Random Sets and Functions

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Outline

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Notation

(1) $\omega^{<\omega}$ is the set of finite strings of elements of ω

(2) $|\sigma|$ equals the length of $\sigma \in \omega^*$

(3) $\sigma \sqsubseteq \tau$ means $|\sigma| \leqslant |\tau|$ and $\sigma(i) = \tau(i) \quad (\forall i < |\sigma|)$

(4) $\sigma^\frown i$ means σ concatenated with i

(5) $x \restriction n$ equals $(x(0), x(1), ..., x(n-1))$

(6) $[\sigma]$ equals $\{x \in \omega^\omega : \sigma \sqsubset x\}$

Closed Sets

Recall, a tree T is a subset of $\omega^{<\omega}$ closed under init. segments. For example, if $63972 \in T$ then $6, 63, 639, 6397 \in T$

Definitions

(i) $\sigma \in T$ is a **dead end** $\iff$ $\sigma^\frown n \notin T$ for all n

(ii) Given a tree $T \subseteq \omega^*$, define $[T] := \{x \in \omega^\omega : (\forall n)\ x\restriction n \in T\}$

Fact: $K \subseteq \omega^\omega$ is **closed** $\iff$ $K = [T]$ for some T

Definitions

(1) $K \subseteq \omega^\omega$ is **effectively closed** $\iff$ $K = [T]$, some comp. T
K^c is said to be a *c.e. open set*.

(2) $K \subseteq \omega^\omega$ is **decidable effectively closed** $\iff$
$K = [T]$ for some comp. T without dead ends

Random Reals

A Definition Based on Measure

One could imagine that a random real $x \in 2^\omega$ should be one that is 'typical' or one that 'lacks any special kind of properties.' Intuitively, it satisfies all statistical laws that hold with prob. 1. We might imagine it as that which is in all meas. 1 sets (or equiv. in all meas. 0 sets). Martin-Löf (1966) defined $x \in 2^\omega$ to be random if it avoids effectively presented measure zero sets.

Definition

$x \in 2^\omega$ is **Martin-Löf random** $\iff$ for any effective sequence $S_1, S_2, \ldots$ of c.e. open sets with $\mu(S_n) \leqslant 2^{-n}$, $x \notin \bigcap_n S_n$

Note

As observed by Solovay, it is suffices (w/ approp. sum bounds) to have $\lim_n \mu(S_n) = 0$. (Instead of $\mu(S_n) \leqslant 2^{-n}$ for all n.)

Outline

Random Closed Sets

Let $\mathfrak{C}$ be the space of nonempty closed $Q \subseteq 2^\omega$.
We will define an eff. bijection $\psi : \mathfrak{C} \to 3^\omega$ and say:

Definition

$Q \in \mathfrak{C}$ is **(Martin-Löf) random** iff $\psi(Q) \in 3^\omega$ is Martin-Löf rand.

First note that if $Q \in \mathfrak{C}$, then there is a unique tree without dead ends $T_Q := \{\sigma : Q \cap [\sigma] \neq \emptyset\}$ such that $Q = [T_Q]$.

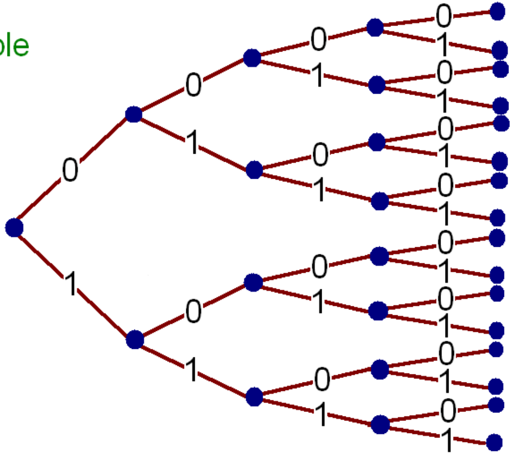
Let $\psi : \mathfrak{C} \to 3^\omega$ be given by $Q = [T_Q] \mapsto x_Q$ as follows.

Order T_Q first by length then lexicographically: $\sigma_0 \prec \sigma_1 \prec \cdots$

Define
$$x_Q(n) = \begin{cases} 0 & \text{if } \sigma_n{}^\frown 0 \in T \text{ and } \sigma_n{}^\frown 1 \notin T \\ 1 & \text{if } \sigma_n{}^\frown 1 \in T \text{ and } \sigma_n{}^\frown 0 \notin T \\ 2 & \text{if } \sigma_n{}^\frown 0 \in T \text{ and } \sigma_n{}^\frown 1 \in T \end{cases}$$

An Example of ψ

Example



An Example of ψ

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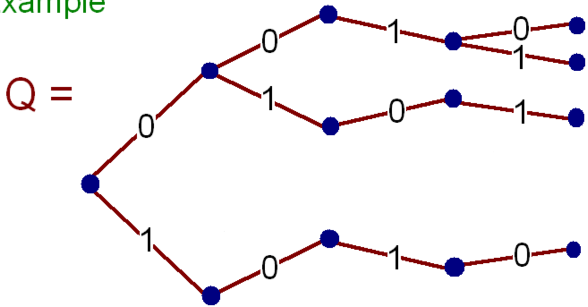
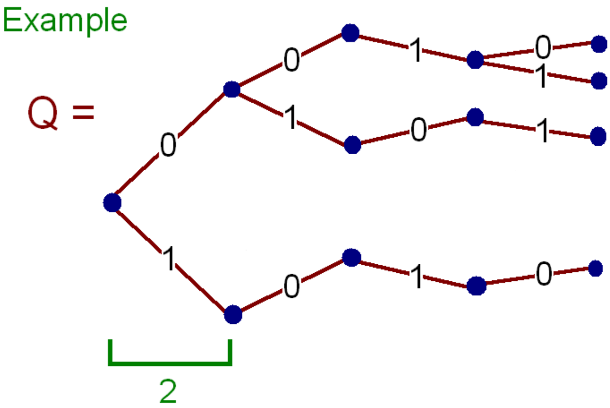
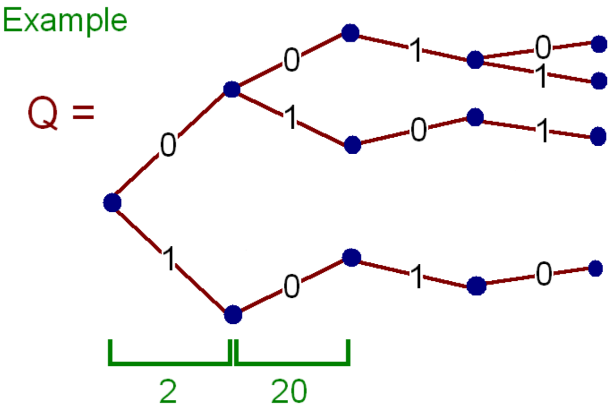


An Example of ψ



An Example of ψ



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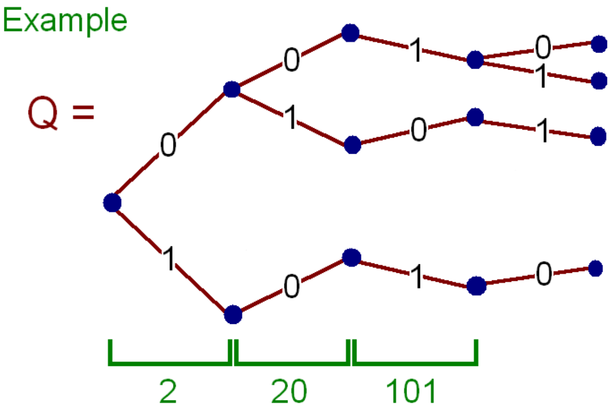

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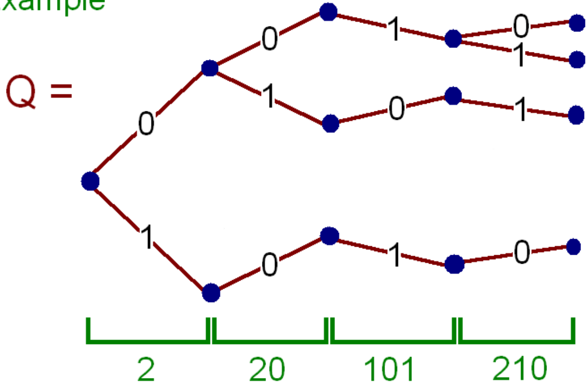

An Example of ψ

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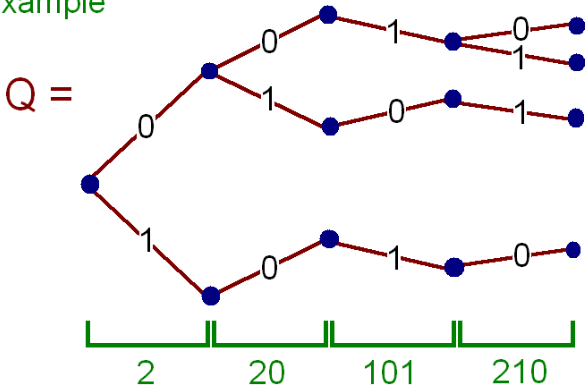


So, $x_Q = 220101210 \cdots$

The Measure

Define a measure $\mu_{\mathfrak{C}}$ on $\mathfrak{C}$ by

$$\mu_{\mathfrak{C}}(S) = \mu(\{x_Q \in \{0,1,2\}^\omega : Q \in S\})$$

where μ is the Lebesgue measure on $\{0,1,2\}^\omega$.

Informal Interpretation

Let $Q = [T_Q]$ be closed and $\sigma \in T_Q$. Then there is a $\frac{1}{3}$ probability that each of the following cases occurs:

- $\sigma^\frown 0 \in T_Q$ and $\sigma^\frown 1 \notin T_Q$
- $\sigma^\frown 1 \in T_Q$ and $\sigma^\frown 0 \notin T_Q$
- $\sigma^\frown 0 \in T_Q$ and $\sigma^\frown 1 \in T_Q$

Formalizing earlier comments, we say $\{S_n\}_{n\geqslant 0}$ is a *Martin-Löf Test* in $\mathfrak{C}$ iff $(\{x_Q : Q \in S_n\})_{n\geqslant 0}$ is an effective sequence of c.e. open sets in 3^ω such that $\lim_{n\to\infty} \mu_{\mathfrak{C}}(S_n) = 0$

Properties of Random Closed Sets

A Summary

Existence Properties

(1) Π^0_2 random closed sets exist.
(2) Random (decidable) eff. closed sets don't exist.

Internal Properties

(1) They contain no n-c.e. or 1-generic Δ^0_2 paths.
(2) The left and rightmost paths are not random.
(3) Each contains a random element.
(Conv., all rand. reals belong to r.c. sets [Miller, Montalbán])

External Properties

They are perfect, have meas. 0, and Hausdorff dim. $\log_2(4/3)$.

Properties of Random Closed Sets

A Summary

Theorem

Let Q be a random closed set. Then,

(1) T_Q *has low degree* $\Rightarrow$ *Q has a low random element*

(2) T_Q *is* Δ^0_2 $\Rightarrow$ *Q has a* Δ^0_2 *element*

(3) T_Q *is* Δ^0_3 $\not\Rightarrow$ *Q has a* Δ^0_2 *element*

Theorem

Let P and Q be closed sets with codes x_P *and* x_Q*, resp. Then* $\{0^\frown x : x \in P\} \cup \{1^\frown x : x \in Q\}$ *is random iff* $x_P \oplus x_Q$ *is random.*

Q: What are the Medvedev or Muchnik deg. of rand. cl. sets?

Next Step: Randomness for closed sets of actual reals.

Outline

Random Continuous Functions

Using the methods used for random closed sets, we consider a notion of randomness for $\mathcal{C}$, the cont. functions $F : 2^\omega \to 2^\omega$.

Representing Continuous Functions

We may represent F by a function $f : 2^{<\omega} \to 2^{<\omega}$ s.t.

(1) f is $\sqsubseteq$-inclusion preserving

(2) f is *output length* bounded by *input length*

(3) for fixed $x \in 2^\omega$, the length of $f(x \restriction n)$ becomes arbitrarily large as n does

(4) $F(x)$ is the unique element in $\cap_n [f(x \restriction n)]$

We are interested in the family $\mathcal{F}$ of all f that satisfy (1) and (2).

Note

Every continuous F has infinitely many representatives $f \in \mathcal{F}$.

Random Continuous Functions

Labeling Functions

To each $f \in \mathcal{F}$, we define a labeling function ℓ_f.

$$\ell_f(\sigma^\frown i) = \begin{cases} f(\sigma^\frown i)(n) & \text{if } n = |f(\sigma^\frown i)| > |f(\sigma)| \\ 2 & \text{otherwise} \end{cases}$$

$\ell_f(\sigma^\frown i)$ is said to *label* the f-output of $\sigma^\frown i$.

Order $2^{<\omega} \setminus \{\emptyset\}$ first by length and then lexic: $\sigma_0 \prec \sigma_1 \prec \dots$
Define $c_f(n) = \ell_f(\sigma_n)$, the code of the label function for f

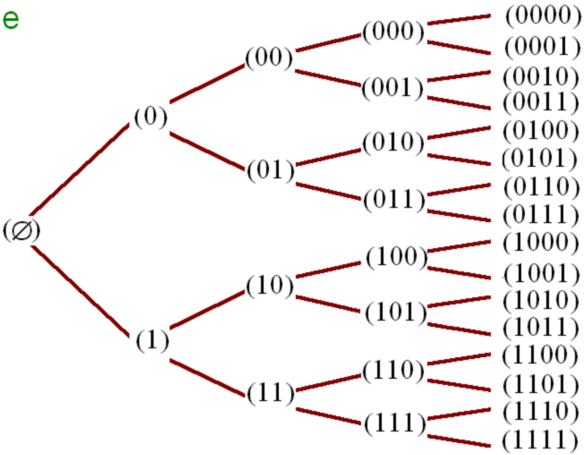
Definition

F is **eff. rand. cont.** iff $(\exists f \in \mathcal{F})$ [f repr. F & $c_f \in 3^\omega$ is rand.]

We define $\mu_{\mathcal{C}}(S) = \mu(\{c_f \in 3^\omega : f \text{ represents some } F \in S\})$

An Example of some c_f

Example



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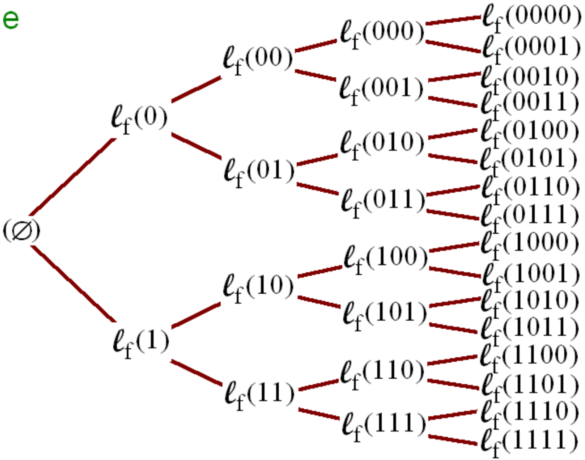
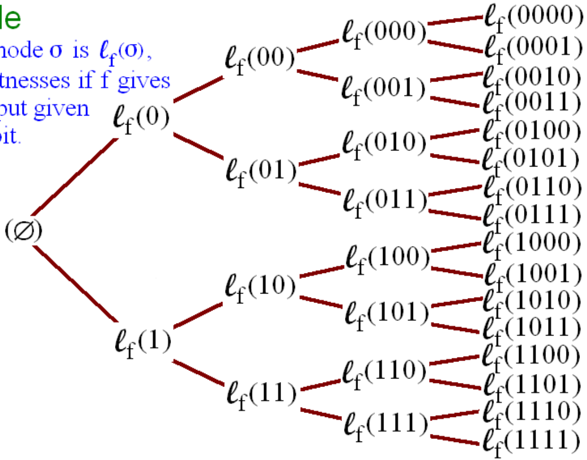


An Example of some c_f

Example

At each node σ is $\ell_f(\sigma)$, which witnesses if f gives more output given σ's new bit.



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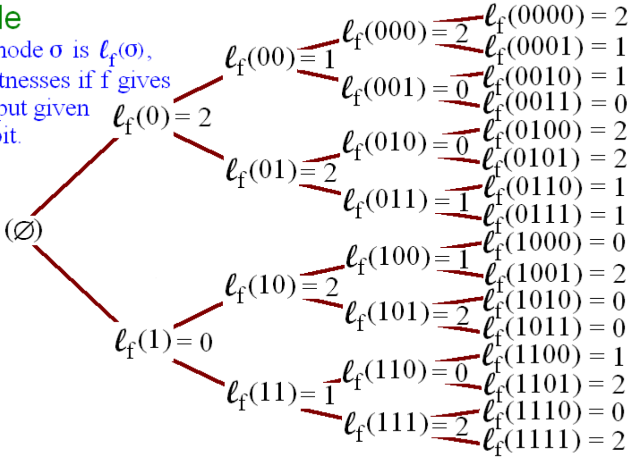

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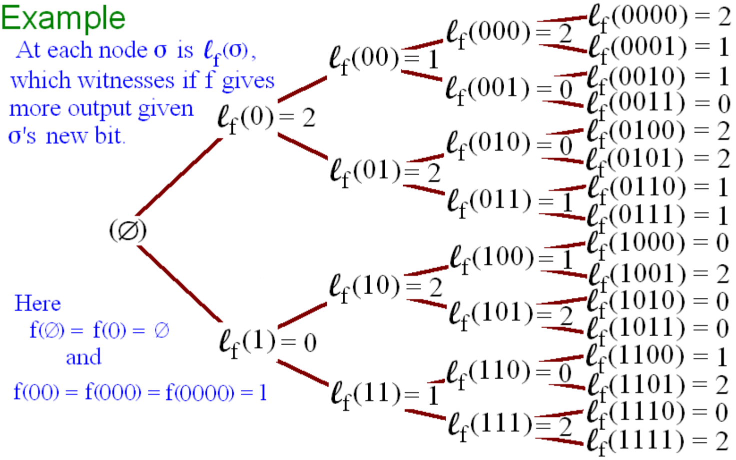

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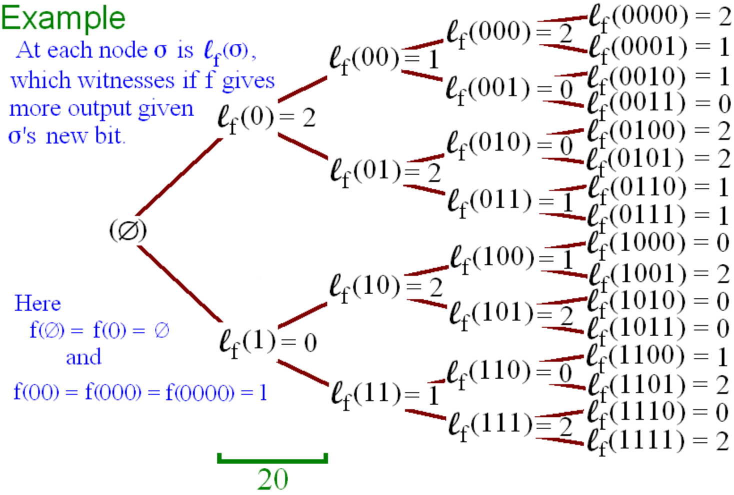
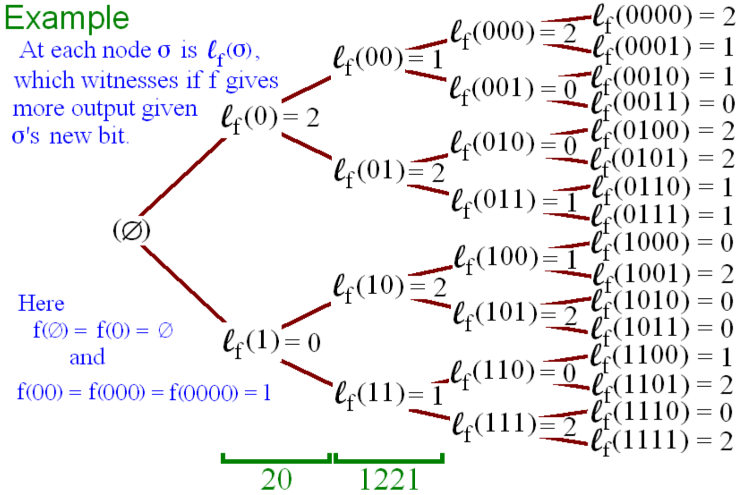
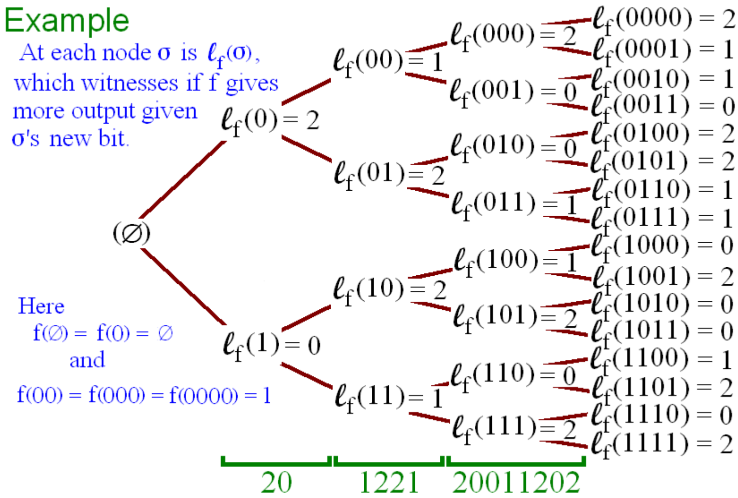


An Example of some c_f



An Example of some c_f

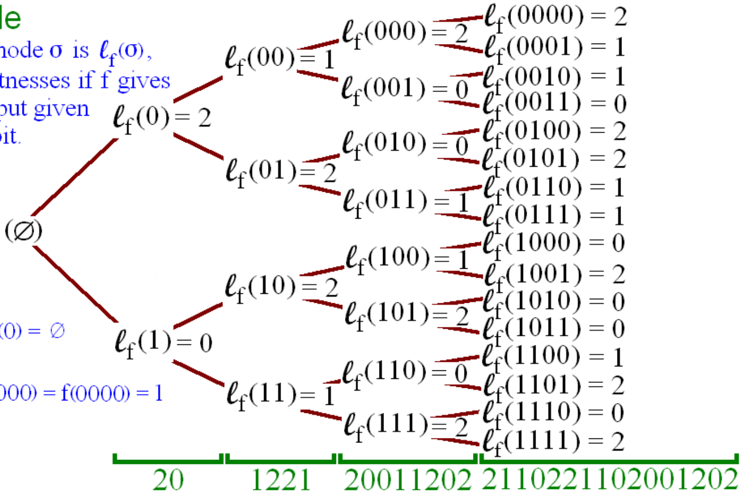


An Example of some c_f

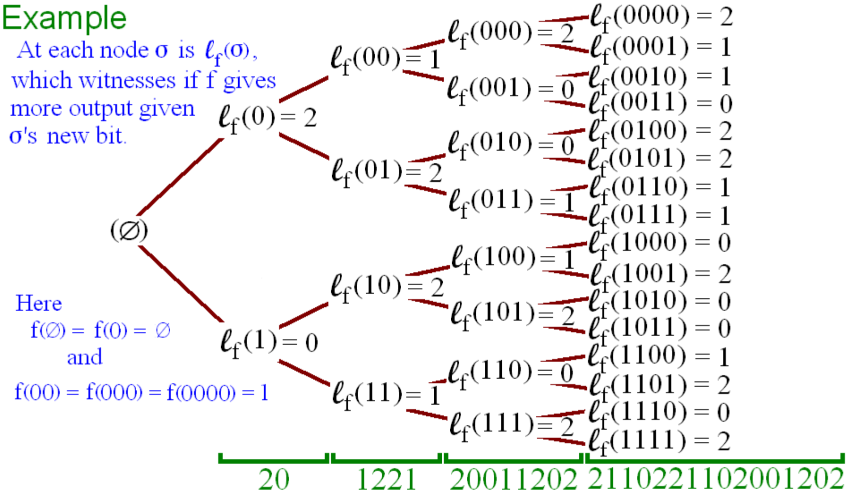
Example

At each node σ is $\ell_f(\sigma)$, which witnesses if f gives more output given σ's new bit.

Here $f(\emptyset) = f(0) = \emptyset$ and $f(00) = f(000) = f(0000) = 1$



An Example of some c_f



So, $c_f = 20122120011202211022110200120 2\cdots$

Interpreting Random Continuous Functions

Informal Interpretation

Given a representation $f \in \mathcal{F}$ of F, for each new bit i of input, there is a $\frac{1}{3}$ probability that each of the following occurs:

- f gives a new output of 0 $(\ell_f(\sigma^\frown i) = 0)$
- f gives a new output of 1 $(\ell_f(\sigma^\frown i) = 1)$
- f gives no new output $(\ell_f(\sigma^\frown i) = 2)$

Geometric Interpretation

Given a representation $f \in \mathcal{F}$, the graph GrF of F may be viewed as a decreasing sequence of closed sets in the unit square. Initially, $f((0))$ and $f((1))$ select from the four quadrants.

Ex. $f((0)) = (0) = f((1)) \rightarrow \text{Gr}F \subseteq [0,1] \times [0, \frac{1}{2}]$

Ex. $f((0)) = \emptyset, f((1)) = (1) \rightarrow \text{Gr}F \cap ((\frac{1}{2}, 1] \times [0, \frac{1}{2})) = \emptyset$

Note: $\exists\ \Delta^0_2$-rand. in 3^ω & $\mu(\text{rand.}) = 1$, so $\exists$ a Δ^0_2-comp. F.

Properties of Random Continuous Functions

Lemma Let Σ_1, Σ be a finite alphabets s.t. $\Sigma_1 \subset \Sigma$, $|\Sigma_1| \geq 2$. If $z \in \Sigma^\omega$ is Martin-Löf random and $G(z)$ is the result of removing from z all symbols from $\Sigma \setminus \Sigma_1$, then $G(z)$ is ML-random in Σ_1^ω

Proof. Assume W.L.O.G. that $\Sigma = \{0, 1, 2\}$ and $\Sigma_1 = \{0, 1\}$. Let $\{S_n\}_{n\in\omega}$ be a Martin-Löf test and f computable s.t. $(\forall n)$ $S_n = \bigcup_j [\sigma_{f(i,j)}]$. We may show: $(\forall n)$ $\mu_\Sigma(G^{-1}(S_n)) = \mu_{\Sigma_1}(S_n)$. If $g : 3^{<\omega} \to 2^{<\omega}$ removes all 2's, then $\exists$ comp. h s.t. $\{\sigma_{h(i,j,n)}\}_{n\in\omega}$ enum. $g^{-1}(\{\sigma_{f(i,j)}\})$. We obtain $G^{-1}(S_n) = \bigcup [\sigma_{h(i,j,n)}]$ for all n, a ML-test – which must omit z. So $\{S_n\}$ omits $G(z)$.

Theorem F is rand. cont. $\Rightarrow$ $F(x)$ is random for any comp. x.

Proof. Suppose f repr. F. Define comp. g s.t. $\sigma_{g(n)} = x \upharpoonright n$. By the Von-Mises-Church-Wald Comp. Sel. Theorem, the subseq. $z(n) = c_f(g(n))$ is random. If G removes the 2's, then $F(x) = G(z)$. This is random by the Lemma.

Properties of Random Continuous Functions

Theorem F rand. cont. $\Rightarrow F[2^\omega]$ has no isolated points

Proof. Suppose f repr. F & y is isolated.
Fix $\tau \sqsubset y$ s.t. $[\tau] \cap F[2^\omega] = \{y\}$. Also fix σ so $f(\sigma) = \tau$.
Let S_n be the set of $g \in \mathcal{F}$ s.t. $\forall \rho_1, \rho_2 \in 2^n$:

$$\tau \sqsubseteq g(\sigma^\frown \rho_1) \sqsubseteq g(\sigma^\frown \rho_2) \text{ or } \tau \sqsubseteq g(\sigma^\frown \rho_2) \sqsubseteq g(\sigma^\frown \rho_1).$$

For each $\rho \in 2^m$ ($m < n$), there are at most 7 of 9 choices. So $\mu(S_n) \leq (\frac{7}{9})^n$, and actually $\{S_n\}$ is a ML-test. Hence $\exists n\, F \notin S_n$.
So $\exists$ 2 incompatible length-n extensions, contr. choice of τ.

Hence, F rand. cont. $\Rightarrow F[2^\omega]$ is perfect with $2^{\aleph_0}$ many elts.

Lemma

F is rand. contin. $\Leftrightarrow (\forall \sigma \in 2^{<\omega})\ F_\sigma(x) = F(\sigma^\frown x)$ is rand. cont.

So, rand. cont. fun. not ness. onto: $(\forall \tau)\ \exists F$ with $\mathrm{Im} F \subseteq [\tau]$.

Properties of Random Continuous Functions

A Summary

Existence Properties

Δ^0_2 random cont. functions exist, but computable ones don't.

Domain Properties

(1) Computable elements map to a random elements.
(2) Conj: (Decidable) eff. closed sets map to random sets.

Image Properties

(1) The image is a perfect set (with continuum many elements).
(2) Rand. cont. funct. not ness. onto: $(\forall \tau)\ \exists F$ with $\mathrm{Im}F \subseteq [\tau]$.
(3) If $y \in 2^\omega$, then $y \in \mathrm{Im}F$ for some random cont. function F.
Corollary: Image not necessarily a random closed set.
(4) The set of elements mapping to 0^ω is a random cl. set.

Random Continuous Functions

and pseudo-distance functions

Definition

$\Delta : 2^\omega \to 2^\omega$ is a **pseudo-distance** function for a set $Q \subseteq 2^\omega$ if Q is the set of elements that Δ maps to 0^ω.

The name comes from a modif. of the dist. funct. for a cl. set Q:

$$\mathrm{dist}_Q(x) = \begin{cases} 0 & \text{if } x \in Q \\ 2^n & \text{if } n = (\mu m)\ x \restriction m \notin T_Q \end{cases}$$

Replace 0, 2^ω by 0^ω, $0^n 1 0^\omega$ to obtain a pseudo-dist. fun. for Q.

Facts

(1) Q is closed $\Leftrightarrow \exists$ pseudo-dist funct. for Q

(2) Q is eff. closed $\Leftrightarrow \exists$ comp. pseudo-dist funct. for Q

(3) Q is rand. closed $\Leftarrow \exists$ rand. cont. pseudo-dist funct. for Q

Conjecture: '$\Rightarrow$'

n-Random Continuous Functions

and n-random closed sets

Recall

(1) A Σ^0_n **test** is a comp. collection $\{V_n\}$ of Σ^0_n classes with $\mu(V_n) \leq 2^{-n}$.
(2) $x \in 2^\omega$ is n**-random** iff it passes all Σ^0_n tests.

Fact x is $n+1$-random $\Leftrightarrow$ x is random relative to $0^{(n)}$.

Idea. Use a result of Kurtz and Kautz using the Leb. meas. μ.

We can obtain an analogue of the same result in 3^ω with $\mu_{\mathfrak{C}}$, $\mu_{\mathcal{C}}$.
This allows us to define:

(1) $Q \subset 2^\omega$ closed is $n+1$-random $\Leftrightarrow$ it is rand. rel. to $0^{(n)}$.
(2) $F : 2^\omega \to 2^\omega$ cont. is $n+1$-random $\Leftrightarrow$ it is rand. rel. to $0^{(n)}$.

Now relativize the previous results for n-rand. cont. functions.

Random Continuous Functions
and open questions

Q: Suppose $F(x) \in [\sigma]$, then $\text{Prob}(F(y) \in [\tau]) = ?$

Q: Does composition preserve randomness?

Q: Is there a notion of triviality? If so, what happens when a random function is composed with a trivial one?

Q: What geom. oper. can be devised that pres. randomness?

Next Step: Randomness for cont. $F : [0, 1] \to [0, 1]$ and $F : \mathbb{R} \to \mathbb{R}$ (by repr. fun. in terms of images of subintervals).

Conjecture: F rand. cont. $\Rightarrow$ F not left, right, weakly comp.

Conjecture: F rand. cont. $\Rightarrow$ F is nowhere differentiable.

Recognitions and References

Much recognition and appreciation goes to all conference organizers for their time, attention, and efforts.

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