e. set. However, this is atypical: the cone above a non-computable set is null.

Two questions:

1. Which random sets can compute non-computable c.e. sets?
2. Which c.e. sets can be computed from an incomplete random set? [Recall that Stephan says that incomplete randoms are not PA, and in that sense quite far from being complete.]

Computing c.e. sets

Theorem (Kučera)

Every Δ^0_2 random set computes a non-computable c.e. set.

Theorem (Hirschfeldt,Miller)

A random set computes a non-computable c.e. set if and only if it is not an element of any null Π^0_2 class if and only if it forms a minimal pair with $\emptyset'$.

Hirschfeldt-Miller

One direction is easy: the upper cone of a Δ^0_2 set is Σ^0_3 and null.

In the other direction, let $V = \bigcap V_n$ be a null Π^0_2 class, with $\langle V_n \rangle$ uniformly Σ^0_1 and nested. Enumerate a c.e. set A and a functional Γ. If Y enters V_n at stage s, set $\Gamma^Y(n) = A_s(n)$. If $Y \in V$, then $\Gamma^Y(n)\downarrow$ for all n.

To ensure that A is not computable, when x enters $W_{e,s}$, we want to enumerate x into A as well. This would make $\Gamma^Y(x)$ wrong for oracles $Y \in V_{x,s}$. Do this if $\lambda(V_{x,s}) < 2^{-e}$, in which case let $G_e = V_{x,s}$.

So $\sum \lambda(G_e) \leqslant 1$. For every random set $Y \in V$, for almost all x, $\Gamma^Y(x) = A(x)$.

Why is A non-computable? Because $\lambda(V_x) \to 0$ (and we make A co-infinite).

Computability and compressibility

Theorem (Chaitin)

A set A is computable if and only if $C(A\restriction_n) \leqslant^+ C(n)$.

However, Solovay built a non-computable set A such that $K(A\restriction_n) \leqslant^+ K(n)$, and called these sets K-trivial. Such sets are "anti-random". Chaitin's argument shows that such sets are Δ^0_2.

Theorem (Nies)

1. *Every K-trivial set is superlow.*
2. *A set is K-trivial iff it is low for ML-randomness iff it is low for K.*
3. *Every K-trivial set is computable from a c.e., K-trivial set.*

Theorem (Hirschfeldt,Nies,Stephan)

A set A is K-trivial if and only if it is computable from some random^A set.

ML-covering

Theorem (Hirschfeldt,Nies,Stephan)

If Z is random and incomplete, A is c.e. and $A \leqslant_T Z$, then A is K-trivial.

Proof.

We show that Z is random^A, and then appeal to DHS. Let $\langle U_n^A \rangle$ be a ML^A-test. For $n < \omega$, let s_n be the least stage s such that $Z \in U_n^A[s]$ by an A-correct computation. If Z is captured by $\langle U_n^A \rangle$, then $n \mapsto s_n$ is total and Z-computable.

Since Z is incomplete, there are infinitely many n such that $n \in \varnothing'$ but $n \notin \varnothing'_{s_n}$.

If $n \in \varnothing'$, at stage s, let $G_n = U_n^A[s]$; otherwise $G_n = \varnothing$. Then $\sum_n \lambda(G_n) \leqslant 1$, and for infinitely many n, $Z \in G_n$. $\square$

Stephan asked: is every K-trivial set computable from an incomplete random set?

An extension of the problem

One can also try for more. If covering holds, how far from complete can witnesses be?

- Is there a single incomplete random set which computes all K-trivial sets?
- Is every K-trivial set computable from a low random set?

After all, every K-trivial set is superlow, and the ideal of K-trivial sets has a low$_2$ c.e. bound. Further, Kučera and Slaman showed that there is a low PA set which computes all K-trivial sets.

This can be extended further: if covering holds, how random can witnesses be?

- Is every K-trivial set computable from both halves of some random set?

Cost functions

Discrete measures

A discrete measure is a measure on ω. For a discrete measure ν we have $\nu(\omega) = \sum_{n<\omega} \nu(\{n\})$. We write $\nu(n)$ for $\nu(\{n\})$.
A discrete measure ν is c.e. if $\nu(n)$ is left-c.e., uniformly in n.

Among all c.e. discrete measures there is an optimal one: a c.e. discrete measure μ such that for every discrete c.e. measure ν, $\mu \geqslant^{\times} \nu$.

Indeed we can let $\mu = 2^{-K}$, or what is a bit more natural, $\mu(n)$ is the measure of the set of descriptions of n in the universal prefix-free machine. The coding theorem says these are the same.

So a set X is random if and only if $\mu(X\restriction_n) \leqslant^{\times} 2^{-n}$, and a set A is K-trivial if and only if $\mu(A\restriction_n) \geqslant^{\times} \mu(n)$ iff $\nu(A\restriction_n) \geqslant^{\times} \mu(n)$ for some c.e. discrete measure ν.

Construction of K-trivial sets

Theorem (Zambella)

There is a non-computable, c.e. K-trivial set.

Proof by Downey,Hirschfeldt,Nies,Stephan:

We enumerate a set A and a c.e. discrete measure ν. At stage s we ensure that $\nu(A_s\restriction_n) = \mu_s(n)$ for all $n \leqslant s$. Then

$$\nu(\omega) = \nu\left(\{\sigma : \sigma < A\}\right) + \nu\left(\{A\restriction_n : n < \omega\}\right).$$

For all $n < \omega$, $\nu(A\restriction_n) = \mu(n)$, and so

$$\nu\left(\{A\restriction_n : n < \omega\}\right) = \mu(\omega) < \infty.$$

For the first part we have

$$\nu\left(\{\sigma : \sigma < A\}\right) = \sum_{s<\omega} \sum \mu_s(n) \; [\![A_s\restriction_n \neq A_{s+1}\restriction_n]\!].$$

Construction of K-trivial sets

Let x_s be the number which is enumerated into $A_{s+1} \setminus A_s$. Then

$$\sum \mu_s(n) \ [\![A_s \upharpoonright_n \neq A_{s+1} \upharpoonright_n]\!] = c_s(x_s),$$

where

$$c_s(x) = \sum_{n \in (x,s)} \mu_s(n)$$

is the cost of enumerating x into A_{s+1}. If

$$\sum_s c_s(x_s) < \infty$$

then ν is indeed a c.e. discrete measure, and A would be K-trivial.

Construction of K-trivial sets

We have

- $c_s(x) \leqslant c_{s+1}(x)$;
- $c_s(x) \geqslant c_s(x+1)$.

Let $c(x) = \lim_s c_s(x) = \sum_{n>x} \mu(n)$. Then:

- each $c(x)$ is finite and $\lim_x c(x) = 0$.

The construction of A: suppose at stage s we want to enumerate $x \in W_{e,s}$ into A. If $c_s(x) \leqslant 2^{-e}$, do it.

General framework: cost functions

A cost function is a non-increasing function $c\colon \omega \to \mathbb{R}^+$ with $c(n)$ left-c.e., uniformly in n, such that $\lim_n c(n) = 0$. A cost function is monotone if it has an increasing approximation $\langle c_s \rangle$ such that each c_s is decreasing.

Let $\langle c_s \rangle$ be a monotone approximation for a cost function c, and let $\langle A_s \rangle$ be a computable approximation for a Δ^0_2 set A. Let x_s be the least x such that $A_s(x) \neq A_{s+1}(x)$. We let

$$\sum c_s(A_s) = \sum_s c_s(x_s).$$

Obedience

Lemma (Nies)

The following are equivalent for a monotone cost function c and a Δ^0_2 set A:

- *For some monotone approximation $\langle c_s \rangle$ of c and some computable approximation $\langle A_s \rangle$ of A, $\sum c_s(A_s) < \infty$.*
- *For every monotone approximation $\langle c_s \rangle$ of c there is some computable approximation $\langle A_s \rangle$ of A such that $\sum c_s(A_s) < \infty$.*

We say that A obeys c.

K-triviality and the cost function

The DHNS argument shows:

Lemma

If a Δ^0_2 set A obeys the cost function $c(n) = \sum_{y>n} \mu(y)$, then A is K-trivial.

The existence of a non-computable, c.e. K-trivial set then follows from:

Proposition

If c is a cost function then there is some non-computable c.e. set which obeys c.

The construction is the same as in the DHNS argument: to meet the requirement $A \neq \overline{W_e}$, allot 2^{-e} much capital.

Theorem (Nies)

A Δ^0_2 set A is K-trivial if and only if it obeys the cost function $c(n) = \sum_{y>n} \mu(y)$.

Miller-Hirschfeldt and cost functions

Recall the Miller-Hirschfeldt argument. We have a null Π^0_2 class $V = \bigcap V_n$ and build a non-computable c.e. set A, computable from every random set in V:

Enumerate a functional Γ. If Y enters V_n at stage s, set $\Gamma^Y(n) = A_s(n)$. If $Y \in V$, then $\Gamma^Y(n)\downarrow$ for all n.

To ensure that A is not computable, when x enters $W_{e,s}$, we want to enumerate x into A as well. This would make $\Gamma^Y(x)$ wrong for oracles $Y \in V_{x,s}$. Do this if $\lambda(V_{x,s}) < 2^{-e}$, in which case let $G_e = V_{x,s}$.

So $\sum \lambda(G_e) \leqslant 1$. For every random set $Y \in V$, for almost all x, $\Gamma^Y(x) = A(x)$.

Why is A non-computable? Because $\lambda(V_x) \to 0$ (and we make A co-infinite).

Miller-Hirschfeldt and cost functions

In the language of cost functions: let $c(n) = \lambda(V_n)$ (and $c_s(n) = \lambda(V_{n,s})$). The "limit condition" $\lim_n c(n) = 0$ is the fact that $\lambda(V) = 0$.

Suppose that a Δ^0_2 set A obeys c. Let $H_s = V_{x_s,s}$. So $\sum \lambda(H_s) < \infty$. Suppose that $Y \in V$ is random. For all $s \geqslant s^*$, $Y \notin H_s$. If Y enters $V_{x,s}$ (where $s \geqslant s^*$), then $A_s \upharpoonright_{x+1} = A \upharpoonright_{x+1}$, so $A \leqslant_{\mathrm{T}} Y$.

The Miller-Hirschfeldt theorem then follows from the general proposition (every cost function is obeyed by somebody).

Additive cost functions

For the following notion, it makes more sense to write $c(x, s)$ for $c_s(x)$.

Definition (Nies)

A cost function c is additive if it has a monotone approximation $\langle c_s \rangle$ such that $c(s, s) = 0$ for all s, and for all $x < y < z$, $c(x, y) + c(y, z) = c(x, z)$.

Additive cost functions correspond to approximations of left-c.e. reals: the additive cost functions are the cost functions of the form $c(n) = \beta - \beta_n$, where $\langle \beta_n \rangle$ is a non-decreasing sequence of rational numbers converging to β. As a monotone approximation for c we can choose $c_s(n) = \beta_s - \beta_n$ (with $c_s(n) = 0$ for $s < n$).

Additive cost functions and K-triviality

The cost function $c(n) = \sum_{y>n} \mu(y)$ is not additive, but it is dominated by an additive cost function. Let $\Omega = \mu(\omega) = \sum_n \mu(n)$ and $\Omega_s = \sum_{n<s} \mu_s(n)$. Then

$$\sum_{y>n} \mu(y) \leqslant \sum_{y \leqslant n} (\mu(y) - \mu_n(y)) + \sum_{y>n} \mu(y) = \Omega - \Omega_n.$$

It follows that every set which obeys the additive cost function $\Omega - \Omega_n$ is K-trivial.

Proposition (Nies)

Every K-trivial set obeys every additive cost function.

Additive cost functions and K-triviality

We prove the proposition for a c.e. K-trivial set A (for Δ^0_2 sets the proof requires the golden run). Let $c(n) = \beta - \beta_n$ be an additive cost function.

Since $\sum_n (\beta_n - \beta_{n-1}) = \beta < \infty$, optimality of μ shows that $\mu(n) \geqslant^\times (\beta_n - \beta_{n-1})$. Thus, $\mu(A \upharpoonright_n) \geqslant^\times (\beta_n - \beta_{n-1})$. By speeding up, let $\langle A_s \rangle$ be an enumeration of A such that for all s, for all $n \leqslant s$, $\mu_s(A_s \upharpoonright_n) \geqslant \epsilon(\beta_n - \beta_{n-1})$, where ϵ is fixed.

Let x_s be the least x such that $A_{s+1}(x) \neq A_s(x)$. So

$$\sum_s c_s(A_s) = \sum_s c_s(x_s) = \sum_s (\beta_s - \beta_{x_s})$$

(we ignore the case $x_s \geqslant s$). For each s,

$$\epsilon(\beta_s - \beta_{x_s}) \leqslant \sum_{n \in (x_s, s]} \mu_s(A_s \upharpoonright_n).$$

But by the definition of x_s, and since A is c.e., the sets $\{A_s \upharpoonright_n : \ n \in (x_s, s]\}$ are pairwise disjoint, and so

$$\sum_s (\beta_s - \beta_{x_s}) \leqslant \mu(\omega)/\epsilon.$$

Computing K-trivials

Definition

An Auckland test is a null Π^0_2 class $V = \bigcap_n V_n$ with $\lambda(V_n) \leqslant \beta - \beta_n$ for some left-c.e. real β.

Proposition

Let V be an Auckland test. If $Z \in V$ is random, then Z computes every K-trivial set.

In general, we can use cost functions to calibrate the rate of convergence of $\lambda(V_n)$ to 0. If c is a cost function then a "null c-class" is a Π^0_2-class $V = \bigcap_n V_n$ with $\lambda(V_n) \leqslant c(n)$. For "benign" cost functions c we get nested Demuth tests, and the conclusion is that every random set which is not weakly Demuth random computes all SJT sets.

Moving tests

Demuth and friends

Figueira, Hirschfeldt, Miller, Ng and Nies asked how bad must be computable approximations of Δ^0_2 random sets. (For example, they showed that while easily there are random sets $Z = \lim Z_s$ with $\#\{s : Z_{s+1} \upharpoonright_n \neq Z_s \upharpoonright_n\} \leqslant 2^n$, things like $2^n / \log \log n$ are impossible.)

Changes in the set are captured by moving test components. This was introduced by Demuth. In general, a (nested) limit test is a nested sequence $\langle V_n \rangle$ of Σ^0_1 classes with $\lambda(V_n) \leqslant 2^{-n}$ and $V_n = [W_{f(n)}]$ for some Δ^0_2 function f. If $f = \lim_s f_s$, then we let $V_n\langle s \rangle = [W_{f_s(n)}]$. This is the version of V_n at stage s; if $f_{s+1}(n) \neq f_s(n)$ then the version changes at stage s. [We require that $\langle V_n\langle s \rangle\rangle_n$ is nested for each s, and does not exceed the measure bound, i.e. is a ML-test. In counting, we suppose that when V_n changes, so does V_{n+1}.]

Balanced randomness

Nested limit randomness is equivalent to weak 2 randomness. Intermediate notions between limit randomness and ML-randomness are obtained by considering smaller collections of Δ^0_2 index functions. For example, for weak Demuth randomness we require f to be ω-computably approximable, so $\#\{s : V_n\langle s+1\rangle \neq V_n\langle s\rangle\}$ is bounded by some computable function.

This notion was further restricted by FHMNN. A balanced test is a limit test $\langle V_n\rangle$ with $\#\{s : V_n\langle s+1\rangle \neq V_n\langle s\rangle\} \leqslant^\times 2^n$.

For example, the left-to-right approximation $\langle Z_s\rangle$ of a left-c.e. real Z satisfies $\#\{s : Z_{s+1}\upharpoonright_n \neq Z_s\upharpoonright_n\} \leqslant 2^n$, and so a left-c.e. real cannot be balanced random.

The general fact that every reasonable notion of randomness (stronger than ML) is downwards closed in the Turing degrees of ML-random sets shows that no balanced random set can be Turing complete.

Diversion: Turing completeness

Theorem (Franklin,Ng)

The following are equivalent for a random set Z:

1. *There is a Π^0_1 class P and a uniformly c.e., nested sequence $\langle V_n \rangle$ such that $Z \in P \cap \bigcap V_n$ and $\lambda(P \cap V_n) \leqslant 2^{-n}$ (Z is captured by a difference test.)*
2. *Z is captured by a version-disjoint nested Demuth test: a nested Demuth test $\langle V_n \rangle = \lim_s \langle V_n\langle s \rangle \rangle$ with $Z \in \bigcap V_n$ and such that the distinct versions $V_n\langle s \rangle$ are pairwise disjoint.*
3. *$Z \geqslant_{\mathrm{T}} \varnothing'$.*

The version-disjoint nested Demuth test can be taken to be a balanced test: like the proof of the "general fact" above, we let $V_n = \Phi^{-1}[\Omega \upharpoonright_n]$, where $\Phi(Z) = \Omega$; Levin / Miller-Yu gives the measure bound.

Tracing

Tracing is now everywhere. [Lowness for Schnorr randomness. Strong minimal covers. BLR. Lowness for Demuth. SJTs.] For our purposes:

- For an oracle A, a trace^A is a uniformly c.e.^{A} sequence of sets $T(z)_{z<\omega}$ with each $T(z)$ finite. A function f is traced by T if $f(z) \in T(z)$ for all z. An order function h bounds a trace T if $|T(z)| \leqslant h(z)$ for all z.
- For a class of functions $\mathcal{F}$, we say that an oracle A is $\mathcal{F}$-tracing if there is some order function h such that every $f \in \mathcal{F}$ has some trace^A bounded by h. [If $\mathcal{F}$ is sufficiently closed then any h would do.]

Balanced tracing

Theorem (FHMNN)

If Z is random but not balanced random, then it is ω-computably-approximable tracing.

Suppose that $f = \lim_s f_s$ with $\#\{s : f_{s+1}(n) \neq f_s(n)\}$ bounded by $h(n)$. Suppose that $\langle V_n \rangle$ is a balanced test which captures Z.

Idea: for each n, find some level $m(n)$ sufficiently large. If $f_{s+1}(n) \neq f_s(n)$, enumerate $V_{m(n)}\langle s \rangle$ into a Solovay test. So among randoms, each version of $V_{m(n)}$ corresponds to just one version of $f(n)$; so if $Z \in V_{m(n)}[s]$, we enumerate $f_s(n)$ into $T^Z(n)$.

Since $\lambda(V_{m(n)}\langle t \rangle) \leqslant 2^{-m(n)}$, the weight of the Solovay test is bounded by $\sum_n h(n) 2^{-m(n)}$. Choose $m(n)$ to make this finite.

Balanced tracing

What about $|T^Z(n)|$? This is bounded by the number of versions of $V_{m(n)}$ which contain Z. Enumerate these versions as $V_{m(n),1}, V_{m(n),2}, \ldots, V_{m(n),k}$, so $k \leqslant 2^{m(n)}$ (a multiplicative constant can be ignored).

Then $\sum_{i \leqslant k} \lambda(V_{m(n),i}) \leqslant 1$. The measure of the set of reals which can belong to at least 2^n many of these versions is at most 2^{-n}. [Consider $g = \sum_{i \leqslant k} 1_{V_{m(n),k}}$ and its integral.] So this is another Solovay test which Z will avoid, so $|T^Z(n)| \leqslant 2^n$.

Note that if $\langle V_n \rangle$ were version-disjoint, then Z would compute f (and we don't need the second Solovay test). So we get another proof that a random set captured by a version-disjoint nested Demuth test is Turing complete.

Pushing the argument

Now we want to trace more functions. The trade-off would be a weakening of balanced randomness. Suppose that $f = \lim_s f_s$ is Δ^0_2, but we don't have a computable bound on the number of mind-changes. We don't know how to pick the levels $m(n)$.

What we'll do is change the level $m(n)$ each time $f(n)$ changes: Start with $m_0(n) = n$; if $f_{s+1}(n) \neq f_s(n)$, enumerate $V_{m_s(n)}\langle s\rangle$ into a Solovay test, wait for $V_{m_s(n)}$ to change, and let $m_{s+1}(n) = m_s(n) + 1$. The problem is bounding $\left|T^Z(n)\right|$. With a balanced test, the opponent can "save up" their moves: start changing $V_{m_s(n)}$ only after $m_s(n)$ is reached. We need the moves to be coordinated.

OW randomness

Definition

An Oberwolfach test is a limit test $\langle V_n \rangle = \langle V_n\langle s\rangle\rangle$ such that if $s < t$ are successive stages at which V_{n+1} changes, then V_n changes at either s or t.

An OW test is a balanced test. Furthermore, the version-disjoint balanced test used in the characterisation of incomplete randomness is actually an OW test. So an OW random set is incomplete.

Let $\langle V_n \rangle$ be an OW test. Let $\alpha_t(n) = 0$ if at the previous stage at which V_n changed, V_{n-1} changed as well, 1 otherwise. Then $\alpha_t \leqslant \alpha_{t+1}$, and $\alpha = \lim \alpha_t$ is finite. And the sum of the measures of all versions $V_{m_s(n)}\langle s\rangle$ is bounded by α.

Computational strength

In fact f can be Σ^0_2 (partial): so if Z is random but not OW random, then $\emptyset' \leqslant_{\mathrm{JT}} Z$. Consequently (Simpson), Z is superhigh.

In comparison, all superlow random sets are balanced random, but there is a low, unbalanced random set.

Something peculiar: the bound on the trace T^Z can be any upper c.e. summable function, i.e. a constant multiple of 2^K. There are no incomplete c.e. sets with this property.

Equivalence

Proposition

Auckland randomness and Oberwolfach randomness coincide.

$\Rightarrow$: Let $\langle U_n \rangle$ be an OW test. Let $V_n = \bigcup_{s>n} U_n\langle s \rangle$. Define α as above; let $\hat{\alpha}_n = \alpha_n - 2^{-n}$.

$\Leftarrow$: Let $V = \bigcap V_n$ be an Auckland test, with $\lambda(V_n) \leqslant \beta - \beta_n$. We will let $U_n\langle s \rangle$ copy $V_{m_s(n)}$ for some level $m_s(n)$; we keep at it as long as $\beta_s - \beta_{m_s(n)} \leqslant 2^{-n}$. If this boundary is crossed at stage t, we let $m_t(n) = t$.

Smart K-trivials

A smart K-trivial

Theorem

There is a K-trivial set which is not computable from any Balanced random set.

Corollary

There is a K-trivial set which is not computable from any superlow random set.

Corollary

There is a K-trivial set which is not computable from both halves of a random set.

Proof.

Let $X = X_0 \oplus X_1$ be random. FHMNN showed that if Z is ω-c.a. tracing, then every random^Z set is weakly Demuth random, and so balanced random. So at least one of X_0 or X_1 is balanced random. □

The construction

Fix a universal Turing functional Θ ($\Theta(0^e 1X) = \Phi_e(X)$). Since balanced randomness is invariant under appending finite strings, for any set A, if A is computable from a balanced random set, then $\Theta(Z) = A$ for some balanced random set Z.

So we enumerate a c.e. K-trivial set A and try to put a balanced test $\langle G_n \rangle$ on $\Theta^{-1}\{A\}$. We pick a witness $v_{n,s}$ and let $G_n\langle s \rangle = \Theta^{-1}[A \upharpoonright_{v_n+1} [s]]$. When we decide to change G_n, we pick a new large witness $v_{n,t+1}$.

We need to make sure that $\lambda(G_n) \leqslant 2^{-n}$, so we change G_n when we observe that $\lambda(G_n\langle s \rangle) > 2^{-n}$. We also need to ensure that G_n eventually stabilizes. We could, for example, enumerate $v_{n,s}$ into A each time we change G_n. Then we actually get a version-disjoint test, G_n changes at most 2^n times, but A will be Turing complete. (This gives another way to capture complete random sets by version-disjoint nested Demuth tests.)

Construction of a smart K-trivial

The problem with the naïve attempt is that sometimes the cost $c_s(v_{n,s})$ is too high. To make A K-trivial, we need to ensure that the sum of these costs, for the stages at which we enumerate $v_{n,s}$ into A, is finite.

Let E_s be the "error set": all oracles which have been proven, by stage s, to compute A incorrectly:

$$E_s = \{Z : \exists v \;\; \Theta_s(Z, v) = 0 \;\;\&\;\; v \in A_s\}.$$

The plan is to count the cost of enumerating $v_{n,s}$ into A against the potential rise in $\lambda(E)$. When $v_{n,s}$ is enumerated into A, all of $G_n[s]$ is added to E. But note that some of $G_n[s]$ may already be in E due to the action of G_m for $m > n$.

Construction of a smart K-trivial

So when we see that $\lambda(G_n[s]) > 2^{-n}$, we enumerate $v_{n,s}$ into A if $c(v_n)[s] \leqslant 2^{-n}/2$ and $\lambda(E \cap G_n)[s] \leqslant 2^{-n}/2$. Otherwise, we change G_n without enumerating $v_{n,s}$ into A.

Suppose that $v_{n,s}$ is enumerated into A. The cost $c(v_n)[s]$ is at most $2^{-n}/2$; we have $\lambda(G_n[s]) > 2^{-n}$ and $\lambda(E \cap G_n)[s] \leqslant 2^{-n}/2$; so $\lambda(E_{s+1} - E_s) \geqslant 2^{-n}/2 \geqslant c(v_n)[s]$. So $\lambda(E)$ is a global bound on the cost.

Note that unlike other constructions, we cannot in advance bound the contribution to the cost of any particular G_n.

Construction of a smart K-trivial

We still need to show that G_n only changes $O(2^n)$ many times. We use the fact that c is 2^n-benign: there can be at most 2^n many stages $s_1 < s_2 < \ldots$ such that $c_{s_{i+1}}(s_i) \geqslant 2^{-n}$. We then check what can prompt a change in G_n at some stage s:

1. $v_{n,s}$ is enumerated into A. We observed that in this case, $\lambda(E_{s+1} - E_s) \geqslant 2^{-n}/2$. This can happen at most $2 \cdot 2^n$ many times.
2. $c(v_n)[s] > 2^{-n}/2$. Since the new v_n is chosen fresh, benignity shows this can happen at most $2 \cdot 2^n$ times.
3. $\lambda(E \cap G_n)[s] > 2^{-n}/2$. Let $t < s$ be the stage at which $v_{n,s}$ was chosen. $v_{n,s}$ is large at stage t. So all of $E \cap G_n[s]$ was added to E after stage t. So again this can happen at most $2 \cdot 2^n$ times.

A smartest K-trivial

Theorem
There is a K-trivial set which is not computable from any OW random set.

Corollary
A random set computes all K-trivial sets if and only if it is not OW random.

The smartest K-trivial set A satisfies:

- No low, indeed non-superhigh, random set can compute A.
- A random set computes A if and only if it computes all K-trivial sets.

So we see that if every K-trivial set is computable from an incomplete random set, then one incomplete random set computes all K-trivial sets.

Construction of a smartest K-trivial

We now need to make use of c being additive. To make the movements of test components cohere, we need to sometimes change G_n even if $\lambda(G_n[s]) \leqslant 2^{-n}$. We change G_n if:

1. either $\lambda(G_n[s]) > 2^{-n}$;
2. or $\lambda(E \cap G_n) + c(v_n)\ [s] \geqslant 2^{-n}$.

If (1) holds but (2) fails, we can enumerate $v_{n,s}$ into A.

(Enumerating $v_{n,s}$ into A may prompt $\lambda(E \cap G_m) + c(v_m)$ to rise beyond 2^{-m} for some $m < n$. We need to act for such m.)

Another weak reducibility

Let $A \leqslant_{\mathrm{ML}} B$ if every random set which computes B also computes A.

- There is a greatest K-trivial ML degree.

There are infinitely many K-trivial ML degrees.

We conjecture that there is no greatest SJT ML degree.

Martingale convergence

Background

A programme in algorithmic randomness is to match levels of randomness with effective versions of almost-everywhere theorems of analysis. (Demuth randomness was introduced with this programme in mind). A prominent example is differentiability. For example,

Theorem (Brattka,Miller,Nies)

A point $x \in [0,1]$ is computably random if and only if every computable monotone function $f : [0,1] \to \mathbb{R}$ is differentiable at x.

Lebesgue and Doob

We investigate two related theorems.

Lebesgue Density Theorem

Let $A \subseteq 2^\omega$ be measurable. For almost all $X \in A$,

$$\lim_{n\to\infty} \lambda\big(A \big| [X\restriction_n]\big) = 1.$$

Doob Martingale Convergence Theorem

Let $M\colon 2^{<\omega} \to \mathbb{R}^+$ be a martingale. For almost all $X \in 2^\omega$,

$$\lim_n M\big(X\restriction_n\big)$$

exists.

Effective versions

An effective version of a classical theorem is obtained by restricting it to a countable, effective collection of objects (e.g. "monotone function" $\mapsto$ "computable monotone function").

For density, we choose the collection of Π^0_1 classes (Σ^0_1 classes would make the notion trivial).

Definition

A real $X \in 2^\omega$ is a (dyadic) density point if for every Π^0_1 class P, if $X \in P$, then

$$\rho_2(P|X) = \liminf_{n\to\infty} \lambda(P|[X\upharpoonright_n]) = 1.$$

For martingale convergence, we choose the collection of (lower-)c.e. martingales.

Martingale convergence and density

They are not equivalent in general: a 1-generic set is a density point, whereas martingale convergence implies randomness.

Proposition

If every c.e. martingale converges on $X \in 2^\omega$, then X is a density point.

The converse holds for random sets [Andrews, Cai, Diamondstone, Miller, Lempp.]

Martingale convergence and density

Let P be a Π^0_1 class. For $\sigma \in 2^{<\omega}$, let $M(\sigma) = \lambda(P|\sigma)$. M is a right-c.e. martingale. If $1 - M$ converges on X then so does M, in which case $\lim_n \lambda(P|[X\restriction_n])$ exists.
But what is that value?

Lemma
If X is random, P is a Π^0_1 class and $X \in P$, then

$$\limsup_n \lambda(P|[X\restriction_n]) = 1.$$

Proof.
Construct U_n, a c.e. antichain, by induction. Given U_n, U_{n+1} is the union of $P_t \cap [\sigma]$ where $\sigma \in U_n$ and t is least such that $\lambda(P_t|\sigma) < q$. $\square$

Martingale convergence and OW randomness

Theorem

If X is Oberwolfach random, then every c.e. martingale converges on X.

Up-crossings

Suppose that a martingale M does not converge on $X \in 2^\omega$. There are rational numbers $\alpha < \beta$ such that $M(X \upharpoonright_n)$ is infinitely often below α and infinitely often above β. A pair $\sigma \prec \tau$ such that $M(\sigma) < \alpha$ and $M(\tau) > \beta$ is an up-crossing.
Having many up-crossings is rare (hence Doob's theorem):

Lemma

Let U_n be the open set generated by the strings which contain n many disjoint up-crossings. Then $\lambda(U_n) \leqslant (\alpha/\beta)^n$.

This is deduced from Kolmogorov's inequality: if $M(\sigma) < \alpha$, then the measure of extensions τ of σ with $M(\tau) > \beta$ is at most α/β: there is no more capital to spread to more measure.

Capturing up-crossings

In the effective setting, we want to capture up-crossings by an OW test. Since the martingale is c.e. (and not computable), up-crossings eventually appear; but also, fake up-crossings appear and then disappear.

But for a fake up-crossing to disappear, global capital must be spent. Suppose that $M_s(\sigma) < \alpha$ but $M(\sigma) > \gamma$ (where $\gamma > \alpha$). Then $M(\langle\rangle) - M_s(\langle\rangle) > (\gamma - \alpha)2^{-|\sigma|}$ (think of the martingale $M - M_s$ and Kolmogorov's inequality).

Capturing up-crossings

Fix $\alpha < \gamma < \beta$ (we assume that the numbers will work nicely). We restart a version of G_n when $M(\langle\rangle)$ crosses a multiple of 2^{-n}. Once a version is started, we enumerate into G_n strings which appear to contain n disjoint up-crossings.

When calculating measure, each G_n is divided into the "good" part: strings which may fail to contain n many (α, β)-up-crossings, but contains n many genuine (γ, β)-up-crossings; and the "junk" part: those strings which appeared to contain n many (α, β)-up-crossings but really, fail to contain even n many (γ, β)-up-crossings.

The measure of the good part is given by Doob's calculation. Also, there cannot be too much junk: once too much junk is accumulated, $M(\langle\rangle)$ must rise and a new version is started (and old junk discarded).

A corollary

Recall:

- A random set which is not OW random computes all K-trivial sets.
- C.e. martingales converge on an OW random set, and so each OW random set is a density point.

Corollary

If X is random but is not a density point, then X computes every K-trivial set.

Over to Joe...