Lebesgue density in Π^0_1 classes (and K-triviality)

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+ many others

Day 1

Lebesgue density

Notation

For $\sigma \in 2^{<\omega}$, let $\mu_\sigma(C) = \dfrac{\mu([\sigma] \cap C)}{\mu([\sigma])}$.

(Here, μ is the standard Lebesgue measure on Cantor space.) So $\mu_\sigma(C)$ is the fraction of $[\sigma]$ occupied by C.

Definition

Let $C \subseteq 2^\omega$ be measurable. The lower density[1] of C at X is

$$\rho(C \mid X) = \liminf_n \mu_{X \upharpoonright n}(C).$$

Lebesgue Density Theorem

If $C \subseteq 2^\omega$ is measurable, then $\rho(C \mid X) = 1$ for almost every $X \in C$.

[1] We use *dyadic* density throughout, not density on the real interval. This simplifies the proofs considerably, but does not change the results.

Lebesgue density for Π^0_1 classes

We want to understand the density points of Π^0_1 classes.

Definition

- $X \in 2^\omega$ is a density-one point if $\rho(C \mid X) = 1$ for every Π^0_1 class C containing X.
- $X \in 2^\omega$ is a positive density point if $\rho(C \mid X) > 0$ for every Π^0_1 class C containing X.

Very basic observations:

1. Density-one $\implies$ positive density.
2. Almost every $X \in 2^\omega$ is a density-one point.
3. Every 1-generic is a density-one point, so it is not a randomness notion.

We avoid 3 by restricting our attention to Martin-Löf random X.

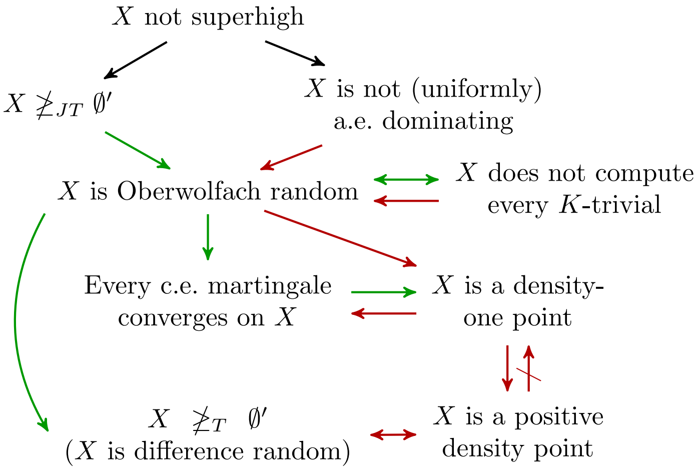
What Noam told us

Let X be Martin-Löf random. The following implications hold:

$$
\begin{tikzcd}
 & X \text{ not superhigh} \arrow[dl] & \\
X \not\geq_{JT} \emptyset' \arrow[dr] & & \\
 & X \text{ is Oberwolfach random} \arrow[r, leftrightarrow] \arrow[d] \arrow[dd, bend right] & \begin{array}{c} X \text{ does not compute} \\ \text{every } K\text{-trivial} \end{array} \\
 & \begin{array}{c} \text{Every c.e. martingale} \\ \text{converges on } X \end{array} \arrow[r] & \begin{array}{c} X \text{ is a density-} \\ \text{one point} \end{array} \\
 & \begin{array}{c} X \not\geq_T \emptyset' \\ (X \text{ is difference random}) \end{array} &
\end{tikzcd}
$$

What I am planning to tell you

Let X be Martin-Löf random. The following implications hold:



Bounding density drops

Lemma

Let $C \subseteq 2^\omega$ be a closed set. Fix $\varepsilon \in (0,1)$ and let

$$U = \{X \colon (\exists k)\ \mu_{X \upharpoonright k}(C) < \varepsilon\}.$$

Then

1. $\mu(U \cap C) \leq \varepsilon$, and
2. $\mu(U) \leq \dfrac{1-\mu(C)}{1-\varepsilon}$.

Proof.

Let D be the minimal strings σ such that $\mu_\sigma(C) < \varepsilon$. Then D is prefix-free and $[D] = U$.

For 1,

$$\mu(U \cap C) = \mu([D] \cap C) = \sum_{\sigma \in D} \mu([\sigma] \cap C) \leq \sum_{\sigma \in D} \mu([\sigma])\varepsilon = \mu([D])\varepsilon \leq \varepsilon.$$

Bounding density drops

For 2,

$$
\begin{aligned}
1-\mu(C) \geq \mu([D]\smallsetminus C) &= \sum_{\sigma\in D}\mu([\sigma]\smallsetminus C) \\
&= \sum_{\sigma\in D}\mu([\sigma])-\mu([\sigma]\cap C) \geq \sum_{\sigma\in D}\mu([\sigma])-\mu([\sigma])\varepsilon \\
&= \sum_{\sigma\in D}\mu([\sigma])(1-\varepsilon) = \mu([D])(1-\varepsilon) = \mu(U)(1-\varepsilon).
\end{aligned}
$$

□

An application of 2:

Theorem (Bienvenu, Greenberg, Kučera, Nies, Turetsky)

If $X \in 2^\omega$ is Oberwolfach random, then it is a density-one point.

Proof. . .

Difference randomness and positive density

Definition

- A difference test is a Π^0_1 class C and an effective sequence $\{V_n\}_{n\in\omega}$ of Σ^0_1 classes such that $\mu(V_n \cap C) \leq 2^{-n}$.
- $X \in 2^\omega$ is difference random if for every difference test $C, \{V_n\}_{n\in\omega}$, there is an n such that $X \notin V_n \cap C$.

Theorem (Franklin, Ng)

X is difference random iff X is Martin-Löf random and $X \not\geq_T \emptyset'$.

Proof. . .

Theorem (Bienvenu, Hölzl, M., Nies)

X is difference random iff X is a ML-random positive density point.

Proof. . .

Aside: Cupping with incomplete random sets

Definition (Kučera 2004)

$A \in 2^\omega$ is weakly ML-cuppable if there is a Martin-Löf random $X \not\geq_T \emptyset'$ such that $A \oplus X \geq_T \emptyset'$. If one can choose $X <_T \emptyset'$, then A is ML-cuppable.

Question (Kučera)

Can the K-trivial sets be characterized as either

1. not weakly ML-cuppable, or
2. $\leq_T \emptyset'$ and not ML-cuppable?

Compare this to:

Theorem (Posner, Robinson)

For every $A >_T \emptyset$ there is a 1-generic X such that $A \oplus X \geq_T \emptyset'$. If $A \leq_T \emptyset'$, then also $X \leq_T \emptyset'$.

Aside: Cupping with incomplete random sets

Question (Kučera 2004)

Can the K-trivial sets be characterized as either

1. not weakly ML-cuppable, or
2. $\leq_T \emptyset'$ and not ML-cuppable?

Answer (Day, M.): Yes, both.

Theorem (Day, M.)

If A is not K-trivial, then it is weakly ML-cuppable. If $A <_T \emptyset'$ is not K-trivial, then it is ML-cuppable.

These are proved by straightforward constructions. It is the other direction I want to focus on.

Theorem (Day, M.)

If A is K-trivial, then it is not weakly ML-cuppable.

We need a Lemma

Assume that A is K-trivial, X is ML-random, and $C \subseteq 2^\omega$ is a $\Pi^0_1[A]$ class containing X. Then there is a Π^0_1 class $D \subseteq C$ containing X.

Proof.

Let $F \subseteq 2^{<\omega}$ be an A-c.e. set such that $2^\omega \setminus C = [F]$. We may assume that F is prefix-free, so by the optimality of K^A, there is a c such that $K^A(\sigma) \leq |\sigma| + c$ for all $\sigma \in F$. But A is low for K, so there is a d such that $K(\sigma) \leq |\sigma| + d$ for all $\sigma \in F$.

Let $G = \{\sigma \colon K(\sigma) \leq |\sigma| + d\}$. Note that

1. G is c.e.,
2. $G \supseteq F$, and
3. $\sum_{\sigma \in G} 2^{-|\sigma|} \leq \sum_{\sigma \in G} 2^{-K(\sigma)+d} \leq 2^d < \infty$.

Because X is Martin-Löf random and G is a *Solovay test*, there are only finitely many $\sigma \in G$ such that $\sigma \prec X$. No such σ is in F, so we may remove them from G while preserving 1-3. Let $D = 2^\omega \setminus [G]$. Note that D is a Π^0_1 class, $D \subseteq C$, $X \in D$. $\square$

Aside: Cupping with incomplete random sets

Theorem (Day, M.)

If A is K-trivial, then it is not weakly ML-cuppable.

Proof.

Let A be K-trivial, X Martin-Löf random, and $A \oplus X \geq_T \emptyset'$. We will show that $X \geq_T \emptyset'$.

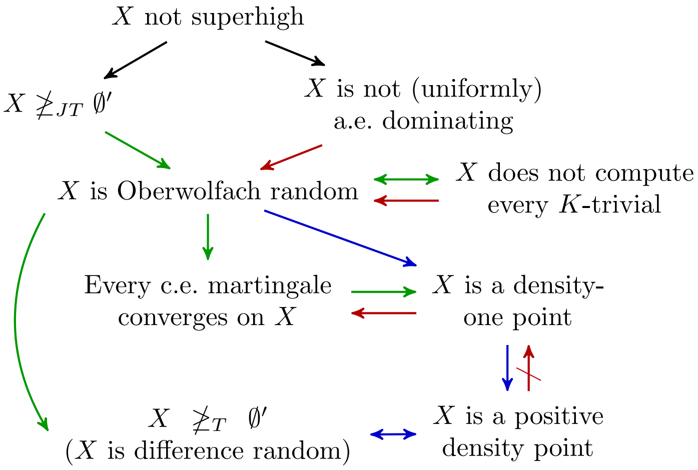
Because A is K-trivial it is low ($\emptyset' \geq_T A'$), hence $A \oplus X \geq_T A'$. It is also low for random, so X is Martin-Löf random relative to A. Therefore, by the Bienvenu et al. result relativized to A, there is a $\Pi^0_1[A]$ class C containing X such that $\rho(C \mid X) = 0$.

By the lemma, there is a Π^0_1 class $D \subseteq C$ containing X. But then $\rho(D \mid X) = 0$, so by the Bienvenu et al. result, $X \geq_T \emptyset'$. In other words, X does not witness the weak ML-cuppability of A. □

Day 2

Where are we now?

Let X be Martin-Löf random. The following implications hold:



Madison tests

Andrews, Cai, Diamondstone, Lempp and M. gave a test characterization of Martin-Löf random density-one points.

Definition

The weight of $U \subseteq 2^{<\omega}$ is $\mathrm{wt}(U) = \sum_{\sigma \in U} 2^{-|\sigma|}$.

Notation: For $\sigma \in 2^{<\omega}$, $[\sigma]^{\prec} = \{\tau \in 2^{<\omega} \mid \sigma \prec \tau\}$.

Definition

A Madison test is a finite weight Δ^0_2 set $U \subseteq 2^{<\omega}$ with a distinguished sequence $\{U_s\}_{s\in\omega}$ of finite approximations such that

1. $\tau \in U_s \smallsetminus U_{s+1} \implies (\exists \sigma \prec \tau)\ \sigma \in U_{s+1} \smallsetminus U_s$, and
2. $\mathrm{wt}(U_s \cap [\sigma]^{\prec}) > 2^{-|\sigma|} \implies \sigma \in U_s$.

$X \in 2^\omega$ passes a Madison test U if at most finitely many prefixes of X are in U.

Madison tests and density

Theorem (Andrews, et al.)

The following are equivalent for $X \in 2^\omega$:

1. X is a Martin-Löf random density-one point,
2. X passes all Madison tests,
3. Every c.e. martingale converges on X.

Proof.

$1 \implies 2$: Suppose that X is ML-random but fails the Madison test U. Uniformly in n, build a Σ^0_1 class S_n such that $\mu(S_n) \leq 2^{-n}\,\mathrm{wt}(U)$ and for each $\sigma \in U$ we have $\mu_\sigma(S_n) \geq 2^{-n}$. To do this, associate every $\tau \in U_s \smallsetminus U_{s+1}$ with a prefix $\sigma \in U_{s+1} \smallsetminus U_s$. When σ enters U_{s+1}, it inherits the part of S_n that is owned by each associated τ. Then add more to S_n until $\mu_\sigma(S_n) \geq 2^{-n}$.

Because $\{S_n\}_{n\in\omega}$ is a Martin-Löf test, there is an n such that $X \notin S_n$. So $C = 2^\omega \smallsetminus S_n$ is a Π^0_1 class containing X and such that $\rho(C \mid X) \leq 1 - 2^{-n}$.

Madison tests and martingale convergence

$2 \implies 3$: Assume that X passes all Madison tests.

Claim: X is Martin-Löf random.

To see this, let $\{U_n\}$ be a Martin-Löf test covering X. We may assume that each U_n is given in the form $[D_n]$, where D_n is c.e. and prefix-free. Let $V_0 = D_0$. If $\tau \in V_n$, enumerate $D_{|\tau|+1} \cap [\tau]^{\prec}$ into V_{n+1}. Then $V = \bigcup_{n\in\omega} V_n$ is a Madison test covering X.

Now let d be a c.e. martingale that diverges along X. Fix a rational $\varepsilon > 0$ such that $\varepsilon < \limsup_n d(X \upharpoonright n) - \liminf_n d(X \upharpoonright n)$. Let $\{d_s\}_{s\in\omega}$ be a nondecreasing sequence of computable martingales with limit d, and let $d^s = d - d_s$. Assume d_0 is the zero martingale, so $d^0 = d$.

Fact: Computable martingales converge on computably random reals.

Therefore, $\varepsilon < \limsup_n d^s(X \upharpoonright n) - \liminf_n d^s(X \upharpoonright n)$. So for every s there are arbitrarily long k such that $d^s(X \upharpoonright k) > \varepsilon$.

Madison tests and martingale convergence

We build a Madison test $V = \bigcup_{n \in \omega} V_n$, where V_0 consists of the minimal strings σ where $d^0(\sigma) > \varepsilon$. If τ is in V_n, then it satisfied some condition of the form $d^s(\tau) = d(\tau) - d_s(\tau) > \varepsilon$. Let t be least such that $d_t(\tau) - d_s(\tau) > \varepsilon$ and let $V_{n+1} \cap [\tau]^{\prec}$ consist of the minimal strings $\sigma \succ \tau$ where $d^t(\sigma) > \varepsilon$.

We claim that V, with the natural sequence of approximations, is a Madison test. We only remove a string σ from $V[s]$ because we realized that it, or some prefix, is not minimal, in which case a prefix of σ appears in $V[s+1]$. So V satisfies condition 1.

Assume $\tau \in V_n$ as witnessed by $d_t(\tau) - d_s(\tau) > \varepsilon$. If $\rho \succeq \tau$, then $d^t(\rho)2^{-|\rho|} \geq \varepsilon\,\mathrm{wt}(V_{n+1} \cap [\rho]^{\prec})$. Applying this inductively, for any $\rho \succ \tau$ we have $d^t(\rho)2^{-|\rho|} \geq \varepsilon\,\mathrm{wt}(V \cap [\rho]^{\prec})$. Assume that τ is the longest proper prefix of ρ in V. Then if $\mathrm{wt}(V \cap [\rho]^{\prec}) > 2^{-|\rho|}$, we have $d^t(\rho) > \varepsilon$, so $\rho \in V_{n+1}$. (The same reasoning holds for the finite approximations.) This proves that V satisfies condition 2, so it is a Madison test.

We have already argued that X fails V, which is a contradiction.

More about martingale convergence

$3 \implies 1$: (Noam gave this proof.) Assume every c.e. martingale converges on X. Clearly, X is ML-random. Let S be a Σ^0_1 class. Then $m(\sigma) = \mu_\sigma(S)$ is a c.e. martingale, so it converges on X. If $X \notin S$ and $m(X \restriction n) \to \varepsilon > 0$, we can build a Martin-Löf test covering X. □

Lemma (Andrews, et al.)

Let X be Martin-Löf random and let d_1, d_2 be c.e. martingales such that $d_1 + d_2$ converges on X. Then both d_1 and d_2 converge on X.

Lemma (Andrews, et al.)

There is a c.e. martingale d that is universal for convergence. I.e., if d converges on $X \in 2^\omega$, then every c.e. martingale converges on X.

Idea.

Let d be the sum of a universal c.e. martingale and a weighted sum, taken over all Σ^0_1 classes S, of the martingales $d_S(\sigma) = \mu_\sigma(S)$. Together, these ensure that X is Martin-Löf random and a density-one point. □

Separating density-one and positive density

Theorem (Day, M.)

There is a Martin-Löf random X that is a positive density point but not a density-one point.

Together with previously discussed work:

- $X \not\geq_T \emptyset'$,
- X is not Oberwolfach random, and so
- X computes every K-trivial.

This solves the covering problem.

Theorem (Day, M.; Bienvenu, Greenberg, Kučera, Nies, Turetsky)

There is a Martin-Löf random $X \not\geq_T \emptyset'$ that computes every K-trivial.

The forcing partial order

Let $P \subseteq 2^\omega$ be a nonempty Π^0_1 class that contains only Martin-Löf random sets. The forcing partial order $\mathbb{P}$ consists of conditions of the form $\langle \sigma, Q \rangle$, where

- $\sigma \in 2^{<\omega}$.
- $Q \subseteq P$ is a Π^0_1 class.
- $[\sigma] \cap Q \neq \emptyset$.
- There is a $\delta < 1/2$ such that

$$(\forall \rho \succeq \sigma)\ [\rho] \cap Q \neq \emptyset \implies \mu_\rho(Q) + \delta \geq \mu_\rho(P).$$

We say that $\langle \tau, R \rangle$ extends $\langle \sigma, Q \rangle$ if $\tau \succeq \sigma$ and $R \subseteq Q$. Let λ be the empty string. Note that $\langle \lambda, P \rangle \in \mathbb{P}$, with $\delta = 0$, so $\mathbb{P}$ is nonempty.

If $G \subseteq \mathbb{P}$ is a filter, let $X_G = \bigcup_{\langle \sigma, Q \rangle \in G} \sigma$.

Properties of the forcing partial order

It is enough to prove that if $G \subseteq \mathbb{P}$ is sufficiently generic, then

1. $X_G \in 2^\omega$. In this case, $X_G \in P$ (hence it is Martin-Löf random).
2. $\rho(P \mid X_G) \leq 1/2$, so X_G is not a density-one point.
3. X_G is a positive density point.

Proof of 1.

Note that if $\langle \sigma, Q \rangle \in \mathbb{P}$ and $\tau \succeq \sigma$ is such that $[\tau] \cap Q \neq \emptyset$, then $\langle \tau, Q \rangle \in \mathbb{P}$. $\square$

Proof of 2: $\rho(P \mid X_G) \leq 1/2$

Fix n. We will show that the conditions forcing

$$(\exists l \geq n)\ \mu_{X_{\dot{G}} \upharpoonright l}(P) < 1/2 \tag{1}$$

are dense in $\mathbb{P}$. Let $\langle \sigma, Q \rangle$ be any condition and let δ witness that $\langle \sigma, Q \rangle \in \mathbb{P}$.

Take m such that $2^{-m} < 1/2 - \delta$. Let Z be the left-most path of $[\sigma] \cap Q$. The set Z is Martin-Löf random and consequently contains arbitrarily long intervals of 1's. Take $\tau \succeq \sigma$ such that $\tau 1^m \prec Z$ and $|\tau| \geq n$. Because Z is the left-most path in Q it follows that $\mu_\tau(Q) \leq 2^{-m}$ and so

$$\mu_\tau(P) \leq \mu_\tau(Q) + \delta \leq 2^{-m} + \delta < \frac{1}{2}.$$

Hence the condition $\langle \tau, Q \rangle$ extends $\langle \sigma, Q \rangle$ and forces (1). $\square$

Proof of 3: X_G is a positive density point

Claim

Let $S \subseteq 2^\omega$ be a Π^0_1 class and let $\langle \sigma, Q \rangle \in \mathbb{P}$. There is an $\varepsilon > 0$ and a condition $\langle \tau, R \rangle$ extending $\langle \sigma, Q \rangle$ such that either

- $[\tau] \cap S = \emptyset$, or
- If $X \in R$, then $\rho(S \mid X) \geq \varepsilon$.

Proof.

If there is a $\tau \succeq \sigma$ such that $[\tau] \cap S = \emptyset$ and $[\tau] \cap Q \neq \emptyset$, then let $\langle \tau, Q \rangle$ be our condition.

Otherwise, $S \cap [\sigma] \supseteq Q \cap [\sigma]$. In this case, let δ witness that $\langle \sigma, Q \rangle \in \mathbb{P}$. Take ε to be a rational greater than 0 and less than $\min\{1/2 - \delta, \mu_\sigma(Q)\}$. (Note that $\mu_\sigma(Q) > 0$ because $[\sigma] \cap Q$ is a non-empty Π^0_1 class containing only Martin-Löf random sets.)

Let Q^ε be the Π^0_1 class $\{X \in Q \cap [\sigma] \mid (\forall n \geq |\sigma|)\ \mu_{X \upharpoonright n}(Q) \geq \varepsilon\}$. We will show that $\langle \sigma, Q^\varepsilon \rangle$ is the required condition.

Proof of 3, continued

Let M be the set of minimal strings in $\{\rho \succeq \sigma \colon \mu_\rho(Q) < \varepsilon\}$. Then M is prefix-free and $Q^\varepsilon = Q \cap [\sigma] \setminus Q \cap [M]$. Summing over M gives us $\mu_\sigma(Q \cap [M]) < \varepsilon$. Hence $\mu_\sigma(Q^\varepsilon) > \mu_\sigma(Q) - \varepsilon > 0$. This proves that $[\sigma] \cap Q^\varepsilon \neq \emptyset$.

If $\tau \succeq \sigma$ and $[\tau] \cap Q^\varepsilon \neq \emptyset$, we can use the same argument to show that $\mu_\tau(Q^\varepsilon) > \mu_\tau(Q) - \varepsilon$. Because $[\tau] \cap Q \neq \emptyset$,

$$\mu_\tau(P) \leq \mu_\tau(Q) + \delta < \mu_\tau(Q^\varepsilon) + \varepsilon + \delta.$$

Hence $\varepsilon + \delta < 1/2$ witnesses that $\langle \sigma, Q^\varepsilon \rangle$ is a condition.

Note that if $X \in Q^\varepsilon$, then $\rho(Q \mid X) \geq \varepsilon$. This implies that $\rho(S \mid X) \geq \varepsilon$ because $S \cap [\sigma] \supseteq Q \cap [\sigma]$, proving the claim. $\square$

It is immediate from the claim that sufficient genericity ensures that X_G is a positive density point. $\square$

Variations

We have finished the proof of:

Theorem

There is a Martin-Löf random $X \ngeq_T \emptyset'$ that computes every K-trivial.

The forcing partial order actually allows us to avoid computing (countably many) non-K-trivials. Hence:

Theorem

There is a Martin-Löf random X such that the hyperarithmetical sets below X are exactly the K-trivials.

Proof...

On the other hand, by carefully effectivizing the forcing:

Theorem

There is a Martin-Löf random $X <_T \emptyset'$ that computes every K-trivial.

Day 3

Where are we now?

Let X be Martin-Löf random. The following implications hold:

$$
\begin{tikzcd}
 & X \text{ not superhigh} \arrow[dl] \arrow[dr] & \\
X \not\geq_{JT} \emptyset' \arrow[dr] & & \begin{array}{c} X \text{ is not (uniformly)} \\ \text{a.e. dominating} \end{array} \arrow[dl] \\
 & X \text{ is Oberwolfach random} \arrow[r, leftrightarrow, shift left] \arrow[d] \arrow[dr] \arrow[dd, bend right=60] & \begin{array}{c} X \text{ does not compute} \\ \text{every } K\text{-trivial} \end{array} \arrow[l, shift left] \\
 & \begin{array}{c} \text{Every c.e. martingale} \\ \text{converges on } X \end{array} \arrow[r, shift left] & \begin{array}{c} X \text{ is a density-} \\ \text{one point} \end{array} \arrow[l, shift left] \arrow[d, shift right] \\
 & \begin{array}{c} X \not\geq_T \emptyset' \\ (X \text{ is difference random}) \end{array} \arrow[r, leftrightarrow] & \begin{array}{c} X \text{ is a positive} \\ \text{density point} \end{array} \arrow[u, shift right, "/" marking]
\end{tikzcd}
$$

Random reals that are not Oberwolfach random

Lemma (Various[2])

Let $X \in 2^\omega$ be Martin-Löf random but not Oberwolfach random. Then X computes a function $f\colon \omega \to \omega$ such that for every oracle A, if X is Martin-Löf random relative to A, then f dominates every A-computable function.

Taking $A = \emptyset$, the lemma says that if X is ML-random but not Oberwolfach random, then $f \leq_T X$ dominates every computable function. In other words, X is high.

We can do significantly better.

Definition (Dobrinen, Simpson)

X is uniformly almost everywhere dominating if there is a function $f \leq_T X$ such that for almost every $A \in 2^\omega$, f dominates every A-computable function.

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[2] Bienvenu, Hölzl, M., Nies proved this assuming that X is not a density-one point. Bienvenu, Greenberg, Kučera, Nies, Turetsky applied essentially the same proof assuming that X is not Oberwolfach random.

Random reals that are not Oberwolfach random

Theorem (Various)

If $X \in 2^\omega$ is Martin-Löf random but not Oberwolfach random, then X is uniformly almost everywhere dominating.

Proof.

Let $f \leq_T X$ be the function from the lemma. Since X is ML-random, it is ML-random relative to almost every A. For such an A, f dominates all A-computable functions. $\square$

We call X LR-hard if every set that is Martin-Löf random relative to X is 2-random (i.e., Martin-Löf random relative to $\emptyset'$). Kjos-Hanssen, M. and Solomon proved that X is (uniformly) almost everywhere dominating if and only if it is LR-hard. Simpson showed that such an X is superhigh ($X' \geq_{tt} \emptyset''$).

Corollary

If a ML-random $X \in 2^\omega$ is not superhigh, then X is Oberwolfach random.

Random reals that are not Oberwolfach random

Theorem (Bienvenu, Greenberg, Kučera, Nies, Turetsky)

If $X \in 2^\omega$ is Martin-Löf random but not Oberwolfach random, then X computes every K-trivial.

Proof.

Assume that A is a c.e. K-trivial set. Then A computes a function g (its settling-time function) such that any function dominating g computes A. Since A is K-trivial and therefore low for ML-randomness, X is Martin-Löf random relative to A. By the lemma, X computes a function dominating g, hence $X \geq_T A$.

Nies proved that every K-trivial is computed by a c.e. K-trivial, which completes the proof. □

This is a very different proof than that given by Bienvenu, et al. In particular, they did not use the fact that every K-trivial is low for random.

Random but not Oberwolfach random: the Lemma

Lemma (Various)

Let $X \in 2^\omega$ be Martin-Löf random but not Oberwolfach random. Then X computes a function $f\colon \omega \to \omega$ such that for every oracle A, if X is Martin-Löf random relative to A, then f dominates every A-computable function.

Proof.

Let $\{U_n\}_{n\in\omega}$, $\{\beta_n\}_{n\in\omega}$ be an Auckland test covering X. Let $\beta = \lim \beta_n$. (Recall that $\mu(U_n) \leq \beta - \beta_n$.) We may assume that $\{U_n\}_{n\in\omega}$ is nested.

We write $\{U_{n,s}\}_{s\in\omega}$ for a fixed effective sequence of clopen approximations to U_n. We may assume that $\mu(U_{n,s}) \leq \beta_s - \beta_n$. We may also assume that $\{U_{n,s}\}_{n\in\omega}$ is nested for each stage s.

Random but not Oberwolfach random: the Lemma

Let $g(n)$ be the least $s > n$ such that $X \in U_{n,s}$. Note that g is total, X-computable, and non-decreasing. Define $f \leq_T X$ by $f(n) = g^{\circ n}(0)$. (I.e., let $f(0) = g(0)$ and, for all $n \in \omega$, let $f(n+1) = g(f(n))$.)

We will show that f satisfies the lemma. To see this, assume that there is an A-computable function h that is not dominated by f. We will use h to build a Solovay test relative to A that captures X. There are two cases.

Case 1: h dominates f. We may assume that $(\forall n)\ h(n) \geq f(n)$. Note that $(\forall n)\ f(n) \geq g(n)$. It is true for $n = 0$. If it holds for n, then $f(n) \geq g(n) \geq n+1$, so $f(n+1) = g(f(n)) \geq g(n+1)$. Therefore, $(\forall n)\ h(n) \geq g(n)$.

Random but not Oberwolfach random: the Lemma

Define $k \leq_T A$ by $k(n) = h^{\circ n}(0)$ and, for all n, let

$$S_n = U_{k(n),k(n+1)} = U_{k(n),h(k(n))} \supseteq U_{k(n),g(k(n))}.$$

Therefore, $X \in S_n$. Also, $\mu(S_n) \leq \beta_{k(n+1)} - \beta_{k(n)}$. Note that $\sum_{n\in\omega} \mu(S_n) \leq \sum_{n\in\omega} \beta_{k(n+1)} - \beta_{k(n)} \leq \beta$. So $\{S_n\}_{n\in\omega}$ is a Solovay test relative to A that covers X.

Case 2: h does not dominate f. For all n, let $S_n = U_{h(n),h(n+1)}$. As in Case 1, $\{S_n\}_{n\in\omega}$ is Solovay test relative to A. We must show that it captures X.

By our assumption, there are infinitely many n such that $h(n) \leq f(n)$ and $h(n+1) \geq f(n+1)$. Fix such an n and note that $h(n+1) \geq f(n+1) = g(f(n)) \geq g(h(n))$. Therefore, $X \in U_{h(n),g(h(n))} \subseteq U_{h(n),h(n+1)} = S_n$. This is true for infinitely many n, so X is not Martin-Löf random relative to A. $\square$

— THANK YOU! —

Oberwolfach randoms are density-one points

Definition

An effective sequence $\{U_n\}_{n\in\omega}$ of Σ^0_1 classes is an Auckland test if there is a left-c.e. real β with a computable nondecreasing sequence of rational approximations $\{\beta_n\}_{n\in\omega}$ such that

- $\beta = \lim_{n\to\infty} \beta_n$, and
- $\mu(U_n) \leq \beta - \beta_n$.

$X \in 2^\omega$ passes an Auckland test if $X \notin \bigcap_{n\in\omega} U_n$. We say that X is Oberwolfach random if it passes all Auckland tests.

Theorem (Bienvenu, Greenberg, Kučera, Nies, Turetsky)

If $X \in 2^\omega$ is Oberwolfach random, then it is a density-one point.

Oberwolfach randoms are density-one points

Proof.

We prove the contrapositive. Assume that X is not a density-one point. There is a rational $\varepsilon \in (0,1)$ and a Π^0_1 class C containing X such that $\rho(C \mid X) < \varepsilon < 1$.

Let D be a prefix-free c.e. set such that $C = 2^\omega \setminus [D]$. Let $D^n = D \setminus D_n$ and let $U_n = \{X : (\exists k)\ \mu_{X \upharpoonright k}(2^\omega \setminus [D^n]) < \varepsilon\}$. It is clear that $X \in \bigcap_{n\in\omega} U_n$. Note that

$$\mu(U_n) \leq \frac{1 - \mu(2^\omega \setminus [D^n])}{1-\varepsilon} = \frac{\mu[D^n]}{1-\varepsilon} = \frac{\mu[D] - \mu[D_n]}{1-\varepsilon}.$$

Therefore, $\{U_n\}_{n\in\omega}$ is an Auckland test, as witnessed by the c.e. real $\beta = \dfrac{\mu[D]}{1-\varepsilon}$ with approximations $\beta_n = \dfrac{\mu[D_n]}{1-\varepsilon}$. Hence X is not Oberwolfach random. □

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Characterizing difference randomness

Theorem (Franklin, Ng)

X is difference random iff X is Martin-Löf random and $X \not\geq_T \emptyset'$.

Proof.

Assume that X fails the difference test consisting of C and $\{V_n\}_{n\in\omega}$. We may slow down the enumeration of each V_n to ensure that $(\forall s)\ \mu(V_n \cap C[s]) \leq 2^{-n}$. Define a function $f \leq_T X$ by letting $f(n)$ be the least s such that $X \in V_n \cap C[s]$.

Now if n enters $\emptyset'$ at stage s, let $G_n = V_n \cap C[s]$. Otherwise, $G_n = \emptyset$. So $\{G_n\}_{n\in\omega}$ is a ML-test (that we treat as a Solovay test). If $(\exists^\infty n)\ n \in \emptyset' \smallsetminus \emptyset'_{f(n)}$, then this test covers X, so X is not ML-random. Otherwise, $X \geq_T \emptyset'$.

For the other direction, first note that if X is difference random, then it is Martin-Löf random. Assume that $X \geq_T \emptyset'$. Fix a Turing functional Γ such that $\Gamma^X = \emptyset'$.

Characterizing difference randomness

We build a difference test C, $\{V_n\}_{n\in\omega}$ as follows. Let

$$C = 2^\omega \setminus \{X : (\exists n)\ \Gamma^X(n)\downarrow = 0 \text{ and } n \in \emptyset'\}.$$

By the recursion theorem, we control an infinite computable set R of positions of $\emptyset'$. Partition R into finite sets $R_0, R_1, \ldots$ such that $|R_n| = 2^n - 1$.

- Whenever we see $\Gamma^\sigma \upharpoonright R_n \downarrow = \emptyset' \upharpoonright R_n[s]$, we put $[\sigma]$ into V_n.
- Whenever we see $\mu(C \cap V_n[s]) > 2^{-n}$, we enumerate an element of R_n into $\emptyset'$. (This has the effect of putting $V_n[s]$ into the complement of C, hence can only happen $2^n - 1$ times.)

The construction ensures that $X \in \bigcap_{n\in\omega} V_n \cap C$ and $\mu(V_n \cap C) \leq 2^{-n}$, proving that X is not difference random. $\square$

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Characterizing ML-random positive density points

Theorem (Bienvenu, Hölzl, M., Nies)

X is difference random iff X is a ML-random positive density point.

Proof.

Assume that X is not a positive density point. Let C be a Π^0_1 class containing X such that $\rho(C \mid X) = 0$. For each n, let $U_n = \{Z\colon (\exists k)\ \mu_{Z\upharpoonright k}(C) < 2^{-n}\}$. Then $\mu(C \cap U_n) \leq 2^{-n}$, so C and $\{U_n\}_{n\in\omega}$ form a difference test covering X.

For the other direction, let C, $\{V_n\}_{n\in\omega}$ be a difference test covering X. Assume that X is ML-random and fix $r \in \omega$. We will show that there is a $\sigma \prec X$ for which $\mu_\sigma(C) \leq 2^{-r}$.

We define an effective sequence of Σ^0_1 classes $\{G_m\}_{m\in\omega}$ with $\mu(G_m) \leq (1 - 2^{-r-1})^m$. Let $G_0 = 2^\omega$. Suppose that G_m has been defined. Let B_m be a prefix-free c.e. set such that $G_m = [B_m]$.

Characterizing ML-random positive density points

We define G_{m+1} as follows. When a string σ enters B_m, we put

$$\left(V_{|\sigma|+r+1} \cap [\sigma]\right)^{(\leq 2^{-|\sigma|}(1-2^{-r-1}))}$$

into G_{m+1}. (If W is a Σ^0_1 class, $W^{(\leq \varepsilon)}$ is the same class except restricted to measure ε.) It is not hard to see that

$$\mu(G_{m+1}) \leq (1 - 2^{-r-1})\mu(G_m) \leq (1 - 2^{-r-1})^{m+1}.$$

Since X is ML-random, there is a minimal m such that $X \notin G_m$. The minimality of m implies that there is a $\sigma \in B_{m-1}$ with $\sigma \prec X$. Let $V = V_{|\sigma|+r+1}$. Note that $\mu_\sigma(V) > 1 - 2^{-r-1}$, otherwise X would enter G_m. Also $\mu_\sigma(C \cap V) \leq 2^{|\sigma|}\mu(C \cap V) \leq 2^{-r-1}$ by the definition of a difference test. But $\mu_\sigma(C) + \mu_\sigma(V) - \mu_\sigma(C \cap V) \leq 1$, which implies that $\mu_\sigma(C) \leq 2^{-r}$, as required.

Since r was arbitrary, $\rho(C \mid X) = 0$. $\square$

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If A is not K-trivial, we can force $X_G \not\geq_T A$

Claim

Assume that $A \in 2^\omega$ is not K-trivial, $\langle \sigma, Q \rangle \in \mathbb{P}$, and Φ is a Turing functional. There is a $\tau \in 2^{<\omega}$ such that $\langle \tau, Q \rangle$ extends $\langle \sigma, Q \rangle$ and

$$(\forall X \in [\tau] \cap Q)[\ \Phi^X = A \implies X \text{ is not difference random }].$$

Proof.

If there is a $\rho \succeq \sigma$ and an n such that $\Phi^\rho(n) \downarrow \neq A(n)$ and $[\rho] \cap Q \neq \emptyset$, then take $\tau = \rho$. Assume that no such ρ and n exist.

Define $V_n = \{X \colon X \in U_n[\Phi^X]\}$, where $U_n[Z]$ is the nth level of the universal Martin-Löf test relative to Z. If $X \in V_n \cap [\sigma] \cap Q$, then because Φ^X is not incompatible with A, we have $X \in U_n[\Phi^X] \subseteq U_n[A]$. Hence $\mu(V_n \cap [\sigma] \cap Q) \leq \mu(U_n[A]) \leq 2^{-n}$. In other words, Q and $\{V_n \cap [\sigma]\}_{n \in \omega}$ form a difference test.

If A is not K-trivial, we can force $X_G \not\geq_T A$

Now assume that $X \in [\sigma] \cap Q$ and $\Phi^X = A$. Because A is not a base for randomness, $X \in U_n[A] = U_n[\Phi^X]$ for all n. Therefore, $X \in \bigcap_{n \in \omega} V_n \cap [\sigma] \cap Q$, so X is not difference random. Hence the claim is satisfied by taking $\tau = \sigma$. $\square$

We have already shown that if $G \subseteq \mathbb{P}$ is sufficiently generic, then X_G is a positive density point, hence it is difference random.

So the claim shown that if $G \subseteq \mathbb{P}$ is sufficiently generic relative to A, then X_G does not compute A. We can build G to ensure that X_G does not compute any member of a countable set of non-K-trivials (e.g., all non-K-trivial hyperarithmetical sets).

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