

Compatible Functions by Priestley Duality

Leonardo M. Cabrer and Marta Sagastume

Introduction

- SOFRONIE-STOKKERMANS, V. *Duality and canonical extensions of bounded distributive lattices with operators, and applications to the semantics of non-classical logics I*. Studia Logica 64 (2000), 93-132.
- SOFRONIE-STOKKERMANS, V. *Duality and canonical extensions of bounded distributive lattices with operators, and applications to the semantics of non-classical logics II*. Studia Logica 64 (2000), 151-192.

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- 2 $f(a_1, \dots, a_{i-1}, b_1 \vee b_2, a_{i+1}, \dots, a_n) = f(a_1, \dots, a_{i-1}, b_1, a_{i+1}, \dots, a_n) \vee f(a_1, \dots, a_{i-1}, b_2, a_{i+1}, \dots, a_n)$.

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Definition

Given \mathbf{L} a bounded lattice and $\varepsilon_1, \dots, \varepsilon_n, \varepsilon \in \{-1, +1\}$. A function $f : L^n \rightarrow L$ is an **operator of type** $\varepsilon_1, \dots, \varepsilon_n \rightarrow \varepsilon$ if $f : L^{\varepsilon_1} \times \dots \times L^{\varepsilon_n} \rightarrow L^\varepsilon$ is a join hemimorphism.

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$$(P_1, \dots, P_n, P) \in R_f \text{ iff } f(P_1^{\varepsilon_1}, \dots, P_n^{\varepsilon_n}) \subseteq P^\varepsilon.$$

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- URQUHART, A. *Distributive Lattices with a Dual Homomorphic Operation*. *Studia Logica* 39 (1979) 201-209.
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- CELANI, S.A. *Distributive lattices with fusion and implication*. *Southeast Asian Bulletin of Mathematics* 28 (2004) 999-1010.

Congruences

Definition

Let $P \in X(\mathbf{L})$. $(P_1, \dots, P_n) \in \text{Max}R_f^{-1}(P)$ if and only the following statements hold:

- 1 $(P_1, \dots, P_n, P) \in R_f$.
- 2 (P_1, \dots, P_n) is maximal between the elements of $X(\mathbf{L})^{\varepsilon_1} \times \dots \times X(\mathbf{L})^{\varepsilon_n}$ with the property 1.

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Definition

A set closed $Y \subseteq X(\mathbf{L})$ is called R_f -**closed** if and only if $\text{Max}R_f^{-1}(P) \subseteq Y^n$ for each $P \in Y$.

Theorem

The lattice of R_f -closed subsets of $X(\mathbf{L})$ is dually isomorphic to the lattice of congruences of $\langle L, \wedge, \vee, 0, 1, f \rangle$.

Subdirectly irreducible and simple algebras

If $Y \subseteq X(\mathbf{L})$, we define Y_n by

$$Y_0 = Y$$
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$$Y_\omega = \bigcup_{n \in \mathbb{N}} Y_n.$$

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An algebra $\langle L, \wedge, \vee, 0, 1, f \rangle$ is simple if and only if for each $P \in X(\mathbf{L})$, $\{P\}_\omega$ is dense in $\mathcal{X}(\mathbf{L})$.

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- 2 There exists $P \in X(\mathbf{L})$, such that $P \notin \text{Cl}(\text{Max}R_f^{-1}(P)_\omega) \neq \emptyset$ and $X(\mathbf{L}) = \{P\} \cup \text{Cl}((\text{Max}R_f^{-1}(P))_\omega)$.*

Applications: Compatible operators on distributive lattices

Theorem

Let $f : L^n \rightarrow L$ be an operator of type $\varepsilon_1, \dots, \varepsilon_n \rightarrow \varepsilon$. The following are equivalent:

- 1 *f is compatible with the congruences of the lattice \mathbf{L} .*

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- 3 $\text{Max}R_f^{-1}(P) \subseteq \{P\}^n$ for each $P \in X(\mathbf{L})$.
- 4 $(P_1, \dots, P_n, P) \in R_f$ if and only if $(P_1, \dots, P_n) \leq (P, \dots, P)$ in $X(\mathbf{L})^{\varepsilon_1} \times \dots \times X(\mathbf{L})^{\varepsilon_n}$ and $(P, \dots, P, P) \in R_f$.

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Let $f : L^n \rightarrow L$ be a join hemimorphism. Then the following are equivalent:

- 1 f is compatible.
- 2 $f(a_1, \dots, a_n) = a_1 \wedge \dots \wedge a_n \wedge f(1, \dots, 1)$.

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Theorem

Let $f : L^n \rightarrow L$ be a meet hemimorphism (type $-1, \dots, -1 \rightarrow -1$). Then the following are equivalent:

- 1 f is compatible.
- 2 $f(a_1, \dots, a_n) = a_1 \vee \dots \vee a_n \vee f(0, \dots, 0)$.

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Let f be compatible operator of type $\varepsilon_1, \dots, \varepsilon_n \rightarrow \varepsilon$ over a distributive lattice \mathbf{L} such that $X_{1,f}(\mathbf{L}) = X(\mathbf{L})$, thus if there exists $1 \leq i \leq n$ such that $\varepsilon_i \neq \varepsilon$, then \mathbf{L} is a boolean lattice.

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Corollary

Let $\langle L, \wedge, \vee, \neg, 0, 1 \rangle$ be a pseudocomplemented lattice. Thus the following are equivalent:

- 1 \neg is compatible with the congruences of its lattice reduct.
- 2 $\langle L, \wedge, \vee, \neg, 0, 1 \rangle$ is a boolean algebra

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- 4 If $(P_1, \dots, P_n, P) \in R_f$, then the following propositions hold:
 - 1 If $\varepsilon_i = -1$, then $P \subseteq P_i$ and
 - 2 if $\varepsilon_i = 1$, then there exists P'_i such that $P_i \cup P \subseteq P'_i$ and $(P_1, \dots, P'_i, \dots, P_n, P) \in R_f$.

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Let $f : H \rightarrow H$ function of type $+1 \rightarrow -1$. Then the following are equivalent:

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Theorem

Let $f : H \rightarrow H$ function of type $-1 \rightarrow +1$. Then the following are equivalent:

- 1 f is a compatible function.
- 2 For every $a \in H$, $a \wedge f(a) = 0$.

Applications: Meet hemimorphisms

Theorem

Let $f : H^n \rightarrow H$ a meet hemimorphism. Then the following are equivalent:

- 1 f is a compatible function.
- 2 For every $a_1, \dots, a_n, b \in H$, $a_1 \vee \dots \vee a_n \leq f(a_1, a_2, \dots, a_n)$.

Open Problems

- Compatibility between operators.
- Compatible operators on residuated lattices.
- Subalgebras.

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