ω-Degree Spectra

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CCR 2013
Buenos Aires
06.02.13

[1] Supported by Sofia University Science Fund and Master Program Logic and Algorithms

Outline

- Degree spectra and jump spectra
- ω-enumeration degrees
- ω-degree spectra
- ω-co-spectra
- A minimal pair theorem
- Quasi-minimal degrees

Enumeration of a Structure

Let $\mathfrak{A} = (\mathbb{N}; R_1, \dots, R_k, =, \neq)$ be a countable abstract structure.

- An enumeration f of $\mathfrak{A}$ is a total mapping from $\mathbb{N}$ onto $\mathbb{N}$.
- for any $A \subseteq \mathbb{N}^a$ let $f^{-1}(A) = \{\langle x_1, \dots, x_a \rangle : (f(x_1), \dots, f(x_a)) \in A\}$.
- $f^{-1}(\mathfrak{A}) = f^{-1}(R_1) \oplus \dots \oplus f^{-1}(R_k) \oplus f^{-1}(=) \oplus f^{-1}(\neq)$.

Degree Spectra

Definition (Richter)

The Turing degree spectrum of $\mathfrak{A}$

$$\mathrm{DS}_{\mathrm{T}}(\mathfrak{A}) = \{d_{\mathrm{T}}(f^{-1}(\mathfrak{A})) \mid f \text{ is an injective enumeration of } \mathfrak{A}\}$$

- J. Knight, Ash, Jockush, Downey, Slaman.

Enumeration reducibility

Definition

We say that $\Gamma : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is an *enumeration operator* iff for some c.e. set W_i for each $B \subseteq \mathbb{N}$

$$\Gamma(B) = \{x | (\exists D)[\langle x, D\rangle \in W_i \,\&\, D \subseteq B]\}.$$

The index i of the c.e. set W_i is an index of Γ and write $\Gamma = \Gamma_i$.

Definition

The set A is *enumeration reducible to* the set B ($A \leq_e B$), if $A = \Gamma_i(B)$ for some e-operator Γ_i.
The enumeration degree of A is $d_e(A) = \{B \subseteq \mathbb{N} | A \equiv_e B\}$.
The set of all enumeration degrees is denoted by $\mathcal{D}_e$.

The enumeration jump

Definition

Given a set A, denote by $A^+ = A \oplus (\mathbb{N} \setminus A)$.

Theorem

For any sets A and B:

1. *A is c.e. in B iff $A \leq_e B^+$.*
2. *$A \leq_T B$ iff $A^+ \leq_e B^+$.*
3. *A is Σ^0_{n+1} relatively to B iff $A \leq_e (B^+)^{(n)}$.*

Definition

For any set A let $K_A = \{\langle i, x\rangle | x \in \Gamma_i(A)\}$. Set $A' = K_A^+$.

Definition

A set A is called *total* iff $A \equiv_e A^+$.

Let $d_e(A)' = d_e(A')$. The enumeration jump is always a total degree and agrees with the Turing jump under the standard embedding $\iota : \mathcal{D}_T \to \mathcal{D}_e$ by $\iota(d_T(A)) = d_e(A^+)$.

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Enumeration Degree Spectra and Co-spectra

Definition (Soskov)

- The enumeration degree spectrum of $\mathfrak{A}$

$$\mathrm{DS}(\mathfrak{A}) = \{d_{\mathrm{e}}(f^{-1}(\mathfrak{A})) \mid f \text{ is an enumeration of } \mathfrak{A}\}.$$

If $\mathbf{a}$ is the least element of $\mathrm{DS}(\mathfrak{A})$, then $\mathbf{a}$ is called the *degree of* $\mathfrak{A}$.

- The co-spectrum of $\mathfrak{A}$

$$\mathrm{CS}(\mathfrak{A}) = \{\mathbf{b} : (\forall \mathbf{a} \in \mathrm{DS}(\mathfrak{A}))(\mathbf{b} \le \mathbf{a})\}.$$

If $\mathbf{a}$ is the greatest element of $\mathrm{CS}(\mathfrak{A})$ then we call $\mathbf{a}$ the *co-degree* of $\mathfrak{A}$.

Jump spectra

Definition

The nth jump spectrum of $\mathfrak{A}$ is the set

$$\mathrm{DS}_n(\mathfrak{A}) = \{d_e(f^{-1}(\mathfrak{A})^{(n)}) : f \text{ is an enumeration of } \mathfrak{A}\}.$$

If **a** is the least element of $DS_n(\mathfrak{A})$, then **a** is called the *nth jump degree of* $\mathfrak{A}$.

Definition

The set $\mathrm{CS}_n(\mathfrak{A})$ of all lower bounds of the nth jump spectrum of $\mathfrak{A}$ is called nth jump co-spectrum of $\mathfrak{A}$.

If $\mathrm{CS}_n(\mathfrak{A})$ has a greatest element then it is called the *nth jump co-degree of* $\mathfrak{A}$.

Some examples

Example (Richter)

Let $\mathfrak{A} = (A; <)$ be a linear ordering. $\mathrm{DS}(\mathfrak{A})$ contains a minimal pair of degrees and hence $\mathrm{CS}(\mathfrak{A}) = \{\mathbf{0}_e\}$. $\mathbf{0}_e$ is the co-degree of $\mathfrak{A}$. So, if $\mathfrak{A}$ has a degree $\mathbf{a}$, then $\mathbf{a} = \mathbf{0}_e$.

Example (Knight)

For a linear ordering $\mathfrak{A}$, $\mathrm{CS}_1(\mathfrak{A})$ consists of all e-degrees of Σ^0_2 sets. The first jump co-degree of $\mathfrak{A}$ is $\mathbf{0}'_e$.

Example (Slaman,Whener)

There exists a structure $\mathfrak{A}$ s.t.

$$\mathrm{DS}(\mathfrak{A}) = \{\mathbf{a} : \mathbf{a} \text{ is total and } \mathbf{0}_e < \mathbf{a}\}.$$

Clearly the structure $\mathfrak{A}$ has co-degree $\mathbf{0}_e$ but has not a degree.

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Some examples

Example (Downey, Jockusch)

Let G be a torsion free abelian group of rank 1, i.e. G is a subgroup of Q. There exists a set called the standard type of the group $S(G)$ with the following property:
The Turing degree spectrum of G is precisely $\{d_T(X) \mid S(G) \in \Sigma^0_1(X)\}$.

Example (Coles, Downey, Slaman)

Let $A \subseteq \mathbb{N}$. Consider $\mathcal{C}(A) = \{X \mid A \in \Sigma^0_1(X)\}$. By Richter there is a set A such that $\mathcal{C}(A)$ has not a member of least Turing degree.
For every sets A the set: $\mathcal{C}(A)' = \{X' \mid A \in \Sigma^0_1(X)\}$ has a member of least degree.
Every torsion free abelian group of rank 1 has a first jump degree.

Representing the principle countable ideals as co-spectra

Example (Soskov)

Let G be a torsion free abelian group of rank 1.
Let $\mathbf{s}_G$ be an enumeration degree of $S(G)$.

- $\mathrm{DS}(G) = \{\mathbf{b} : \mathbf{b}$ is total and $\mathbf{s}_G \leq_e \mathbf{b}\}$.
- The co-degree of G is $\mathbf{s}_G$.
- G has a degree iff $\mathbf{s}_G$ is a total e-degree.
- If $1 \leq n$, then $\mathbf{s}_G^{(n)}$ is the n-th jump degree of G.

For every $\mathbf{d} \in \mathcal{D}_e$ there exists a G, s.t. $\mathbf{s}_G = \mathbf{d}$.

Corrolary

Every principle ideal of enumeration degrees is $\mathrm{CS}(G)$ for some G.

Representing the principle countable ideals as co-spectra

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Let $\mathbf{s}_G$ be an enumeration degree of $S(G)$.

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- The co-degree of G is $\mathbf{s}_G$.
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Corrolary

Every principle ideal of enumeration degrees is $\mathrm{CS}(G)$ for some G.

Representing non-principle countable ideals as co-spectra

Example (Soskov)

Let $B_0, \dots, B_n, \dots$ be a sequence of sets of natural numbers. Set $\mathfrak{A} = (\mathbb{N}; f; \sigma)$,

$$f(\langle i, n\rangle) = \langle i+1, n\rangle;$$
$$\sigma = \{\langle i, n\rangle : n = 2k+1 \vee n = 2k \ \&\ i \in B_k\}.$$

Then $CS(\mathfrak{A}) = I(d_e(B_0), \dots, d_e(B_n), \dots)$

Spectra with a countable base

Definition

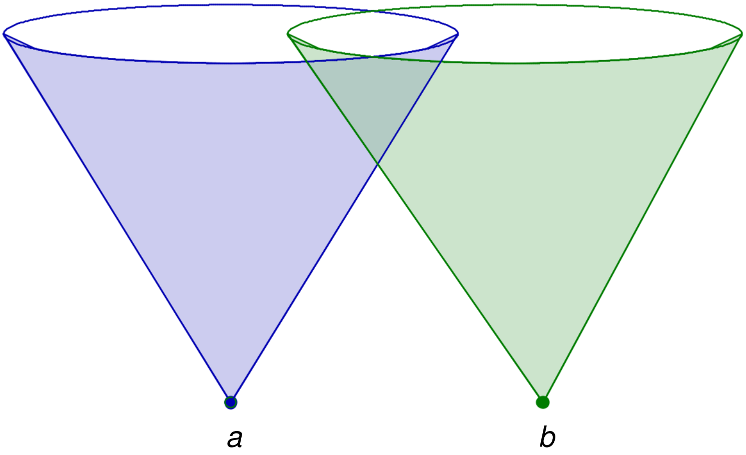
Let $\mathcal{B} \subseteq \mathcal{A}$ be sets of degrees. Then $\mathcal{B}$ is a base of $\mathcal{A}$ if

$$(\forall \mathbf{a} \in \mathcal{A})(\exists \mathbf{b} \in \mathcal{B})(\mathbf{b} \leq \mathbf{a}).$$

Theorem (Soskov)

A structure $\mathfrak{A}$ has a degree if and only if $DS(\mathfrak{A})$ has a countable base.

An upwards closed set of degrees which is not a degree spectra of a structure



Upwards closed sets

Definition

Let $\mathcal{A} \subseteq \mathcal{D}_e$. $\mathcal{A}$ is *upwards closed with respect to total enumeration degrees*, if

$$\mathbf{a} \in \mathcal{A}, \mathbf{b} \text{ is total and } \mathbf{a} \leq \mathbf{b} \Rightarrow \mathbf{b} \in \mathcal{A}.$$

The degree spectra are upwards closed with respect to total enumeration degrees.

Properties of upwards closed sets (Soskov)

Let $\mathcal{A} \subseteq \mathcal{D}_e$ be upwards closed with respect to total enumeration degrees. Denote by

$$co(\mathcal{A}) = \{b : b \in \mathcal{D}_e \ \& \ (\forall a \in \mathcal{A})(b \leq_e a)\}.$$

- (Selman) $\mathcal{A}_t = \{\mathbf{a} : \mathbf{a} \in \mathcal{A} \ \& \ \mathbf{a} \text{ is total}\}$ $\Longrightarrow co(\mathcal{A}) = co(\mathcal{A}_t)$.
- Let $\mathbf{b} \in \mathcal{D}_e$ and $n > 0$.

$$\mathcal{A}_{\mathbf{b},n} = \{\mathbf{a} : \mathbf{a} \in \mathcal{A} \ \& \ \mathbf{b} \leq \mathbf{a}^{(n)}\} \Longrightarrow co(\mathcal{A}) = co(\mathcal{A}_{\mathbf{b},n}).$$

Properties of degree spectra and co-spectra (Soskov)

- Let $\mathbf{c} \in \mathrm{DS}_n(\mathfrak{A})$ and $n > 0$. Then

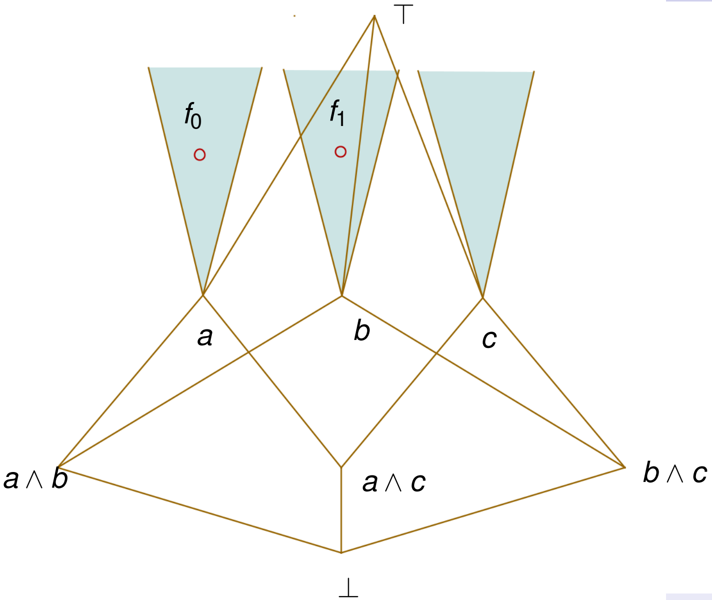
$$\mathrm{CS}(\mathfrak{A}) = co(\{\mathbf{a} \mid \mathbf{a} \in \mathrm{DS}(\mathfrak{A}) \ \&\ \mathbf{a}^{(n)} = \mathbf{c}\}).$$

- A minimal pair theorem:
  There exist **f** and **g** in $\mathrm{DS}(\mathfrak{A})$:

$$(\forall \mathbf{a} \in \mathcal{D}_e)(\forall k)(\mathbf{a} \leq_e \mathbf{f}^{(k)} \ \&\ \mathbf{a} \leq_e \mathbf{g}^{(k)} \Rightarrow \mathbf{a} \in \mathrm{CS}_k(\mathfrak{A})).$$

- Quasi-minimal degree:
  There exists $\mathbf{q_0}$ quasi-minimal for $\mathrm{DS}(\mathfrak{A})$
  - $\mathbf{q_0} \notin \mathrm{CS}(\mathfrak{A})$;
  - for every total e-degree **a**: $\mathbf{a} \geq_e \mathbf{q_0} \Rightarrow \mathbf{a} \in \mathrm{DS}(\mathfrak{A})$ and $\mathbf{a} \leq_e \mathbf{q_0} \Rightarrow \mathbf{a} \in \mathrm{CS}(\mathfrak{A})$.

An upwards closed set with no minimal pair



Relative Spectra

Let $\mathfrak{A}_1, \dots, \mathfrak{A}_n$ be given structures.

Definition

The relative spectrum $\mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1 \dots, \mathfrak{A}_n)$ of the structure $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \dots, \mathfrak{A}_n$ is the set

$$\{d_{\mathrm{e}}(f^{-1}(\mathfrak{A})) \mid \begin{array}{l} f \text{ is an enumeration of } \mathfrak{A} \text{ \&} \\ (\forall k \leq n)(f^{-1}(\mathfrak{A}_k) \leq_{\mathrm{e}} f^{-1}(\mathfrak{A})^{(k)})\} \end{array}$$

It turns out that all properties of the degree spectra remain true for the relative spectra.

Relatively intrinsically Σ^0_α sets

Let $\alpha < \omega^{CK}$.

Definition

A set A is *intrinsically relatively* Σ^0_α *on* $\mathfrak{A}$ if for every enumeration f of $\mathfrak{A}$ the set $f^{-1}(A)$ is Σ^0_α relative to $f^{-1}(\mathfrak{A})$.

Theorem (Ash, Knight, Manasse, Slaman, Chisholm)

A set A is intrinsically relatively Σ^0_α on $\mathfrak{A}$ iff the set A is definable on $\mathfrak{A}$ by a Σ^c_α formula with parameters.

Relatively α-intrinsic sets

Let $\mathcal{B} = \{B_\gamma\}_{\gamma<\xi}$ be a sequence of sets, $\xi < \omega_1^{CK}$.

Definition

A set A is *relatively α-intrinsic on $\mathfrak{A}$ with respect to $\mathcal{B}$* if for every enumeration f of $\mathfrak{A}$ such that
$(\forall\gamma < \xi)(f^{-1}(B_\gamma) \leq_e f^{-1}(\mathfrak{A})^{(\gamma)})$ uniformly in $\gamma < \xi$
$f^{-1}(A) \leq_e f^{-1}(\mathfrak{A})^{(\alpha)}$.

Theorem (Soskov, Baleva)

A set A is relatively α-intrinsic on $\mathfrak{A}$ with respect to $\mathcal{B}$ iff A is definable on $\mathfrak{A}, \mathcal{B}$ by specific kind of positive Σ^c_α formula with parameters, analogue of Ash's recursive infinitary propositional sentences applied for abstract structures.

ω-Enumeration Degrees - background

Theorem (Selman)

$A \leq_e B$ iff $(\forall X)(B$ is c.e. in $X \Rightarrow A$ is c.e. in $X)$.

Theorem (Case)

$A \leq_e B \oplus \emptyset^{(n)}$ iff $(\forall X)(B \in \Sigma^X_{n+1} \Rightarrow A \in \Sigma^X_{n+1})$.

Theorem (Ash)

Formally describes the relation:
$\mathcal{R}^n_k(A, B_0, \dots, B_k)$ iff
$(\forall X)[B_0 \in \Sigma^X_1 \ \& \ \dots \ \& \ B_k \in \Sigma^X_{k+1} \Rightarrow A \in \Sigma^X_{n+1}]$.

ω-Enumeration Reducibility

- Uniform reducibility on sequences of sets
- $\mathcal{S}$ the set of all sequences of sets of natural numbers
- For $\mathcal{B} = \{B_n\}_{n<\omega} \in \mathcal{S}$ call *the jump class of* $\mathcal{B}$ the set

$$J_{\mathcal{B}} = \{d_{\mathrm{T}}(X) \mid (\forall n)(B_n \text{ is c.e. in } X^{(n)} \text{ uniformly in } n)\} \quad .$$

Definition (Soskov)

$\mathcal{A} \leq_\omega \mathcal{B}$ ($\mathcal{A}$ is *ω-enumeration reducible to* $\mathcal{B}$) if $J_{\mathcal{B}} \subseteq J_{\mathcal{A}}$

- $\mathcal{A} \equiv_\omega \mathcal{B}$ if $J_{\mathcal{A}} = J_{\mathcal{B}}$.

ω-Enumeration Degrees

- $\equiv_\omega$ is an equivalence relation on $\mathcal{S}$.
- $d_\omega(\mathcal{B}) = \{\mathcal{A} \mid \mathcal{A} \equiv_\omega \mathcal{B}\}$
- $\mathcal{D}_\omega = \{d_\omega(\mathcal{B}) \mid \mathcal{B} \in \mathcal{S}\}$.
- If $A \subseteq \mathbb{N}$ denote by $A \uparrow \omega = \{A, \emptyset, \emptyset, \dots\}$.
- For every $A, B \subseteq \mathbb{N}$:

$$A \leq_{\mathrm{e}} B \iff J_{B\uparrow\omega} \subseteq J_{A\uparrow\omega} \iff A \uparrow \omega \leq_\omega B \uparrow \omega.$$

- The mapping $\kappa(d_{\mathrm{e}}(A)) = d_\omega(A \uparrow \omega)$ gives an isomorphic embedding of $\mathcal{D}_{\mathrm{e}}$ to $\mathcal{D}_\omega$.

ω-Enumeration Degrees

Let $\mathcal{B} = \{B_n\}_{n<\omega} \in \mathcal{S}$.
A jump sequence $\mathcal{P}(\mathcal{B}) = \{\mathcal{P}_n(\mathcal{B})\}_{n<\omega}$:

1. $\mathcal{P}_0(\mathcal{B}) = B_0$
2. $\mathcal{P}_{n+1}(\mathcal{B}) = (\mathcal{P}_n(\mathcal{B}))' \oplus B_{n+1}$

Definition

Let $\mathcal{A} = \{A_n\}_{n<\omega},\ \mathcal{B} = \{B_n\}_{n<\omega} \in \mathcal{S}$.
$\mathcal{A} \leq_{\mathrm{e}} \mathcal{B}$ ($\mathcal{A}$ *is enumeration reducible* $\mathcal{B}$) iff
$A_n \leq_{\mathrm{e}} B_n$ uniformly in n, i.e. there is a computable function h such that $(\forall n)(A_n = \Gamma_{h(n)}(B_n))$.

Theorem (Soskov, Kovachev)

$\mathcal{A} \leq_\omega \mathcal{B} \iff \mathcal{A} \leq_{\mathrm{e}} \mathcal{P}(\mathcal{B})$.

Proposition

$(n < k)$ $\mathcal{R}^n_k(A, B_0, \dots, B_k) \iff A \leq_e \mathcal{P}_n(B_0, \dots, B_n)$.
$(n \geq k)$ $\mathcal{R}^n_k(A, B_0, \dots, B_k) \iff A \leq_e \mathcal{P}_k(B_0, \dots, B_k)^{(n-k)}$.

ω-Enumeration Degrees

Let $\mathcal{B} = \{B_n\}_{n<\omega} \in \mathcal{S}$.
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Proposition

- $\mathcal{B} \leq_e \mathcal{P}(\mathcal{B})$.
- $\mathcal{P}(\mathcal{P}(\mathcal{B})) \leq_e \mathcal{P}(\mathcal{B})$.
- $\mathcal{B} \equiv_\omega \mathcal{P}(\mathcal{B})$.
- $\mathcal{A} \leq_e \mathcal{B} \Rightarrow \mathcal{A} \leq_\omega \mathcal{B}$.

Lemma

Let $\mathcal{A}_0, \dots, \mathcal{A}_r, \dots$ be sequences of sets such that for every r, $\mathcal{A}_r \not\leq_\omega \mathcal{B}$. There is a total set X such that $\mathcal{B} \leq_\omega \{X^{(n)}\}_{n<\omega}$ and $\mathcal{A}_r \not\leq_\omega \{X^{(n)}\}_{n<\omega}$ for each r.

ω-Enumeration Jump

Definition (Soskov)

For every $\mathcal{A} \in \mathcal{S}$ *the* ω*-enumeration jump of* $\mathcal{A}$ is $\mathcal{A}' = \{\mathcal{P}_{n+1}(\mathcal{A})\}_{n<\omega}$
We have that $J'_{\mathcal{A}} = \{\mathbf{a}' \mid \mathbf{a} \in J_{\mathcal{A}}\}$.

Proposition

1. $\mathcal{A} <_\omega \mathcal{A}'$.
2. $\mathcal{A} \leq_\omega \mathcal{B} \Rightarrow \mathcal{A}' \leq_\omega \mathcal{B}'$.

- $d_\omega(\mathcal{A})' = d_\omega(\mathcal{A}')$
- $d_\omega(\mathcal{A})^{(n)} = d_\omega(\mathcal{A}^{(n)})$.

ω- Degree Spectra

Let $\mathfrak{A} = (\mathbb{N}; R_1, \ldots, R_k, =, \neq)$ be an abstract structure and $\mathcal{B} = \{B_n\}_{n<\omega}$ be a fixed sequence of subsets of $\mathbb{N}$.
The enumeration f of the structure $\mathfrak{A}$ is *acceptable with respect to* $\mathcal{B}$, if for every n,

$$f^{-1}(B_n) \leq_{\mathrm{e}} f^{-1}(\mathfrak{A})^{(n)} \text{ uniformly in } n.$$

Denote by $\mathcal{E}(\mathfrak{A}, \mathcal{B})$ - the class of all acceptable enumerations.

Definition

The ω- degree spectrum of $\mathfrak{A}$ with respect to $\mathcal{B} = \{B_n\}_{n<\omega}$ is the set

$$\mathrm{DS}(\mathfrak{A}, \mathcal{B}) = \{d_{\mathrm{e}}(f^{-1}(\mathfrak{A})) \mid f \in \mathcal{E}(\mathfrak{A}, \mathcal{B})\}$$

ω- Degree Spectra and Relative Spectra

The notion of the ω-degree spectrum is a generalization of the relative spectrum:

- $\mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n) = \mathrm{DS}(\mathfrak{A}, \mathcal{B})$, where $\mathcal{B} = \{B_k\}_{k<\omega}$,
- $B_0 = \emptyset$,
- B_k is the positive diagram of the structure $\mathfrak{A}_k$, $k \leq n$
- $B_k = \emptyset$ for all $k > n$.

ω- Degree Spectra and Degree Spectra

It is easy to find a structure $\mathfrak{A}$ and a sequence $\mathcal{B}$ such that $\mathrm{DS}(\mathfrak{A}, \mathcal{B}) \neq \mathrm{DS}(\mathfrak{A})$.

- $\mathfrak{A} = \{\mathbb{N}, S, =, \neq\}$, where
- $S = \{(n, n+1) \mid n \in \mathbb{N}\}$.
- $\mathbf{0}_e \in \mathrm{DS}(\mathfrak{A})$ and then all total enumeration degrees are elements of $\mathrm{DS}(\mathfrak{A})$.
- $B_0 = \emptyset'$, $B_n = \emptyset$ for each $n \geq 1$.
- Let $f \in \mathcal{E}(\mathfrak{A}, \mathcal{B})$ and $f(x_0) = 0$.
- $k \in B_n \iff (\exists x_1)\dots(\exists x_k)(f^{-1}(S)(x_0, x_1)\ \&\dots\&\ f^{-1}(S)(x_{k-1}, x_k)\ \&\ x_k \in f^{-1}(B_n))$.
- $B_n \leq_e f^{-1}(\mathfrak{A}) \oplus f^{-1}(B_n) \leq_e f^{-1}(\mathfrak{A})^{(n)}$.
- Then $\emptyset' \leq_e B_0 \leq_e f^{-1}(\mathfrak{A})$. Thus $\mathbf{0}_e \notin \mathrm{DS}(\mathfrak{A}, \mathcal{B})$.

ω- Degree Spectra

Proposition

$\mathrm{DS}(\mathfrak{A}, \mathcal{B})$ *is upwards closed with respect to total e-degrees.*

Lemma

Let f be an enumeration of $\mathfrak{A}$ and F be a total set such that $f^{-1}(\mathfrak{A}) \leq_{\mathrm{e}} F$ and $f^{-1}(B_n) \leq_{\mathrm{e}} F^{(n)}$ uniformly in n. Then there exists an acceptable enumeration g of $\mathfrak{A}$ with respect to $\mathcal{B}$ such that $g^{-1}(\mathfrak{A}) \equiv_{\mathrm{e}} F$.

ω- Jump Spectra

Definition

The k*th* ω*-jump spectrum of* $\mathfrak{A}$ *with respect to* $\mathcal{B}$ is the set

$$\mathrm{DS}_k(\mathfrak{A},\mathcal{B}) = \{\mathbf{a}^{(\mathbf{k})} \mid \mathbf{a} \in \mathrm{DS}(\mathfrak{A},\mathcal{B})\}.$$

Proposition

$\mathrm{DS}_k(\mathfrak{A},\mathcal{B})$ *is upwards closed with respect to total e-degrees.*

Lemma (Soskov)

Let $Q \subseteq \mathbb{N}$ *be a total set,* $B_0, \dots, B_k \subseteq \mathbb{N}$*, such that* $\mathcal{P}_k(\{B_0, \dots, B_k\}) \leq_{\mathrm{e}} Q$*. There is a total set* F *such that:*

- $F^{(k)} \equiv Q$.
- $(\forall i \leq k)(B_i \leq_{\mathrm{e}} F^{(i)})$.

ω-Co-Spectra

For every $\mathcal{A} \subseteq \mathcal{D}_\omega$ let
$co(\mathcal{A}) = \{\mathbf{b} \mid \mathbf{b} \in \mathcal{D}_\omega \;\&\; (\forall \mathbf{a} \in \mathcal{A})(\mathbf{b} \leq_\omega \mathbf{a})\}$.

Definition

The ω-co-spectrum of $\mathfrak{A}$ with respect to $\mathcal{B}$ is the set

$$\mathrm{CS}(\mathfrak{A}, \mathcal{B}) = co(\mathrm{DS}(\mathfrak{A}, \mathcal{B})).$$

For every enumeration f of $\mathcal{E}(\mathfrak{A}, \mathcal{B})$ consider the sequence

- $f^{-1}(\mathcal{B}) = \{f^{-1}(\mathfrak{A}) \oplus f^{-1}(B_0), f^{-1}(B_1), \dots, f^{-1}(B_n), \dots\}$
- $\mathcal{P}(f^{-1}(\mathcal{B})) \equiv_\omega \{f^{-1}(\mathfrak{A})^{(n)}\}_{n<\omega} \equiv_\omega f^{-1}(\mathfrak{A}) \uparrow \omega$.
- So $f \in \mathcal{E}(\mathfrak{A}, \mathcal{B})$ iff $\mathcal{P}(f^{-1}(\mathcal{B})) \leq_\omega f^{-1}(\mathfrak{A}) \uparrow \omega$.

kth ω-Co-Spectrum

Proposition

For each $\mathcal{A} \in \mathcal{S}$ it holds that $d_\omega(\mathcal{A}) \in \mathrm{CS}(\mathfrak{A}, \mathcal{B})$ if and only if $\mathcal{A} \leq_\omega \mathcal{P}(f^{-1}(\mathcal{B}))$ for every $f \in \mathcal{E}(\mathfrak{A}, \mathcal{B})$.

Actually the elements of the ω-co-spectrum of $\mathfrak{A}$ with respect to $\mathcal{B}$ form a countable ideal in $\mathcal{D}_\omega$.

Definition

The kth ω-co-spectrum of $\mathfrak{A}$ with respect to $\mathcal{B}$ is the set

$$\mathrm{CS}_k(\mathfrak{A}, \mathcal{B}) = co(\mathrm{DS}_k(\mathfrak{A}, \mathcal{B})).$$

We will see that the kth ω-co-spectrum of $\mathfrak{A}$ with respect to $\mathcal{B}$ is the least ideal containing all kth ω-enumeration jumps of the elements of $\mathrm{CS}(\mathfrak{A}, \mathcal{B})$.

Normal Form Theorem

Let $\mathcal{L}$ be the language of the structure $\mathfrak{A}$. For each n let P_n be a new unary predicate representing the set B_n.

- An elementary Σ_0^+ formula is an existential formula of the form $\exists Y_1 \dots \exists Y_m \Phi(W_1, \dots, W_r, Y_1, \dots, Y_m)$, where Φ is a finite conjunction of atomic formulae in $\mathcal{L} \cup \{P_0\}$;
- A Σ_n^+ formula is a c.e. disjunction of elementary Σ_n^+ formulae;
- An elementary Σ_{n+1}^+ formula is a formula of the form $\exists Y_1 \dots \exists Y_m \Phi(W_1, \dots, W_r, Y_1, \dots, Y_m)$, where Φ is a finite conjunction of atoms of the form $P_{n+1}(Y_j)$ or $P_{n+1}(W_i)$ and Σ_n^+ formulae or negations of Σ_n^+ formulae in $\mathcal{L} \cup \{P_0\} \cup \dots \cup \{P_n\}$.

Normal Form Theorem

Definition

The sequence $\mathcal{A} = \{A_n\}_{n<\omega}$ of sets of natural is *formally k-definable* on $\mathfrak{A}$ with respect to $\mathcal{B}$ if there exists a computable function $\gamma(x, n)$ such that for each $n, x \in \omega$ $\Phi^{\gamma(n,x)}(W_1, \ldots, W_r)$ is a Σ^+_{n+k} formula, and elements $t_1, \ldots, t_r$ of $|\mathfrak{A}|$ such that for every $n, x \in \omega$, the following equivalence holds:

$$x \in A_n \iff (\mathfrak{A}, \mathcal{B}) \models \Phi^{\gamma(n,x)}(W_1/t_1, \ldots, W_r/t_r).$$

Theorem

The sequence $\mathcal{A}$ of sets of natural numbers is formally k-definable *on $\mathfrak{A}$ with respect to $\mathcal{B}$ iff $d_\omega(\mathcal{A}) \in \mathrm{CS}_k(\mathfrak{A}, \mathcal{B})$.*

Properties of upwards closed sets

Let $\mathcal{A} \subseteq \mathcal{D}_e$ be an upwards closed set with respect to total e-degrees.
We remind that
$co(\mathcal{A}) = \{\mathbf{b} \mid \mathbf{b} \in \mathcal{D}_\omega \ \& \ (\forall \mathbf{a} \in \mathcal{A})(\mathbf{b} \leq_\omega \mathbf{a})\}$.

Proposition

$co(\mathcal{A}) = co(\{\mathbf{a} : \mathbf{a} \in \mathcal{A} \ \& \ \mathbf{a}$ is total$\})$.

Corrolary

$\mathrm{CS}(\mathfrak{A}, \mathcal{B}) = co(\{\mathbf{a} \mid \mathbf{a} \in \mathrm{DS}(\mathfrak{A}, \mathcal{B}) \ \& \ \mathbf{a}$ *is a total e-degree*$\})$.

Negative results (Vatev)

Let $\mathcal{A} \subseteq \mathcal{D}_e$ be an upwards closed set with respect to total e-degrees and $k > 0$.

Proposition

There exists $\mathbf{b} \in \mathcal{D}_e$ *such that*

$$co(\mathcal{A}) \neq co(\{\mathbf{a} : \mathbf{a} \in \mathcal{A} \,\&\, \mathbf{b} \leq \mathbf{a}^{(k)}\}).$$

- Let $\mathrm{d}_e(A) \in \mathcal{A}$ and a set $B \not\leq_{\mathrm{e}} A^{(k)}$.
- Consider $\mathcal{B} = \{\emptyset, \ldots, \emptyset^{(k-1)}, B, B', \ldots, \}$.
- $\mathcal{B} \not\leq_\omega A \uparrow \omega \Rightarrow \mathbf{d}_\omega(\mathcal{B}) \notin co(\mathcal{A})$.
- $\mathcal{B} \leq_\omega C \uparrow \omega$ for each C s.t. $B \leq_{\mathrm{e}} C^{(k)}$.

Negative results (Vatev)

Proposition

Let $n > 0$. There is a structure $\mathfrak{A}$, a sequence $\mathcal{B}$ and $\mathbf{c} \in \mathrm{DS}_n(\mathfrak{A}, \mathcal{B})$ such that if $\mathcal{A} = \{\mathbf{a} \in \mathrm{DS}(\mathfrak{A}, \mathcal{B}) \mid \mathbf{a}^{(n)} = \mathbf{c}\}$ then

$$\mathrm{CS}(\mathfrak{A}, \mathcal{B}) \neq co(\mathcal{A}).$$

- Consider a linear order $\mathfrak{A}$ which has no n-jump degree, $\mathcal{B} = \emptyset \uparrow \omega$ and $\mathrm{d}_e(C) \in \mathrm{DS}_n(\mathfrak{A})$.
- Consider $\mathcal{C} = \{\emptyset, \dots, \emptyset^{(n-1)}, C, C', \dots, \}$.
- $\mathbf{d}_\omega(\mathcal{C}) \notin \mathrm{CS}(\mathfrak{A})$, otherwise $\mathrm{d}_e(C)$ will be an n-jump degree of $\mathfrak{A}$.
- $\mathbf{d}_\omega(\mathcal{C}) \in co(\mathcal{A})$.

Minimal pair theorem

Theorem

For every structure $\mathfrak{A}$ and every sequence $\mathcal{B} \in \mathcal{S}$ there exist total enumeration degrees **f** *and* **g** *in* $\mathrm{DS}(\mathfrak{A}, \mathcal{B})$ *such that for every ω-enumeration degree* **a** *and $k \in \mathbb{N}$:*

$$\mathbf{a} \leq_\omega \mathbf{f}^{(k)} \ \& \ \mathbf{a} \leq_\omega \mathbf{g}^{(k)} \Rightarrow \mathbf{a} \in \mathrm{CS}_k(\mathfrak{A}, \mathcal{B}) \ .$$

Minimal pair theorem

Proof.

Case $k = 0$.

- Let $f \in \mathcal{E}(\mathfrak{A}, \mathcal{B})$ and $F = f^{-1}(\mathfrak{A})$ is a total set.
- Denote by $\mathcal{X}_0, \mathcal{X}_1, \dots \mathcal{X}_r \dots$ all sequences ω-enumeration reducible to $\mathcal{P}(f^{-1}(\mathcal{B}))$.
- Consider $\mathcal{C}_0, \mathcal{C}_1, \dots, \mathcal{C}_r \dots$ among them which are not formally definable on $\mathfrak{A}$ with respect to $\mathcal{B}$.
- There is an enumeration h such that $\mathcal{C}_r \not\leq_\omega \mathcal{P}(h^{-1}(\mathcal{B}))$, $r \in \omega$.
- There is a total set G such that $\mathcal{P}(h^{-1}(\mathcal{B})) \leq_\omega G \uparrow \omega$ and $\mathcal{C}_r \not\leq_\omega G \uparrow \omega$, $r \in \omega$.
- There is a $g \in \mathcal{E}(\mathfrak{A}, \mathcal{B})$ such that $g^{-1}(\mathfrak{A}) \equiv_{\mathrm{e}} G$. Thus $d_{\mathrm{e}}(G) \in \mathrm{DS}(\mathfrak{A}, \mathcal{B})$.
- If $\mathcal{A} \leq_\omega F \uparrow \omega$ and $\mathcal{A} \leq_\omega G \uparrow \omega$ then $\mathcal{A} = \mathcal{X}_r$ and $\mathcal{A} \neq \mathcal{C}_l$ for all $l \in \omega$. So $d_\omega(\mathcal{A}) \in \mathrm{CS}(\mathfrak{A}, \mathcal{B})$.

Minimal pair theorem

Proof.

$I(\mathbf{a}) = \{\mathbf{b} \mid \mathbf{b} \in \mathcal{D}_\omega \ \& \ \mathbf{b} \leq_\omega \mathbf{a}\} = co(\{\mathbf{a}\})$.

- $\mathrm{CS}(\mathfrak{A}, \mathcal{B}) = I(\mathbf{f}) \cap I(\mathbf{g})$ where $\mathbf{f} = d_e(F)$ and $\mathbf{g} = d_e(G)$.
- We shall prove now that $I(\mathbf{f}^{(k)}) \cap I(\mathbf{g}^{(k)}) = \mathrm{CS}_k(\mathfrak{A}, \mathcal{B})$ for every k.
- $\mathbf{f}^{(k)}, \mathbf{g}^{(k)} \in \mathrm{DS}_k(\mathfrak{A}, \mathcal{B}) \Rightarrow \mathrm{CS}_k(\mathfrak{A}, \mathcal{B}) \subseteq I(\mathbf{f}^{(k)}) \cap I(\mathbf{g}^{(k)})$.
- Suppose that $\mathcal{A} = \{A_n\}_{n<\omega}$, $\mathcal{A} \leq_\omega F^{(k)} \uparrow \omega$ and $\mathcal{A} \leq_\omega G^{(k)} \uparrow \omega$.
- Denote by $\mathcal{C} = \{C_n\}_{n<\omega}$ the sequence such that $C_n = \emptyset$ for $n < k$, and $C_{n+k} = A_n$ for each n.
- $\mathcal{A} \leq_\omega \mathcal{C}^{(k)}$, $\mathcal{C} \leq_\omega F \uparrow \omega$ and $\mathcal{C} \leq_\omega G \uparrow \omega \Rightarrow d_\omega(\mathcal{C}) \in \mathrm{CS}(\mathfrak{A}, \mathcal{B})$.
- Let $h \in \mathcal{E}(\mathfrak{A}, \mathcal{B})$. Then $\mathcal{C} \leq_\omega h^{-1}(\mathfrak{A}) \uparrow \omega$ and thus $\mathcal{C}^{(k)} \leq_\omega (h^{-1}(\mathfrak{A}) \uparrow \omega)^{(k)}$.
- Hence $d_\omega(\mathcal{A}) \in \mathrm{CS}_k(\mathfrak{A}, \mathcal{B})$.

Countable ideals of ω-enumeration degrees

Corrolary

$\mathrm{CS}_k(\mathfrak{A}, \mathcal{B})$ is the least ideal containing all kth ω-jumps of the elements of $\mathrm{CS}(\mathfrak{A}, \mathcal{B})$.

- $I = \mathrm{CS}(\mathfrak{A}, \mathcal{B})$ is a countable ideal;
- $\mathrm{CS}(\mathfrak{A}, \mathcal{B}) = I(\mathbf{f}) \cap I(\mathbf{g})$;
- $I^{(k)}$ - the least ideal, containing all kth ω-jumps of the elements of I;
- (Ganchev) $I = I(\mathbf{f}) \cap I(\mathbf{g}) \Longrightarrow I^{(k)} = I(\mathbf{f}^{(k)}) \cap I(\mathbf{g}^{(k)})$ for every k;
- $I(\mathbf{f}^{(k)}) \cap I(\mathbf{g}^{(k)}) = \mathrm{CS}_k(\mathfrak{A}, \mathcal{B})$ for each k
- Thus $I^{(k)} = \mathrm{CS}_k(\mathfrak{A}, \mathcal{B})$.

Countable ideals of ω-enumeration degrees

There is a countable ideal I of ω-enumeration degrees for which there is no structure $\mathfrak{A}$ and sequence $\mathcal{B}$ such that $I = \mathrm{CS}(\mathfrak{A}, \mathcal{B})$.

- $\mathcal{A} = \{\mathbf{0}, \mathbf{0}', \mathbf{0}'', \dots, \mathbf{0}^{(n)}, \dots\}$;
- $I = I(\mathcal{A}) = \{\mathbf{a} \mid \mathbf{a} \in \mathcal{D}_\omega \;\&\; (\exists n)(\mathbf{a} \leq_\omega \mathbf{0}^{(n)})\}$ - a countable ideal generated by $\mathcal{A}$.
- Assume that there is a structure $\mathfrak{A}$ and a sequence $\mathcal{B}$ such that $I = \mathrm{CS}(\mathfrak{A}, \mathcal{B})$
- Then there is a minimal pair $\mathbf{f}$ and $\mathbf{g}$ for $\mathrm{DS}(\mathfrak{A}, \mathcal{B})$, so $I^{(n)} = I(\mathbf{f}^{(n)}) \cap I(\mathbf{g}^{(n)})$ for each n.
- $\mathbf{f} \geq \mathbf{0}^{(n)}$ and $\mathbf{g} \geq \mathbf{0}^{(n)}$ for each n.
- Then by Enderton and Putnam [1970], Sacks [1971]: $\mathbf{f}'' \geq \mathbf{0}^{(\omega)}$ and $\mathbf{g}'' \geq \mathbf{0}^{(\omega)}$.
- Hence $I'' \neq I(\mathbf{f}'') \cap I(\mathbf{g}'')$. A contradiction.

Quasi-Minimal Degree

Theorem

For every structure $\mathfrak{A}$ and every sequence $\mathcal{B}$, there exists $F \subseteq \mathbb{N}$, such that $\mathbf{q} = d_\omega(F \uparrow \omega)$ and:

1. $\mathbf{q} \notin \mathrm{CS}(\mathfrak{A}, \mathcal{B})$;
2. *If* $\mathbf{a}$ *is a total e-degree and* $\mathbf{a} \geq_\omega \mathbf{q}$ *then* $\mathbf{a} \in \mathrm{DS}(\mathfrak{A}, \mathcal{B})$
3. *If* $\mathbf{a}$ *is a total e-degree and* $\mathbf{a} \leq_\omega \mathbf{q}$ *then* $\mathbf{a} \in \mathrm{CS}(\mathfrak{A}, \mathcal{B})$.

Quasi-Minimal Degree

Proof.

- (Soskov) There is a partial generic enumeration f of $\mathfrak{A}$ such that $d_e(f^{-1}(\mathfrak{A}))$ is quasi-minimal with respect to $\mathrm{DS}(\mathfrak{A})$ and $f^{-1}(\mathfrak{A}) \not\leq_e D(\mathfrak{A})$.
- (Ganchev) There is a set F such that $f^{-1}(\mathfrak{A}) <_e F$, $f^{-1}(\mathcal{B}) \leq_\omega F \uparrow \omega$ and for total X: $X \leq_e F \Rightarrow X \leq_e f^{-1}(\mathfrak{A})$.
- Set $\mathbf{q} = d_\omega(F \uparrow \omega)$ and let X be a total set.
- If $\mathbf{q} \in \mathrm{CS}(\mathfrak{A}, \mathcal{B})$ then $d_\omega(f^{-1}(\mathfrak{A}) \uparrow \omega) \in \mathrm{CS}(\mathfrak{A}, \mathcal{B})$. Then $f^{-1}(\mathfrak{A}) \leq_e D(\mathfrak{A})$. A contradiction.
- If $X \leq_e F$ then $X \leq_e f^{-1}(\mathfrak{A})$. Thus $d_e(X) \in \mathrm{CS}(\mathfrak{A})$. But $\mathrm{DS}(\mathfrak{A}, \mathcal{B}) \subseteq \mathrm{DS}(\mathfrak{A})$. So $d_\omega(X \uparrow \omega) \in \mathrm{CS}(\mathfrak{A}, \mathcal{B})$.
- If $X \geq_e F$ then $X \geq_e f^{-1}(\mathfrak{A})$. Hence $\mathrm{dom}(f)$ is c.e. in X. Let ρ be a computable in X enumeration of $\mathrm{dom}(f)$. Set $h = \lambda n.f(\rho(n))$. So $h^{-1}(\mathcal{B}) \leq_e X \uparrow \omega$. Then $d_e(X) \in \mathrm{DS}(\mathfrak{A}, \mathcal{B})$.

$\square$

ω-degree spectra

- Questions:
  - Is it true that for every structure $\mathfrak{A}$ and every sequence $\mathcal{B}$ there exists a structure $\mathfrak{B}$ such that $\mathrm{DS}(\mathfrak{B}) = \mathrm{DS}(\mathfrak{A}, \mathcal{B})$?
  - If for a countable ideal $I \subseteq \mathcal{D}_\omega$ there is an exact pair then are there a structure $\mathfrak{A}$ and a sequence $\mathcal{B}$ so that $\mathrm{CS}(\mathfrak{A}, \mathcal{B}) = I$?

- Ganchev, H., Exact pair theorem for the ω-enumeration degrees, *LNCS*, (B. Löwe S. B. Cooper and A. Sorbi, eds.), **4497**, 316–324 (2007)
- Soskov I. N., Degree spectra and co-spectra of structures. *Ann. Univ. Sofia*, **96** 45–68 (2003)
- Soskov, I. N., Kovachev, B. Uniform regular enumerations *Mathematical Structures in Comp. Sci.* **16** no. 5, 901–924 (2006)
- Soskov, I. N. The ω-enumeration degrees, *J. Logic and Computation* **17** no. 6, 1193-1214 (2007)
- Soskov, I. N., Ganchev H. The jump operator on the ω-enumeration degrees. *Annals of Pure and Applied Logic*, **160**, 289–301 (2009)
- Soskova, A. A. Relativized degree spectra. J. Logic and Computation **17**, no. 6, 1215-1233 (2007)