Partial orders and reverse mathematics

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Outline

1. **Linear extensions preserving finiteness properties**

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2. Decomposing initial intervals

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2. **Decomposing initial intervals**
3. **Counting initial intervals**

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Linear extensions preserving finiteness properties

Some finiteness properties

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- ω-like if every element of P has finitely many predecessors;
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- ζ-like if for every pair of elements $x, y \in P$ there exist finitely many z such that $x <_P z <_P y$.

Linear extensions preserving finiteness properties

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Every ω-like partial order has a linear extension which is also ω-like.

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Every ζ-like partial order has a linear extension which is also ζ-like.

Reverse mathematics results: I

Theorem

Over RCA_0*, the following are pairwise equivalent:*

1. $\mathsf{B}\mathbf{\Sigma}^0_2$: $\forall i < n \, \exists m \, \varphi(i,n,m) \implies \exists k \, \forall i < n \, \exists m < k \, \varphi(i,n,m)$ *where* φ *is any* $\mathbf{\Sigma}^0_2$ *formula;*

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Reverse mathematics results: II

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Over RCA_0*, the following are equivalent:*

1. ACA_0;
2. *every* $\omega + \omega^*$*-like partial order has a linear extension which is* $\omega + \omega^*$*-like.*

Decomposing initial intervals

Initial intervals and ideals

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- $S \subseteq P$ is a strong antichain in P if
$$\forall x, y \in S(x \neq y \implies \neg\exists z \in P\, x, y \leq_P z);$$

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$$\forall x, y \in S(x \neq y \implies \neg \exists z \in P\, x, y \leq_P z);$$
- $I \subseteq P$ is an initial interval of P if
$$\forall x, y \in P(x \leq_P y \wedge y \in I \implies x \in I);$$

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- $I \subseteq P$ is an initial interval of P if
$$\forall x, y \in P(x \leq_P y \wedge y \in I \implies x \in I);$$
- An initial interval A of P is an ideal if
$$\forall x, y \in A\, \exists z \in A(x \leq_P z \wedge y \leq_P z).$$

Three theorems

Theorem (Bonnet, 1975)

A partial order P is FAC if and only if every initial interval of P is a finite union of ideals.

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Theorem (Erdös-Tarski, 1943)

If a partial order P has no infinite strong antichains then there is a finite bound on the size of strong antichains in P.

Theorem

A partial order has no infinite strong antichains if and only if it is a finite union of ideals.

Reverse mathematics results

Theorem

Over RCA_0*, the following are pairwise equivalent:*

1. ACA_0*;*
2. *every partial order with no infinite strong antichains has a finite bound on the size of strong antichains;*
3. *every partial order with no infinite strong antichains is a finite union of ideals;*
4. *if a partial order is FAC then every initial interval is a finite union of ideals.*

Initial interval separation

Initial interval separation

Lemma

Over RCA_0*, the following are equivalent:*

1. WKL_0*;*
2. $\mathbf{\Sigma}^0_1$ *initial interval separation Let* P *be a partial order and* $\varphi(x)$*,* $\psi(x)$ *be* $\mathbf{\Sigma}^0_1$ *formulas with one distinguished free number variable.*
   If $(\forall x, y \in P)(\varphi(x) \land \psi(y) \implies y \not\leq_P x)$*, then there exists an initial interval* I *of* P *such that*
   $$(\forall x \in P)(\varphi(x) \implies x \in I) \text{ and } (\forall x \in P)(\psi(x) \implies x \notin I).$$
3. *initial interval separation Let* P *be a partial order and suppose* $A, B \subseteq P$ *are such that* $(\forall x \in A)(\forall y \in B)(y \not\leq_P x)$*. Then there exists an initial interval* I *of* P *such that* $A \subseteq I$ *and* $B \cap I = \emptyset$*.*

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Lemma

Over RCA_0*, the following are equivalent:*

1. WKL_0*;*
2. *every antichain D of a partial order P is contained in an initial interval I such that $\forall x \in D \, \forall y \in I \; x \not<_P y$.*

Unprovability in RCA_0

Theorem

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There exists a computable partial order P *with an infinite computable antichain such that any infinite computable initial interval of* P *contains an element* y *such that* $P \setminus \downarrow y$ *is finite.*

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Proof of Theorem from Lemma.

Let I be a computable initial interval of P.

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Let I be a computable initial interval of P.
If I is finite then $I = \bigcup_{x \in I} {\downarrow} x$.

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Proof of Theorem from Lemma.

Let I be a computable initial interval of P.
If I is finite then $I = \bigcup_{x \in I} {\downarrow} x$.
If I is infinite then fix $y \in I$ as in Lemma: then $I = {\downarrow} y \cup \bigcup_{x \in I \setminus {\downarrow} y} {\downarrow} x$. □

Counting initial intervals

$\mathcal{I}(P)$ and its size

Let $\mathcal{I}(P)$ the collection of initial intervals of P.

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P has countably many initial intervals
if there exists $\{I_n : n \in \mathbb{N}\}$ such that $\forall I \in \mathcal{I}(P)\, \exists n \in \mathbb{N}\, I = I_n$.

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P has uncountably many initial intervals
if it does not have countably many initial intervals.

P has perfectly many initial intervals
if there exists a nonempty perfect tree $T \subseteq 2^{<\mathbb{N}}$ such that $[T] \subseteq \mathcal{I}(P)$.

The tree $T(P)$

The tree of finite approximations of initial intervals of P is $T(P) \subseteq 2^{<\mathbb{N}}$: $\sigma \in T(P)$ iff for all $x, y < |\sigma|$:

- $\sigma(x) = 1$ implies $x \in P$;
- $\sigma(y) = 1$ and $x \leq_P y$ imply $\sigma(x) = 1$.

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"P has perfectly many initial intervals" is provably $\mathbf{\Sigma}^1_1$ within RCA_0;

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"P has perfectly many initial intervals" is provably $\mathbf{\Sigma}^1_1$ within RCA_0;

"P has uncountably many initial intervals" is provably $\mathbf{\Sigma}^1_1$ within ATR_0.

The main theorem

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If an infinite partial order P is scattered (no copy of $\mathbb{Q}$ in P) and FAC, then $|\mathcal{I}(P)| = |P|$.

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If an infinite partial order P is scattered (no copy of $\mathbb{Q}$ in P) and FAC, then $|\mathcal{I}(P)| = |P|$.

Theorem

A countable partial order P is scattered and FAC if and only if $\mathcal{I}(P)$ is countable.

Provability in WKL_0

Lemma

RCA_0 *proves that both* $\mathbb{Q}$ *and the infinite antichain have perfectly many initial intervals.*

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Over RCA_0*, the following are equivalent:*

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2. *If* $Q \subseteq P$ *then* $\mathcal{I}(Q) = \{J \cap Q \colon J \in \mathcal{I}(P)\}$.

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Theorem

WKL_0 *proves that if a partial order has countably many initial intervals, then it is scattered and FAC.*

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RCA_0 *does not prove that if a partial order has countably many initial intervals, then it is FAC.*

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There exists a computable partial order P with an infinite computable antichain such that any infinite computable initial interval of P contains an element y such that $P \setminus \downarrow y$ is finite.

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Any computable initial interval of P is either finite or cofinite in P.

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Proof of Theorem from Lemma.

Any computable initial interval of P is either finite or cofinite in P.
Let $\{I_n : n \in N\}$ computably list all finite and cofinite subsets of P. □

A classic reverse mathematics result

Theorem (Clote, 1989)

Over ACA_0*, the following are equivalent:*

1. ATR_0*;*
2. *linear orders have either countably many or perfectly many initial intervals;*
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Thus "FAC scattered partial orders have countably many initial intervals" implies ATR_0.

A preliminary lemma

Lemma

ACA_0 *proves that if* P *has perfectly many initial intervals, then there exists* $x \in P$ *such that either*

- $x^\perp$ *has uncountably many initial intervals, or*
- *both* $\downarrow x$ *and* $\uparrow x$ *have uncountably many initial intervals.*

A tree construction

Suppose P has uncountably many initial intervals.
Let $\mathrm{Fin}(P)$ the set of finite subsets of P. If $F, G, H \in \mathrm{Fin}(P)$, let

$$P_{F,G,H} = \bigcap_{x \in F} \downarrow x \cap \bigcap_{x \in G} \uparrow x \cap \bigcap_{x \in H} x^{\perp}.$$

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The statement is $\mathbf{\Pi}^1_2$ and cannot imply $\mathbf{\Pi}^1_1$-CA$_0$.

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ATR_0 *proves that for all* X *and* Y *there exists a countable coded* ω*-model* M *such that* $X, Y \in M$*, and* M *satisfies both* $\mathbf{\Sigma}_1^1$-DC_0 *and* ATR_0^X*.*

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The key observation is that each $T(P_\sigma)$ is P-computable.

The reverse mathematics result

Theorem

Over ACA_0*, the following are equivalent:*

1. ATR_0*;*
2. *FAC scattered partial orders have countably many initial intervals;*
3. *scattered linear orders have countably many initial intervals.*

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Is "every non-scattered partial order has uncountably many initial intervals" provable in RCA_0?