# Low upper bounds of ideals

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## The main result

- There is a low $T$-upper bound for the class of $K$-trivials
- Ideals in $\Delta_2^0$ degrees which have a low $T$-upper bound

## Standard notation

$2^\omega$ denotes the set of infinite binary sequences (the Cantor set)
$2^{<\omega}$ denotes the set of finite binary strings
$\Sigma^0_1$ classes, $\Pi^0_1$ classes
relativized $\Pi^0_1$ classes ($\Pi^{0,A}_1$ classes)

## Algorithmic randomness

$K$ denotes prefix-free Kolmogorov complexity
$\{\mathcal{U}_n : n \in \omega\}$ denotes universal ML test
concepts of 1-randomness and their relativizations

## Algorithmic weakness

There are several notions of computational weakness related to 1-randomness

### Definition

1. $\mathcal{L}$ denotes the class of sets which are low for 1-randomness, i.e. sets $A$ such that every 1-random set is also 1-random relative to $A$.
2. $\mathcal{K}$ denotes the class of $K$-trivial sets, i.e. the class of sets $A$ such that for all $n$, $K(A \upharpoonright n) \leq K(0^n) + O(1)$.
3. $\mathcal{M}$ denotes the class of sets that are low for $K$, i.e. sets $A$ such that for all $\sigma$, $K(\sigma) \leq K^A(\sigma) + O(1)$.
4. A set $A$ is a basis for 1-randomness if $A \leq_T Z$ for some $Z$ such that $Z$ is 1-random relative to $A$. The collection of such sets is denoted by $\mathcal{B}$.

**Theorem (Nies, Hirschfeldt, Stephan)**

$\mathcal{K} = \mathcal{L} = \mathcal{M} = \mathcal{B}$

More precisely:

- Nies: $\mathcal{L} = \mathcal{M}$
- Hirschfeldt, Nies: $\mathcal{K} = \mathcal{M}$
- Hirschfeldt, Nies, Stephan: $\mathcal{K} = \mathcal{B}$

Four different characterizations of the same class!
However, these characterizations yield different information content

## Basic facts about $\mathcal{K}$

- $\mathcal{K} \subseteq \Delta^0_2$
- $\mathcal{K} \subseteq L_1$ (i.e. $K$-trivials are low)

More precisely:

- Chaitin: $\mathcal{K} \subseteq \Delta^0_2$
- A.K.: $\mathcal{L} \subseteq GL_1$ (thus, $\mathcal{L} = \mathcal{K} \subseteq L_1$)

Nowadays there are easier ways to prove lowness of $K$-trivials

## Theorem (Nies; Downey,Hirschfeldt,Nies,Stephan)

- *r.e. $K$-trivial sets induce a $\Sigma^0_3$ ideal in the r.e. $T$-degrees*
- *$K$-trivial sets induce an ideal in the $\omega$-r.e. $T$-degrees generated by its r.e. members (in fact, a $\Sigma^0_3$ ideal in the $\omega$-r.e. $T$-degrees)*

Theorem (Downey,Hirschfeldt,Nies,Stephan; Nies)

- *There is an effective sequence* $\{B_e, d_e\}_e$ *of all the r.e.* $K$*-trivial sets and of constants such that each* $B_e$ *is* $K$*-trivial via* $d_e$
- *There is no effective sequence* $\{B_e, c_e\}_e$ *of all the r.e. low for* $K$ *sets with appropriate constants*
- *There is no effective way to obtain from a pair* $(B, d)$*, where* $B$ *is an r.e. set that is* $K$*-trivial via* $d$*, a constant* $c$ *such that* $B$ *is low for* $K$ *via* $c$
- *There is no effective listing of all the r.e.* $K$*-trivial sets together with their low indices*

### Theorem (Nies)

*For each low r.e. set $B$, there is an r.e. $K$-trivial set $A$ such that $A \not\leq_T B$.*

Thus, no low r.e. set can be a $T$-upper bound for the class $\mathcal{K}$.

### Comment

The proof uses Robinson low guessing technique which is compatible for r.e. sets with a technique do what is cheap.

Cheap is defined

- either by a cost function in case of $K$-trivials,
- or by having a small measure in case of low for random sets.

However, in the more general case of $\Delta_2^0$ instead of r.e. sets, the Robinson low guessing technique does not seem to be compatible with a technique do what is cheap. In fact, it is not.

Since all $K$-trivials are low and every $K$-trivial set is recursive in some r.e. $K$-trivial set, we have, as a corollary, that the ideal (induced by) $\mathcal{K}$ is nonprincipal (in the $\Delta_2^0$ $T$-degrees)

A more general result.

### Theorem (Nies)

*For any effective listing* $\{B_e, z_e\}_e$ *of low r.e. sets and of their low indices there is an r.e.* $K$*-trivial set* $A$ *such that* $A \not\leq_T B_e$ *for all* $e$*.*

This result is, in fact, used to prove that there is no effective way to obtain low indices of (r.e.) $K$-trivial sets

## Theorem (Nies)

- *There is a low$_2$ r.e. set which is a $T$-upper bound for the class of $K$-trivials.*
- *Any proper $\Sigma^0_3$ ideal in the r.e. $T$-degrees has a low$_2$ r.e. $T$-upper bound*

## Question

Is there a low $\Delta^0_2$ $T$-upper bound for the class $\mathcal{K}$ ?

The problem is mentioned:

- Nies, Computability and randomness, draft: question 5.46
- Miller, Nies, Open Questions: Question 4.3.
- AIM/ARCC Open Problem List: Problem 3.6.

Several attempts: Nies, Downey, A.K., Barmpalias, ...

### Theorem (Yates)

*For any r.e. set A TFAE:*

1. $A'' \equiv_T \emptyset''$
2. $\{x : W_x \leq_T A\}$ *is a* $\Sigma^0_3$ *set*
3. *the class* $\{W_x : W_x \leq_T A\}$ *is uniformly r.e.*

Together with a result of Nies, this provides information about the existence of low$_2$ r.e. $T$-upper bounds for $\Sigma^0_3$ ideals in the r.e. $T$-degrees

### Open

A characterization of $\Sigma^0_3$ ideals in the r.e. $T$-degrees for which there is a low $T$-upper bound, not necessarily r.e.(!)
(similarly for ideals in $\Delta^0_2$ $T$-degrees)

**Theorem**

*Let $\mathcal{C}$ be a $\Sigma_3^0$ ideal in the r.e. $T$-degrees. Then TFAE:*

1. *there is a function $F$ recursive in $\emptyset'$ which dominates all partial functions recursive in any member of the ideal $\mathcal{C}$,*
2. *there is a low $T$-upper bound for $\mathcal{C}$*

A slightly more general result.

**Theorem**

*Let $\mathcal{C}$ be an ideal in $\Delta_2^0$ $T$-degrees such that there is a uniformly recursively in $\emptyset'$ a sequence of sets $\{A_n\}_n$ which generate the ideal $\mathcal{C}$. Then TFAE:*

1. *there is a function $F$ recursive in $\emptyset'$ which dominates all partial functions recursive in any member of the ideal $\mathcal{C}$,*
2. *there is a low $T$-upper bound for $\mathcal{C}$*

## Corollary

*There is a low $T$-upper bound for the class $\mathcal{K}$ (the class of $K$-trivials).*

## Proof

Nies proved that the ideal (induced by) $\mathcal{K}$ is generated by its r.e. members and r.e. $K$-trivial sets induce a $\Sigma^0_3$ ideal in the r.e. $T$-degrees.
A.K. proved that there is a function $F$ recursive in $\emptyset'$ which dominates all partial functions recursive in any member of $\mathcal{K}$.
Thus, the result follows from the previous Theorem.

### Remark

We may equivalently require that a low $T$-upper bound (in the above) is $PA$ since every low set has a low $PA$ set $T$-above it. Thus, $T$-upper bounds which are $PA$ are the most general case in this characterization

## Proof

$1 \to 2$ (the opposite direction is trivial)

Given $\mathcal{C}$, $\{A_n\}_n$, and $F$ recursive in $0'$ eventually dominating every function recursive in an $A_n$, we want a $T$-upper bound $A$ of $\mathcal{C}$.

## Obstacle

Lowness of $A$

*versus*

$A \geq_T A_n$ without having low indices for the $A_n$.

- A possible solution: use of $\Pi_1^{0,A_n}$-classes.
- Question: how to code into $\Pi_1^{0,A_n}$-classes.
- Answer: use rich $\Pi_1^{0,A_n}$-classes.

We replace the missing lowness indices of $A_n$ by guesses relative to the function $F$. Properties of $F$ guarantee that in a finite injury style an infinite path through $\Pi_1^{0,A_n}$ classes is found (more details on that later)

### Definition

Let $\mathcal{PA}(B)$ denote the class of $\{0,1\}$-valued $B$-DNR functions, i.e. the class of functions $f \in 2^\omega$ such that $f(x) \neq \Phi_x(B)(x)$ for all $x$ . If $B$ is $\emptyset$ we simply speak of $\mathcal{PA}$.

### Definition (Simpson)

$\mathbf{b} << \mathbf{a}$ means that every infinite tree $T \subseteq 2^{<\omega}$ of degree $\leq \mathbf{b}$ has an infinite path of degree $\leq \mathbf{a}$.

### Theorem (D. Scott and others)

*The following conditions are equivalent:*

1. $\mathbf{a}$ *is a degree of a* $\{0,1\}$*-DNR function*
2. $\mathbf{a} >> \mathbf{0}$
3. $\mathbf{a}$ *is a degree of a complete extension of PA*
4. $\mathbf{a}$ *is a degree of a set separating some effectively inseparable pair of r.e. sets.*

### Remark

1. $\mathcal{PA}$ is a kind of a "universal" $\Pi^0_1$ class
2. $\{0, 1\}$-valued DNR functions are also called PA sets and degrees $>> \mathbf{0}$ are called PA degrees.
3. Similarly for $\mathcal{PA}(B)$.

### Fact (Simpson)

1. The partial ordering $<<$ is dense
2. $\mathbf{a} << \mathbf{b}$ implies $\mathbf{a} < \mathbf{b}$

## Definition

Let $M$ be an infinite set and $\{m_0, m_1, m_2, \ldots\}$ be an increasing list of all members of $M$.

- If $f \in 2^\omega$ then by $Restr(f, M)$ we denote $g \in 2^\omega$ defined for all $i$ by $g(i) = f(m_i)$
- Similarly, if $\mathcal{A} \subseteq 2^\omega$ then by $Restr(\mathcal{A}, M)$ we denote a class of functions $\{g : g = Restr(f, M) \ \wedge \ f \in \mathcal{A}\}$

## Lemma (A.K.)

- *For every* $\Pi^0_1$ *class* $\mathcal{A} \subseteq \mathcal{PA}$ *there is an infinite recursive set* $M$ *such that if* $\mathcal{A}$ *is nonempty then* $Restr(\mathcal{A}, M) = 2^\omega$, *i.e. for every* $g \in 2^\omega$ *there is a function* $f \in \mathcal{A}$ *such that* $Restr(f, M) = g$.
- *For every* $\Pi^{0,B}_1$ *class* $\mathcal{A} \subseteq \mathcal{PA}(B)$ *there is an infinite recursive set* $M$ *such that if* $\mathcal{A}$ *is nonempty then* $Restr(\mathcal{A}, M) = 2^\omega$, *where (an index of)* $M$ *can be found uniformly from an index of* $\mathcal{A}$*, i.e. it does not depend on* $B$.

## Remark

This is basically Gödel incompleteness phenomenon.

The Lemma is crucial for coding into $\Pi^{0,B}_1$ classes which are subclasses of $\mathcal{PA}(B)$

We may

- code either an individual set $C$ (by $Restr(\mathcal{A}, M) = \{C\}$ )
- or nest another class $\mathcal{B} \subseteq 2^\omega$ (by $Restr(\mathcal{A}, M) = \mathcal{B}$ )

Nesting in this way a $\Pi^{0,C}_1$ class into a $\Pi^{0,B}_1$ class we obtain $\Pi^{0,B\oplus C}_1$ class.

Key ingredients of the proof ( $1 \to 2$ ).
WLOG: $A_0 = \emptyset$, $A_e \leq_T A_{e+1}$ for all $e$
We construct sequences of

- $\Pi_1^{0,A_e}$ classes $\mathcal{A}_e$, $\mathcal{A}_0 = \mathcal{PA}$, $\mathcal{A}_{e+1} \subseteq \mathcal{A}_e$ for all $e$
- infinite recursive sets $M_e$, $M_0 = \omega$, $M_{e+1} \subseteq M_e$ for all $e$,

such that $Restr(\mathcal{A}_e, M_e) = \mathcal{PA}(A_e)$ for all $e$.

Condition $e \in A'$.
Take $\Phi_e$ (a T.r.f.), provided we have a low index of $A_e$, use LBT technique to find a $\Pi_1^{0,A_e}$ class $\mathcal{A}_e^* \subseteq \mathcal{A}_e$ which decides this condition.
Next find an infinite recursive set $M_{e+1} \subseteq M_e$ such that $Restr(\mathcal{A}_e^*, M_{e+1}) = 2^\omega$.
Then take a $\Pi_1^{0,A_{e+1}}$ class $\mathcal{A}_{e+1} \subseteq \mathcal{A}_e^*$ so that $Restr(\mathcal{A}_{e+1}, M_{e+1}) = \mathcal{PA}(A_{e+1})$.

**Remark**

We do not code directly sets $A_e$ into our constructed set $A$.
Instead, we nest $\mathcal{PA}(A_{e+1})$ into $\mathcal{A}_e$ (or $\mathcal{A}_e^*$) , thus ensuring that a $\{0,1\}$-valued $A_{e+1}$-DNR function $f_{e+1}$ is coded in $A$, and obviously $A_{e+1} \leq_T f_{e+1}$.
This way of nesting $\mathcal{PA}(A_e)$ classes at each level leaves enough space for later coding requirements.

Let $A$ be the only set in all classes $\mathcal{A}_e$.

Obviously, $A_e \leq_T A$,

since for $f_e = Restr(A, M_e)$ we have $f_e \in \mathcal{PA}(A_e)$,
$A_e$ is recursive in any $\{0,1\}$-valued $A_e$-DNR function,
and, thus, $A_e \leq_T f_e$ and $A_e \leq_T A$.

Here: low indices of $A_e$ are uniformly recursive in $\emptyset'$ (a trivial case).

However, in a more general case with function $F$ instead of low inidices of $A_e$, we replace questions about $\omega$-extendability of a string on a tree recursive in $A_e$ by its finite-extendability, where the depth to which extendability is required is computed by $F$.
We have to be able, for each $A_e$, to construct a $A_e$-recursive (partial) function which $F$ has to dominate and which will ensure that guesses computed by $F$ are, eventually, correct guesses, i.e. yield $\omega$-extendability. We adapt a finite injury style to do that.
R.sp., whenever necessary, we leave the current version of a $\Pi_1^{0,A_e}$ class and nest a completely new copy of $\mathcal{PA}(A_e)$ in the currently available $\Pi_1^{0,A_{e-1}}$ class (in an appropriate way).
Observe, $\mathcal{A}_0$ is a $\Pi_1^0$ class $\mathcal{PA}$ so that by means of $\emptyset'$ we always have correct answers about $\omega$-extandability here and, thus, there is no injury at the 0-level.

As a corollary of a result of Shore there is an exact pair for the class $\mathcal{K}$ in $\Delta^0_2$ $T$-degrees.

### Question

Is there an exact pair for the class $\mathcal{K}$ in the r.e. $T$-degrees? I.e. are there r.e. $T$-degrees $\mathbf{a}$, $\mathbf{b}$ such that $[\mathbf{0}, \mathbf{a}] \cap [\mathbf{0}, \mathbf{b}]$ is equal to $T$-degrees of $\mathcal{K}$?

# Appendix

## Applications of coding into $\mathcal{PA}$

- Posner-Robinson: For every nonrecursive set $Z$
  $\exists A(Z \oplus A \equiv_T A')$ (with $A \leq_T Z \oplus \emptyset'$)
- Shore-Slaman: For every $Z \notin \Delta^0_n$, $n \geq 1$
  $\exists A(Z \oplus A \equiv_T A^{(n)})$ (with $A \leq_T Z \oplus \emptyset^{(n)}$)

In the above, $A$ may be chosen to be a $PA$ set (i.e. $A \in \mathcal{PA}$).

# Appendix

The main idea (for $n = 1$).

Isolated paths through recursive trees are recursive.

Given a $\Pi^0_1$ class $\mathcal{A} \subseteq \mathcal{PA}$,
and an infinite recursive set $M$ with $Restr(\mathcal{A}, M) = 2^\omega$,

find $\sigma \prec Z$ and $\Pi^0_1$ classes $\mathcal{B}_i \subseteq \mathcal{A}$, $i = 0, 1$
with $Restr(\mathcal{B}_i, M) = \sigma * i * 2^\omega$, *(r.sp. $\sigma$ is coded in $\mathcal{A}$)*
such that

- either $\mathcal{B}_i$ forces a $\Sigma^0_1$ property with $|\sigma|$ longer than all $\Sigma^0_1$ witnesses in question, for both $i = 0, 1$
- or for both $i = 0, 1$, $\mathcal{B}_i$ forces its negation, i.e. a $\Pi^0_1$ property.

Then take $\mathcal{B}_i$ for $i \neq Z(|\sigma|)$, i.e. code a difference from $Z$.

# Appendix

For $n = 2$,
a coding into $\omega^{<\omega}$ rather than a linear coding is used,
where $Z$ can find upper bounds of relevant finitely branching subtree of $\omega^{<\omega}$
and, by an analogous technique,
$Z$ can recognize first sufficiently many $\Sigma_1^0$ or $\Pi_1^0$ events,
so that eventually $Z$ can also recognize whether a $\Sigma_2^0$ or a $\Pi_2^0$ event happens.

# Appendix

### Definition

$A \leq_{LR} B$ if every set 1-random in $B$ is also 1-random in $A$.

### Fact

TFAE:

1. $A \leq_{LR} B$
2. for some level $e$ of a universal ML test relative to $A$, i.e. $\mathcal{U}_e^A$, there is a $V^B$ which is $\Sigma_1^{0,B}$ such that $\mu(V^B) < 1$ and $\mathcal{U}_e^A \subseteq V^B$

Proof.

- Terwijn and A.K. $2 \rightarrow 1$ (for $B = \emptyset$)
- Nies and Stephan $1 \rightarrow 2$

# Appendix

### Definition

$B$ is almost complete if $\emptyset'$ is $K$-trivial relative to $B$, i.e.
$\forall n \; ( \; K^B(\emptyset' \restriction n) \leq K^B(0^n) + O(1) \; )$

### Lemma (Nies)

*$B$ is almost complete* $\iff \emptyset' \oplus B \leq_{LR} B$

Thus, for $\Delta^0_2$ sets: $B$ is almost complete $\iff \emptyset' \leq_{LR} B$

# Appendix

Psedo-jump inversion.

### Theorem (Jockusch,Shore,1983)

*For every r.e. operator $W$, there is an r.e. set $B$ such that $B \oplus W^B \equiv_T \emptyset'$.*

### Corollary (Nies)

*There is an almost complete r.e. set $B <_T \emptyset'$.*

Apply Theorem to the r.e. operator obtained by relativizing
- r.e. $K$-trivial set construction, or
- r.e. low for random set construction.

# Appendix

### Theorem

- *There is an almost complete PA set* $A <_T \emptyset'$
- *For every nonrecursive* $Z \leq_T \emptyset'$*, there is an almost complete PA set* $A$ *such that* $A \oplus Z \equiv_T \emptyset'$.

The same technique as in a version of Posner-Robinson theorem for *PA* sets (above) applied to the r.e. operator obtained by relativizing a low for random r.e. set construction

# Appendix

Coding into $\Pi^0_1$ classes of positive measure

## Theorem (A.K,1989)

- *There is an incomplete high* 1*-random set* $A <_T \emptyset'$.
- *For every set* $B$ *r.e.a. in* $\emptyset'$ *and every nonrecursive* $\Delta^0_2$ *set* $C$ *there is a* $\Delta^0_2$ 1*-random set* $A$ *such that* $A' \equiv_T B$ *and* $A \not\geq_T C$.

## Theorem (Nies)

- *There is an almost complete* 1*-random set* $A <_T \emptyset'$
- *For every r.e. operator* $W$ *there is a* 1*-random set* $A \leq_T \emptyset'$ *such that* $A \oplus W^A \equiv_T \emptyset'$.

## Remark

In fact, the jump inversion technique in Theorem (A.K.,1989) yields immediately also a pseudo-jump inversion method.

# Appendix

A direct construction of an r.e. almost complete set $A <_T \emptyset'$.

This was done first by Nies and Shore who used a cost function method (by building an appropriate oracle KC-set).
Not surprisingly, one can use for that also do what is cheap method based on measure as in low for random r.e. set construction (of Terwijn and A.K.).

## Remark

There is a direct construction of an almost complete r.e. set $A <_T \emptyset'$ using a method:

$\emptyset' \leq_{LR} A \Longleftrightarrow$
for some level $e$ of a universal ML test relative to $\emptyset'$, i.e. $\mathcal{U}_e^{\emptyset'}$, there is a $V^A$ which is $\Sigma_1^{0,A}$ such that $\mu(V^A) < 1$ and $\mathcal{U}_e^{\emptyset'} \subseteq V^A$.

# Appendix

## Comment

The idea is to construct an r.e. set $A$ which is, in a sense, very close to $\emptyset'$.
Typically, to make $A <_T \emptyset'$ we sometimes want to put some $x$ into $\emptyset'$ but not into $A$.
To make $A$ almost complete, we are allowed to do that only when such $x$ is cheap relatively to $A$. Nonrecursivness of $A$ makes this more difficult.
Instead of a recursive version *to be cheap* we have here a version *to be cheap* relativized to $A$. This leads to approximations and to more subtle construction.

# Appendix

### Definition

Let $\mathcal{K}_0$ denote the set of r.e. $K$-trivials which are $T$-below all almost complete 1-random sets.

### Remark

Hirschfeldt and Miller proved that $\mathcal{K}_0$ is a subclass of the r.e. members of $\mathcal{K}$, containing also nonrecursive r.e. sets.
Sets in $\mathcal{K}_0$ are ML-noncuppable, i.e. for such sets $A$, $A \oplus Z <_T \emptyset'$ for all $\Delta^0_2$ 1-random sets $Z <_T \emptyset'$.
(Nies was the first proving the existence of a ML-noncuppable r.e. $K$-trivial set).

# Appendix

## Open questions

- Does for every $K$-trivial set $A$ exist a 1-random set $Z$ such that $A \leq_T Z \ \wedge \ \emptyset' \not\leq Z$ ?
- Is $\mathcal{K}_0$ equal to r.e. members of $\mathcal{K}$ ?
- Are there minimal pairs of r.e. almost complete sets ?
- Can a $K$-trivial set be ML-cuppable?

# Appendix

## Comment

Many obstacles in solving the above questions concerning 1-randomness are connected with a problem of coding an information into 1-random sets.
While we can code an infinitary information into *PA* sets
(or into $\Pi^0_1$ subclasses of $\mathcal{PA}$),
coding an information into 1-random sets
(or into $\Pi^0_1$ classes of positive measure) is less powerful and it is still not completely understood.

Among others it should demonstrate

# Splendors and miseries of $\Pi^0_1$ classes

Thank you