Complex isomorphisms of simple computable structures

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Buenos Aires, 2013.

In this talk:

1. all structures are computable,
2. all isomorphisms are Δ^0_2,
3. all our structures are algebraically simple (not far from being sets).

Categoricity

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Describe Δ^0_n-categorical members of a given class K of computable structures.

Even for $n = 1$ the problem is too hard in general (Downey, Kach, Lempp, Lewis, Montalban, and Turetsky).

Computable categoricity

Theorem (Goncharov, Remmel, Nurtazin, LaRoche, Smith et al.)

Computably categorical members can be characterized in the following classes of computable structures:

- Boolean algebras,
- linear orderings,
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Not much is known about Δ_2^0-categorical members of these classes.

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Miller investigated the question in the class of algebraic fields.

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A multi-ciclic group is a direct sum of cyclic and quasi-cyclic abelian p-groups.

Equivalence structures

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In contrast to linear orders and Boolean algebras, both equivalence structures and multi-cyclic groups have nice and simple algebraic classifications.

Case of study: coding a set

Definition

For a set $X \subset \omega$, let $E(X)$ be an equivalence structure with ω-many infinite classes and exactly one class of size n for each $n \in X$.

Say that an infinite Σ^0_2 set X is categorical if the computable $E(X)$ is Δ^0_2-categorical.

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Recall a Σ^0_2 set $S \subseteq \omega$ is semi-low$_{1.5}$ if

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Theorem

1. *Each infinite d.c.e. semi-low$_{1.5}$ set is not categorical.*
2. *Some infinite semi-low$_{1.5}$ set is categorical. Some d-c.e. set is categorical.*

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How much do these notions differ?

Categoricity bounding vs. (none-)l.m. bounding

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Theorem

For a c.e. degree $\mathbf{a}$*, the following are equivalent:*

1. $\mathbf{a}$ *is high.*
2. *There exists an infinite categorical set* $X \leq_T \mathbf{a}$.
3. *(Downey, Kach, Turetsky) There exists an infinite* $X \leq_T \mathbf{a}$ *such that* X *is not limitwise monotonic.*

Thus, c.e. degrees do not see the difference. The proof of $1 \Leftrightarrow 2$ has nothing to do with limitwise monotonicity.

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Proposition

If E is Δ^0_2-categorical, then its condensation is Δ^0_2-categorical as well.

The annoying problem

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Proof.

A $0'''$ argument, to be written up. □

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There are also uniform versions of Δ^0_2-categoricity such as:

1. relative Δ^0_2-categoricity,
2. uniform Δ^0_2-categoricity,
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Theorem (CCHM; Kach and Turetsky; Downey, M., Ng)

All these notions are different in the context of equivalence relations, and all are not the same as (plain) Δ^0_2-categoricity.

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A multi-cyclic group with infinitely many infinite quasi-cyclic summands is effectively Δ^0_2-categorical if, and only if, the naturally associated equivalence structure is effectively Δ^0_2-categorical.

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Corollary

There exists a Δ^0_2-categorical multi-cyclic group having infinitely many quasi-cyclic summands. (Answers a question left open by CCHM)

Multi-cyclic groups

Comments on the proof:

1. (Effective) Δ^0_2-categoricity in such groups is regulated by the complexity of height-function. (The proof uses a refinement of the first half of Kaplansky's book.)
2. We don't know if the theorem holds for plain Δ^0_2-categoricity (conjecture: no).
3. A direct proof of the Corollary, without using the Theorem, would be problematic.

We conclude by giving some further properties of effective Δ^0_2-categoricity in the context of equivalence structures and comparing them to the plain case.

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There are some further nice results that we skip.

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We obtained several (mostly negative) results towards

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... but it is a different paper and a different story.

Thanks (in Russian)

SPASIBO