Profinite topologies on words

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Outline

(1) Preliminaries
(2) Profinite topologies
(3) Equational definitions of varieties
(4) The p-adic case
(5) The group case
(6) Applications to finite automata and semigroups
(7) Conclusion and open problems

Semigroups, monoids and groups

A semigroup is a set equipped with an associative binary operation. A monoid is a semigroup with a unit element. A group is a monoid in which each element has an inverse.

Let p be a prime number. A p-group is a finite group of size p^n for some $n > 0$.

Given an alphabet A, one denotes by A^* [A^+, $FG(A)$] the free monoid [semigroup, group] on A. The empty word is denoted by 1.

Varieties

A Birkhoff variety of semigroups is a class of semigroups closed under taking subsemigroups, quotients (= homomorphic images) and direct products.

A variety of finite semigroups is a class of finite semigroups closed under taking subsemigroups, quotients and finite direct products.

- Similar definitions can be given for monoids and for groups.
- Finite groups form a variety of finite monoids.

Part I

Profinite topologies

Separating words

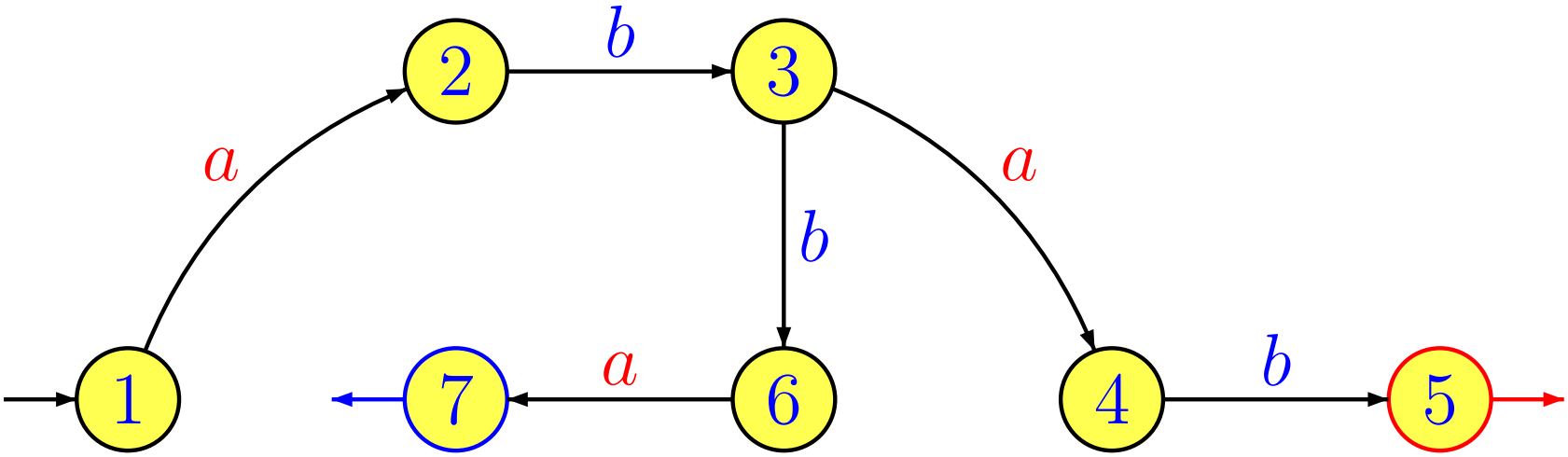
A monoid M separates two words u and v of A^* if there exists a monoid morphism $\varphi : A^* \to M$ such that $\varphi(u) \neq \varphi(v)$.

For instance, the morphism which maps each word onto its length modulo 2 is a morphism from $\{a, b\}^*$ onto $\mathbb{Z}/2\mathbb{Z}$ which separates $abaaba$ and $abaabab$.

Let $M = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\right\}$ and let $\varphi : \{a, b\}^* \to M$ defined by $\varphi(a) = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ and $\varphi(b) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$. Then, for each word u, φ separates ua and ub since $\varphi(ua) = \varphi(a)$ and $\varphi(ub) = \varphi(b)$.

Separating $abab$ and $abba$ with a finite group

(1) Build an automaton with the two words



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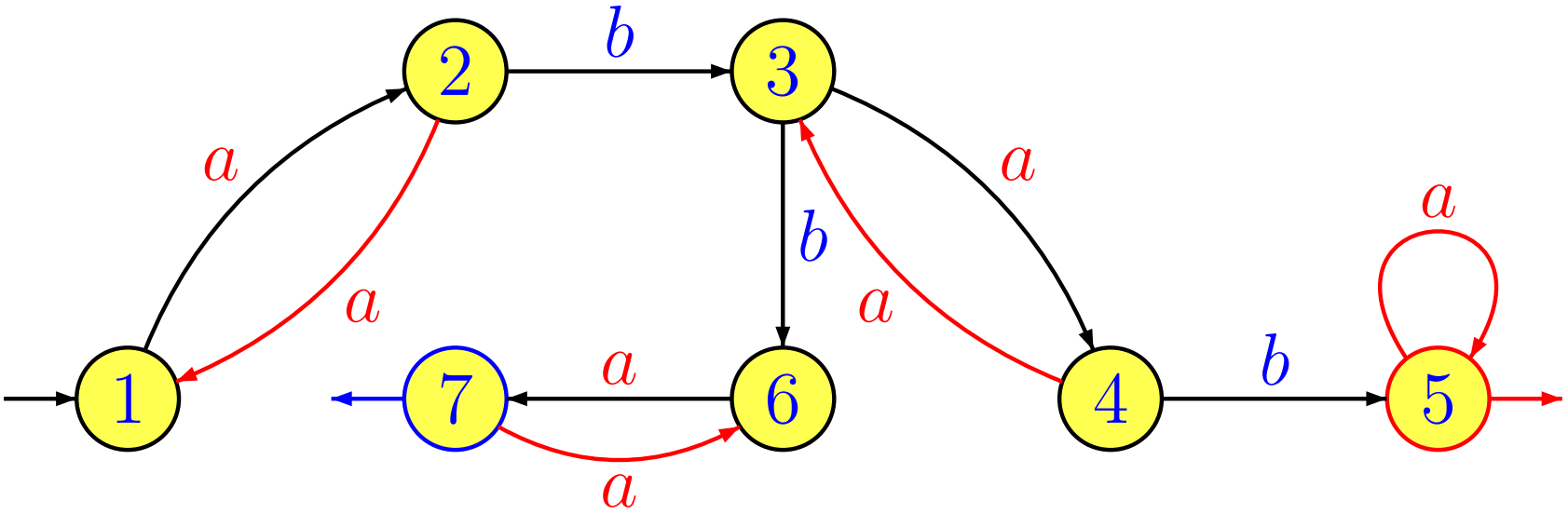

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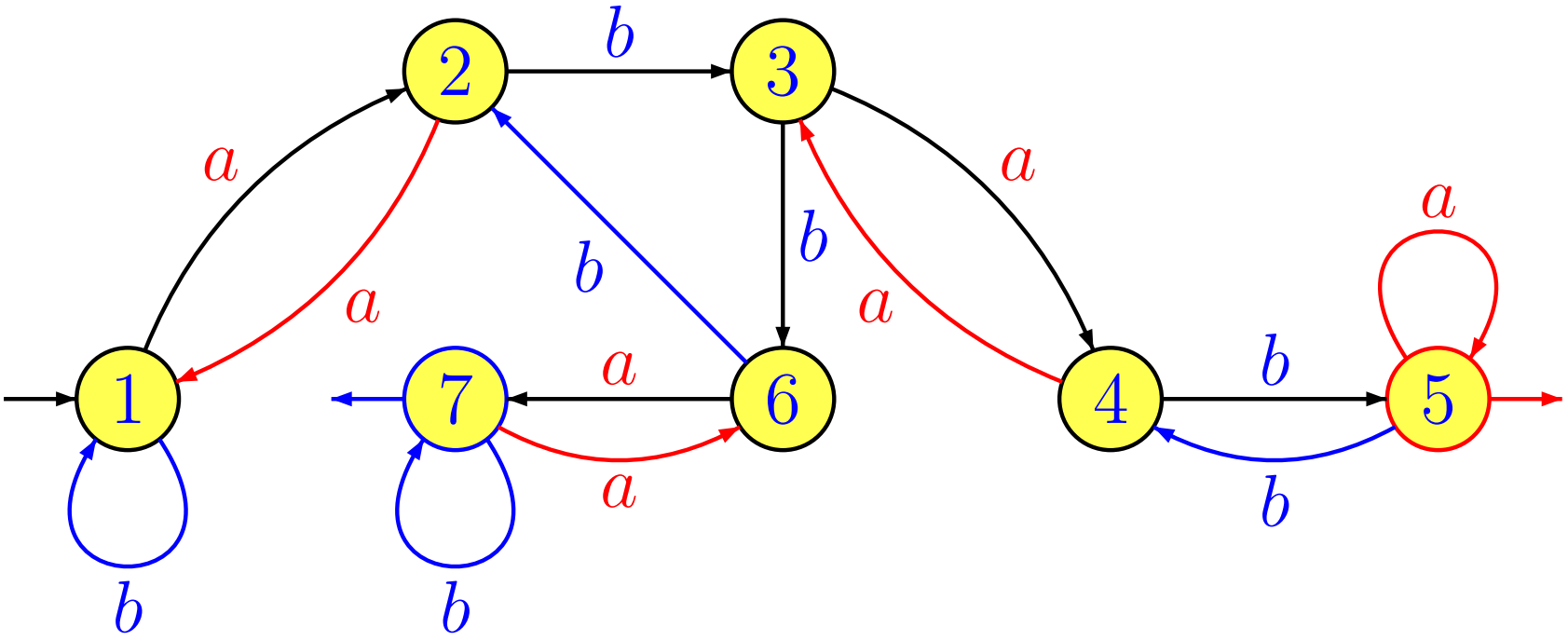

(2) Complete into permutations.

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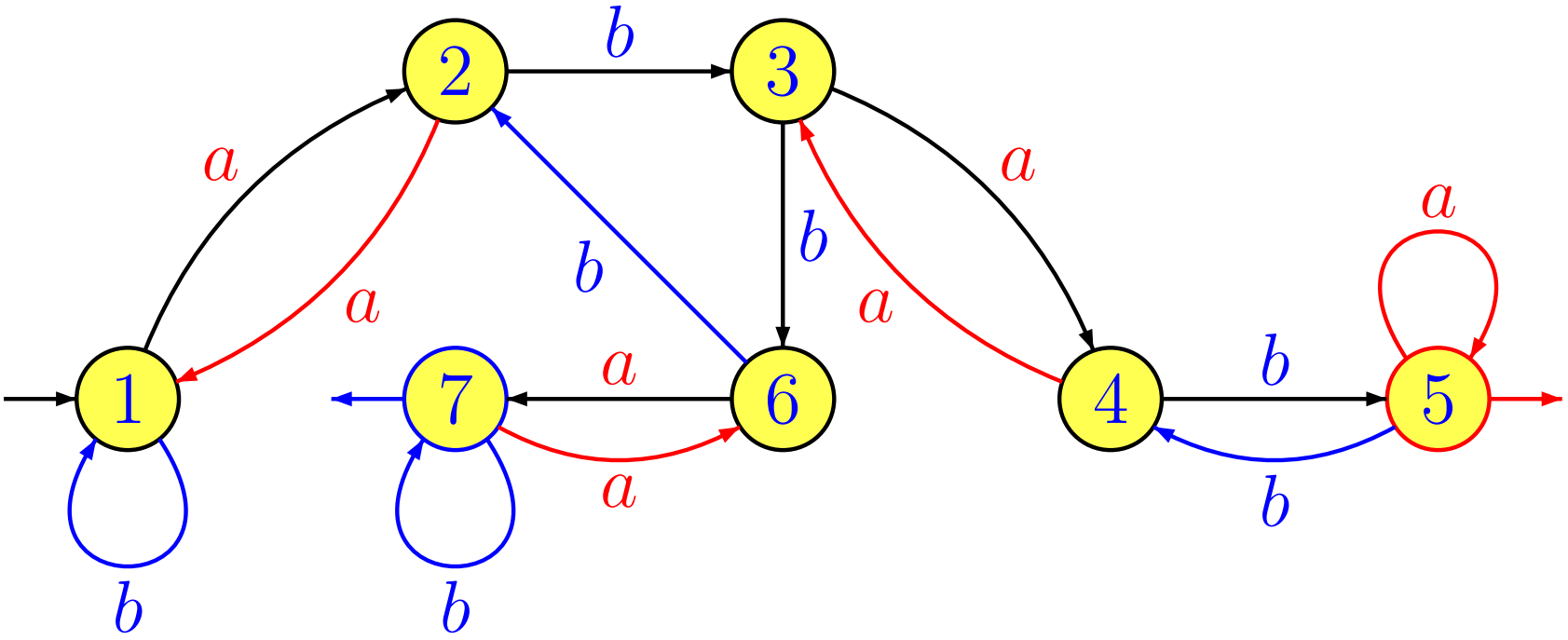


(2) Complete into permutations. The resulting permutation group separates $abab$ and $abba$ since $1 \cdot abab = 5$ and $1 \cdot baba = 7$.

Some metrics on words

Let u and v be two words. Put

$$r(u,v) = \min\{\,|M| \mid M \text{ is a finite monoid that separates } u \text{ and } v\}$$

$$r_{\mathbf{G}}(u,v) = \min\{\,|G| \mid G \text{ is a finite group that separates } u \text{ and } v\}$$

$$d(u,v) = 2^{-r(u,v)} \qquad d_{\mathbf{G}}(u,v) = 2^{-r_{\mathbf{G}}(u,v)}$$

- Intuitively, two words are close for d [$d_{\mathbf{G}}$] if one needs a large monoid [group] to separate them.
- One could give a similar definition for any variety of finite monoids.

Main properties of d and $d_{\mathbf{G}}$

For all $u, v, w \in A^*$,

- $d(u, v) = 0$ iff $u = v$,
- $d(u, v) \leqslant \max\{d(u, w), d(w, v)\}$ and hence d [$d_{\mathbf{G}}$] is an ultrametric,
- $d(uw, vw) \leqslant d(u, v)$, $d(wu, wv) \leqslant d(u, v)$,
- the product of words is uniformly continuous,
- any monoid morphism from A^* to B^* is uniformly continuous for d [$d_{\mathbf{G}}$],
- any monoid morphism from (A^*, d) [$(A^*, d_{\mathbf{G}})$] onto a finite discrete monoid [group] is uniformly continuous.

Main properties of d and $d_{\mathbf{G}}$ (continued)

- A sequence of words $(u_n)_{n\geqslant 0}$ is a Cauchy sequence for d [for $d_{\mathbf{G}}$] iff, for every monoid morphism φ from A^* to a finite monoid [group], the sequence $\varphi(u_n)_{n\geqslant 0}$ is ultimately constant.
- A sequence of words $(u_n)_{n\geqslant 0}$ is converging to a word u (for d [for $d_{\mathbf{G}}$]) iff, for every monoid morphism φ from A^* to a finite monoid [group], the sequence $\varphi(u_n)_{n\geqslant 0}$ is ultimately equal to $\varphi(u)$.

A converging sequence for $d_{\mathbf{G}}$

Theorem (M. Hall 1950)

For each word x, the sequence $(x^{n!})_{n\geqslant 0}$ converges to the empty word in the pro-group topology.

Proof. Let $\varphi : A^* \to G$ be a monoid morphism onto a finite group G and let $g = \varphi(x)$. We claim that the sequence $\varphi(x^{n!}) = g^{n!}$ is ultimately equal to $\varphi(1) = 1$.

By Lagrange's theorem, $g^{|G|} = 1$. Now, for $n \geqslant |G|$, $|G|$ divides $n!$ and hence, $g^{n!} = 1$. □

Free profinite monoids and groups

The metric space (A^*, d) $[(A^*, d_{\mathbf{G}})]$ is not complete. Its completion is the free profinite monoid [profinite group] on A and is denoted by $\widehat{A^*}$ $[\widehat{FG(A)}]$. If A is finite, the completion is compact.

The product is uniformly continuous on A^* and hence can be extended by continuity to the completion of A^*.

For each $x \in A^*$, the sequence $x^{n!-1}$ is a Cauchy sequence, and hence converges in $\widehat{FG(A)}$. Since

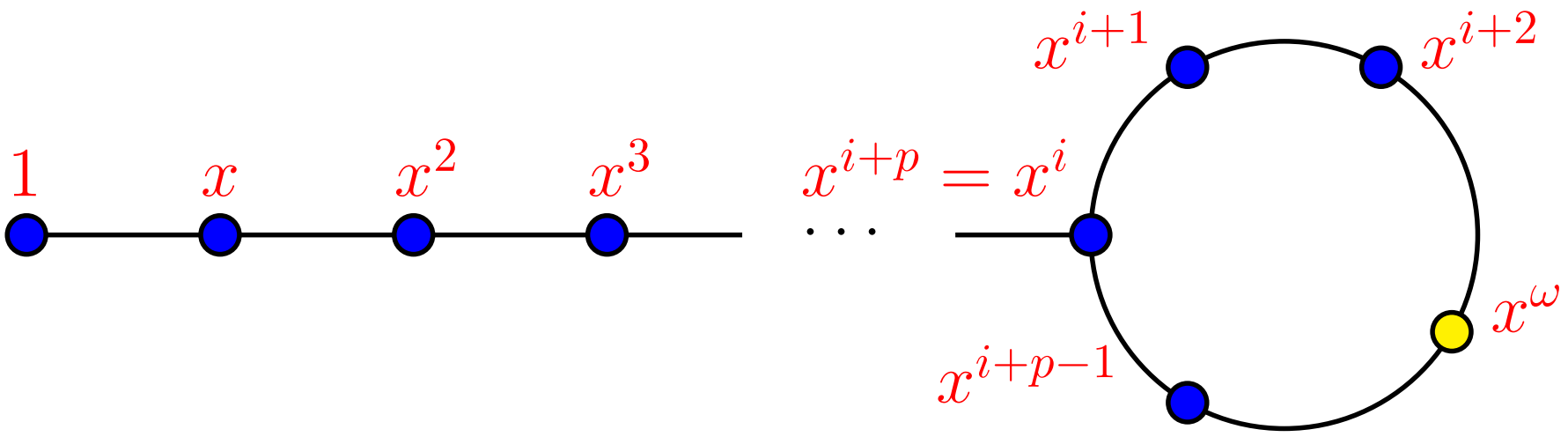
$$x \lim_{n\to\infty} x^{n!-1} = \lim_{n\to\infty} x^{n!-1}x = \lim_{n\to\infty} x^{n!} = 1$$

x has an inverse.

Properties of the profinite metric

Theorem (Reutenauer 1979)

For each $x \in A^*$, the sequence $x^{n!}$ converges in $\widehat{A^*}$ to a limit, denoted by x^ω.



Part II

Varieties

Quotation (M. Stone)

A cardinal principle of modern mathematical research may be stated as a maxim: One must always topologize.

Identities

Let A be an alphabet and let $u, v \in A^+$. A semigroup S satisfies the identity $u = v$ iff, for each semigroup morphism $\varphi : A^+ \to S$, $\varphi(u) = \varphi(v)$.

- A semigroup satisfies the identity $xyx = x$ if, for each $x, y \in S$, $xyx = x$.
- A semigroup is commutative iff it satisfies the identity $xy = yx$.
- A semigroup is idempotent iff it satisfies the identity $x = x^2$.

Birkhoff's theorem

Proposition (Easy!)

Given a set E of identities, the class of all semigroups satisfying the identities of E is a Birkhoff variety of semigroups.

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Theorem (Birkhoff 1935)

A class of semigroups is a Birkhoff variety iff it is defined by a set of identities.

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Theorem (Birkhoff 1935)

A class of semigroups is a Birkhoff variety iff it is defined by a set of identities.

What happens for finite semigroups ?

Reiterman's theorem

Define a profinite identity as a formal equality of the form $u = v$, where u and v are elements of a free profinite monoid.

Theorem (Reiterman 1982)

A class of finite semigroups is a variety iff it is defined by a set of profinite identities.

The variety of finite groups is defined by the single identity $x^\omega = 1$ since, in a finite group, the unique idempotent is the identity.

Part III

The pro-p topology

Let p be a prime number. Put

$$r_p(u,v) = \min\{|G| \mid G \text{ is a finite } p\text{-group that separates } u \text{ and } v\}$$

$$d_p(u,v) = 2^{-r_p(u,v)}$$

The metric d_p defines the pro-p topology on A^*.

Subwords

A word $u = a_1 a_2 \cdots a_k$ is a subword of v if $v = v_0 a_1 v_1 a_2 \cdots a_k v_k$ for some $v_0, v_1, \ldots, v_k \in A^*$.
For instance, Bees is a subword of BuenosAires.

Let $\binom{v}{u}$ be the number of distinct ways to write u as a subword of v. For instance,
$\binom{abab}{ab} = 3$ ($abab$, $abab$, $abab$) and $\binom{a^n}{a^m} = \binom{n}{m}$.

- For every word $u \in A^*$, $\binom{u}{1} = 1$.
- For every word $u \in A^+$, $\binom{1}{u} = 0$.
- If $w = uv$, then $\binom{w}{x} = \sum_{x_1 x_2 = x} \binom{u}{x_1}\binom{v}{x_2}$.

Computing the binomial coefficients modulo p

The function $\tau : A^* \longrightarrow \mathcal{M}_{n+1}(\mathbb{Z}/p\mathbb{Z})$ defined by

$$\tau(u) = \begin{pmatrix} 1 & \binom{u}{a_1} & \binom{u}{a_1a_2} & \binom{u}{a_1a_2a_3} & \cdots & \binom{u}{a_1a_2\cdots a_n} \\ 0 & 1 & \binom{u}{a_2} & \binom{u}{a_2a_3} & \cdots & \binom{u}{a_2\cdots a_n} \\ 0 & 0 & 1 & \binom{u}{a_3} & \cdots & \binom{u}{a_3\cdots a_n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \binom{u}{a_n} \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

is a morphism of monoids and the unitriangular $n \times n$ matrices with entries in $\mathbb{Z}/p\mathbb{Z}$ form a p-group.

Another metric for the pro-p topology

Let

$$r'_p(u,v) = \min\left\{ |x| \mid \binom{u}{x} \not\equiv \binom{v}{x} \bmod p \right\}$$

$$d'_p(u,v) = 2^{-r'_p(u,v)}$$

Then d'_p is a metric which also defines the pro-p topology on A^*.

Thue-Morse word

Let $A = \{a, b\}$, and let $\tau : A^* \to A^*$ be the monoid morphism defined by $\tau(a) = ab$ and $\tau(b) = ba$. The sequence $\tau^n(a)$ is $a, ab, abba, abbabaab, \ldots$. It defines an infinite word $t = abbabaabbaababba \cdots$ called the Thue-Morse word.

For each prefix $t[n]$ of t, we are interested in the value $\bmod p$ of the numbers

$\binom{t[n]}{a}, \binom{t[n]}{b}, \binom{t[n]}{aa}, \binom{t[n]}{ab}, \binom{t[n]}{ba}, \binom{t[n]}{bb}, \binom{t[n]}{aaa}, \binom{t[n]}{aab}, \ldots$

Some binomial coefficients $\binom{t[n]}{v}$

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
a	1	1	1	2	2	3	4	4	4	5	6	6	7	7	7	8
b	0	1	2	2	3	3	3	4	5	5	5	6	6	7	8	8
aa	0	0	0	1	1	3	6	6	6	10	15	15	21	21	21	28
ab	0	1	2	2	4	4	4	8	12	12	12	18	18	25	32	32
ba	0	0	0	2	2	5	8	8	8	13	18	18	24	24	24	32
bb	0	0	1	1	3	3	3	6	10	10	10	15	15	21	28	28

Some converging sequences

For every prime p and $n > 0$, let

$$f(p,n) = \begin{cases} 2^n p^{1+\lfloor log_p n \rfloor} & \text{if } p \neq 2 \\ 2^k \text{ where } F_{k-1} \leqslant n < F_k & \text{if } p = 2 \end{cases}$$

Theorem

Let $m = f(p,n)$. Then for every non-empty word v of length $\leqslant n$, $\binom{t[m]}{v} \equiv 0 \bmod p$. Thus the sequence $(t[f(p,n)])_{n>0}$ converges to the empty word for d_p.

Part IV

Back to the pro-group topology

Let L be a regular language of A^* ($=$ set of words recognized by a finite automaton). Is it decidable whether L is open, closed or clopen in the pro-group topology?

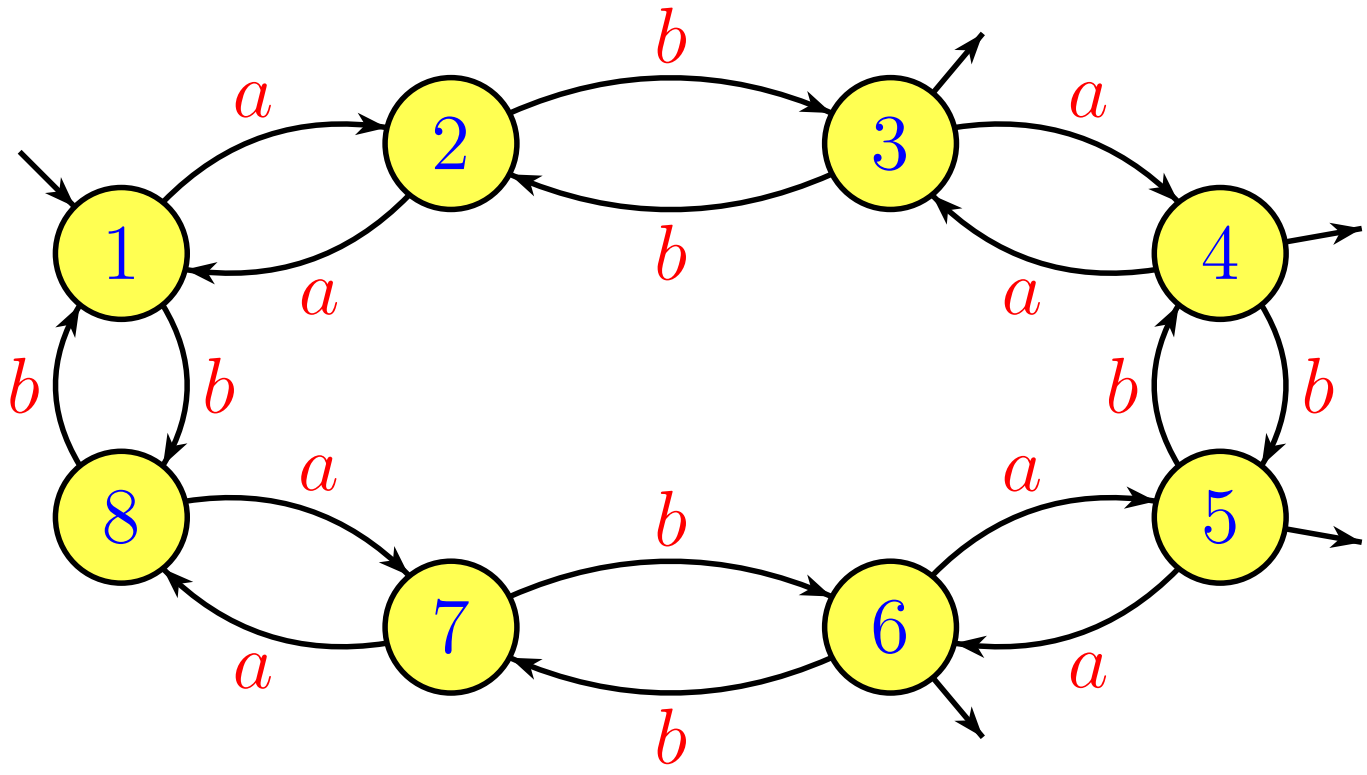
Group languages

- A group automaton is a finite deterministic automaton in which each letter defines a permutation on the set of states.
- A group language is a language accepted by a group automaton.

Proposition

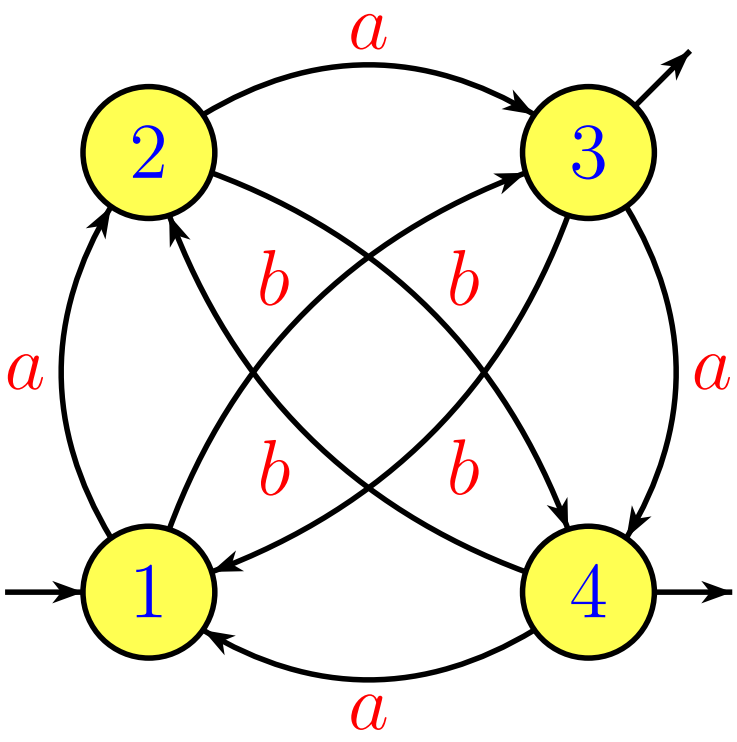
A regular language is clopen in the pro-group topology iff it is a group language.

A group language



This group automaton recognizes the set of words u such that $\binom{u}{ab}$ is an odd number. Its transition monoid is a 2-group of order 8.

Another group automaton



Polynomial closure of the group languages

The polynomial closure of the group languages Pol $\mathcal{G}$ is the set of languages that are finite unions of languages of the form $L_0 a_1 L_1 \cdots a_k L_k$ where $k \geqslant 0$, each $a_i \in A$ and each L_i is a group language.

Theorem

A regular language is open in the pro-group topology iff it belongs to Pol $\mathcal{G}$.

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Is it decidable?

Topology on the free group

Theorem (M. Hall Jr., 1950)

Every finitely generated subgroup of a finitely generated free group is closed.

Theorem (Ribes–Zalesskii, 1993)

Let $H_1, \ldots, H_n$ be f.g. subgroups of $FG(A)$. Then $H_1 \cdots H_n = \{h_1 \cdots h_n \mid h_1 \in H_1, \ldots, h_n \in H_n\}$ is closed.

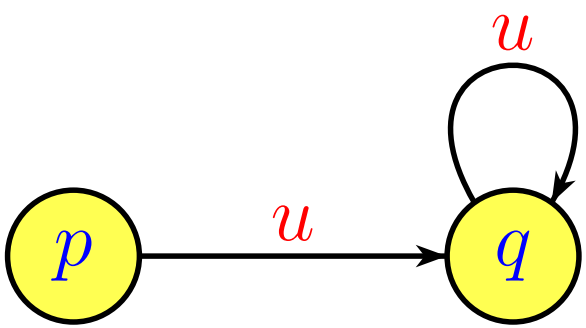
Several proofs, including one via model theory.

Closure of a regular language

Theorem (Pin 1991, Pin-Reutenauer 1991)

The closure of a regular language for $d_{\mathbf{G}}$ is still regular and can be computed effectively.

Intuitively, the only rule to apply is $\lim_{n\to\infty} x^{n!} = 1$.
In practice, just two rules, where u is any word.



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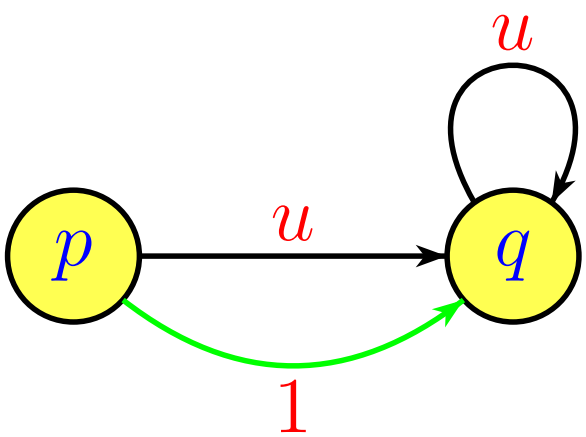

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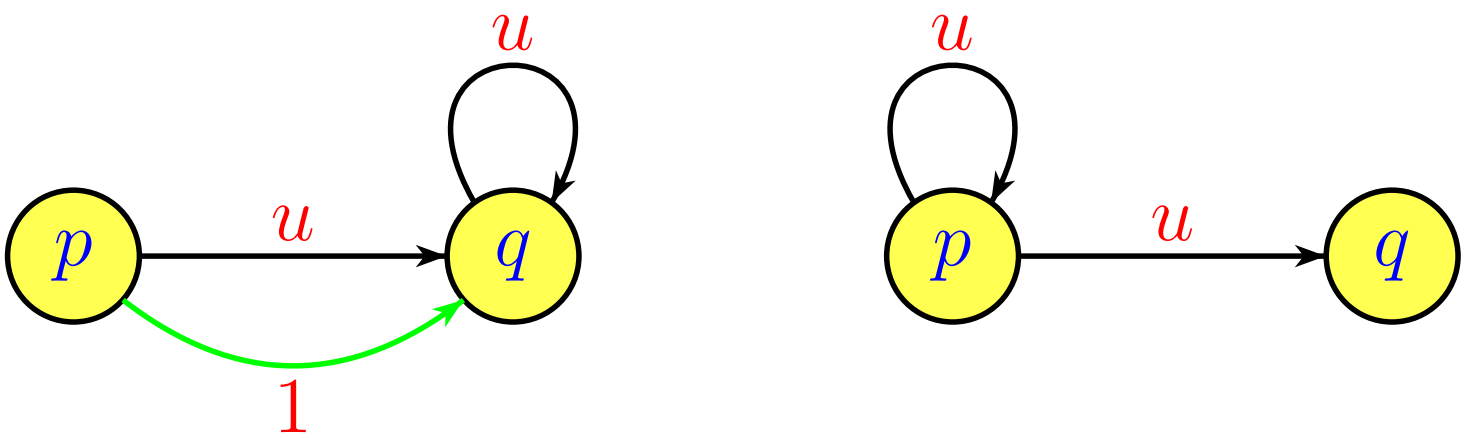

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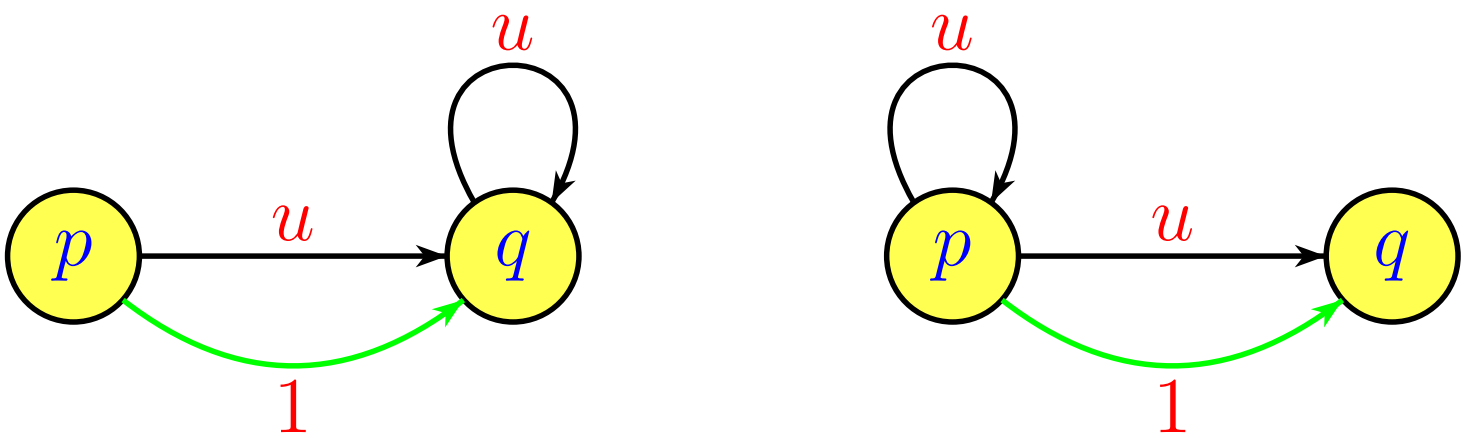
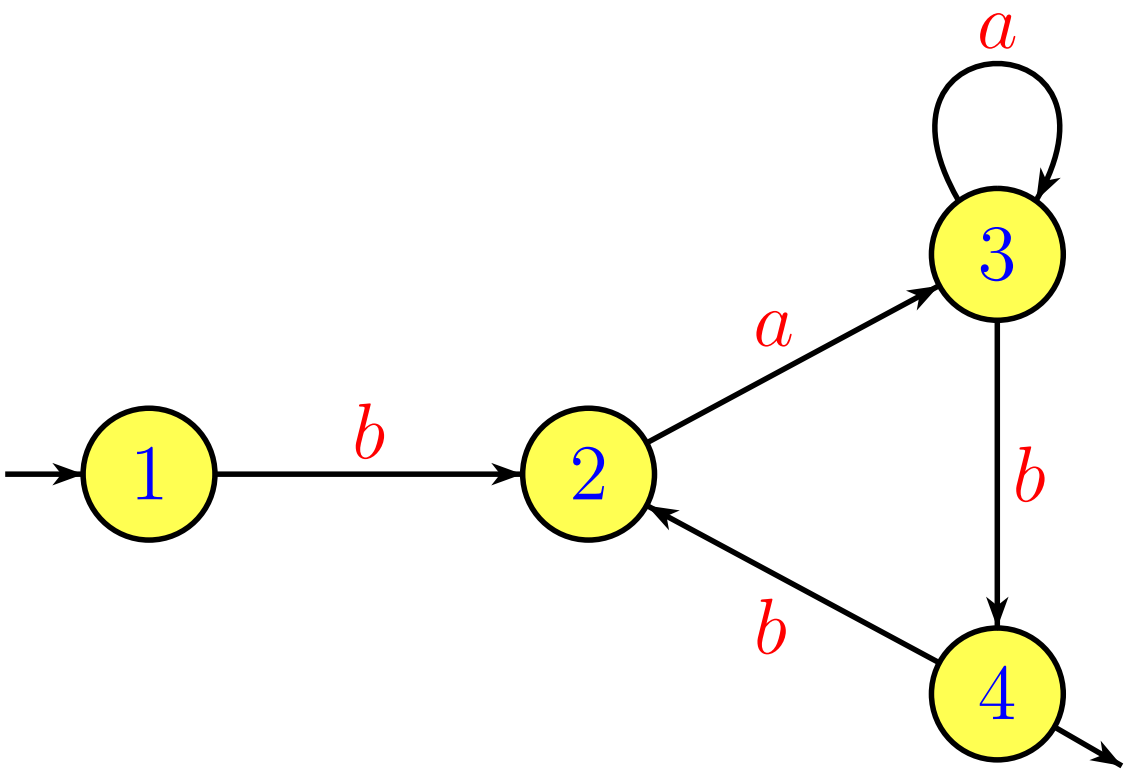


An example: $L = (ba^+b)^+$



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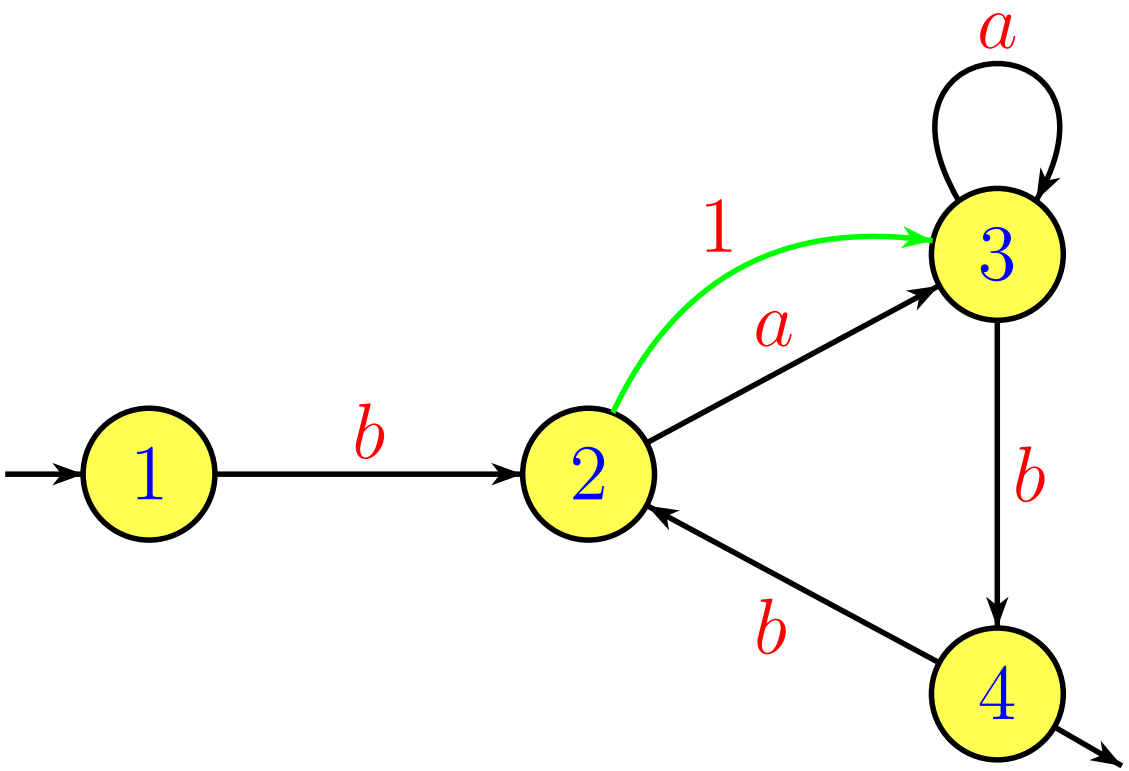

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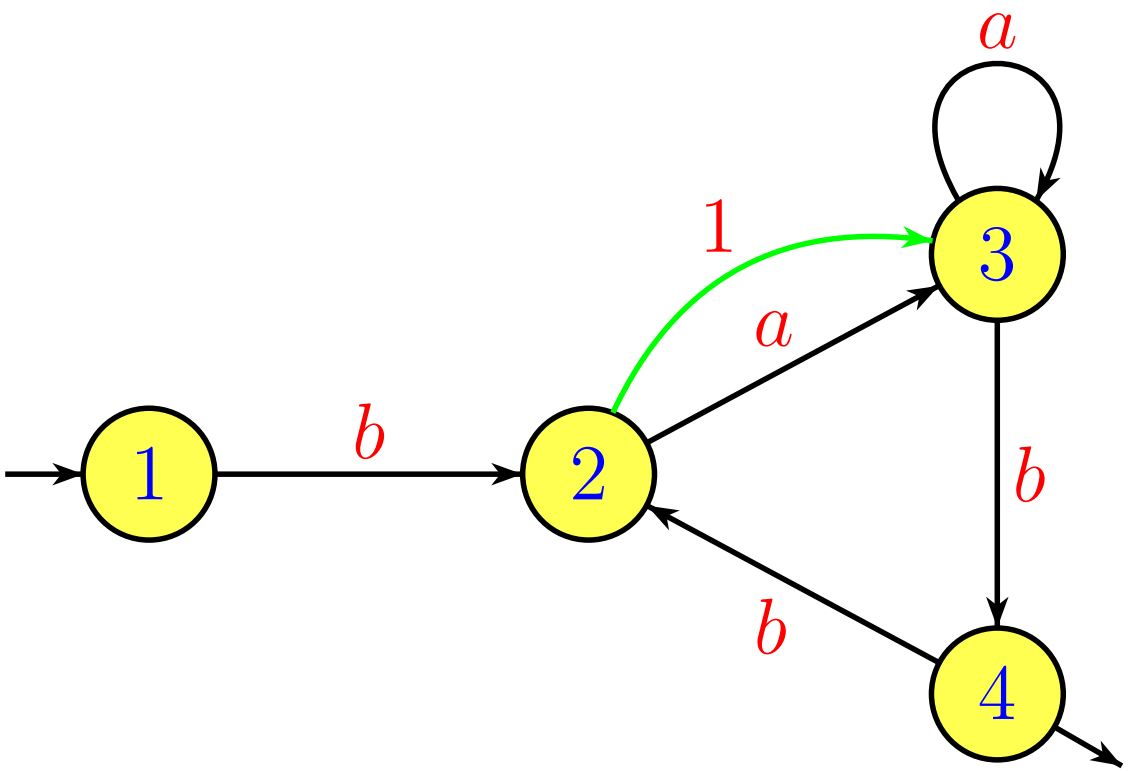

Now, there are a path from 1 to 4 and a loop around 4, both labeled by bb.

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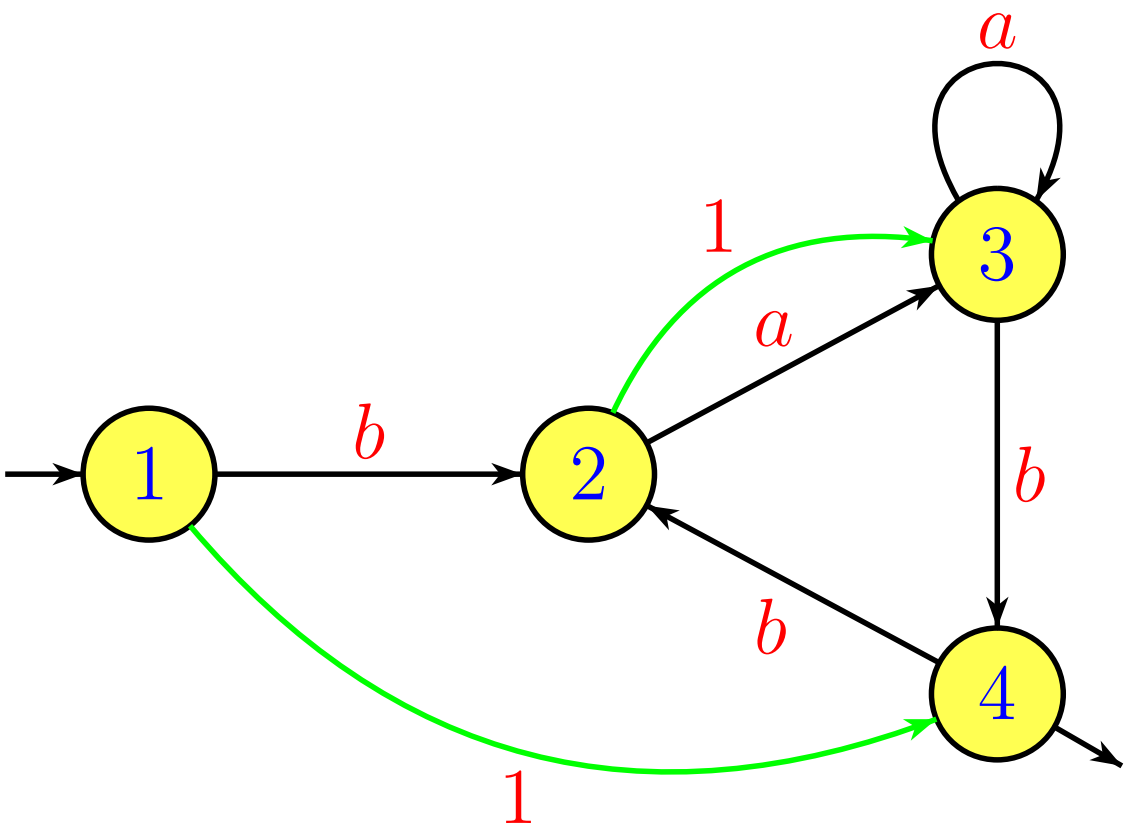

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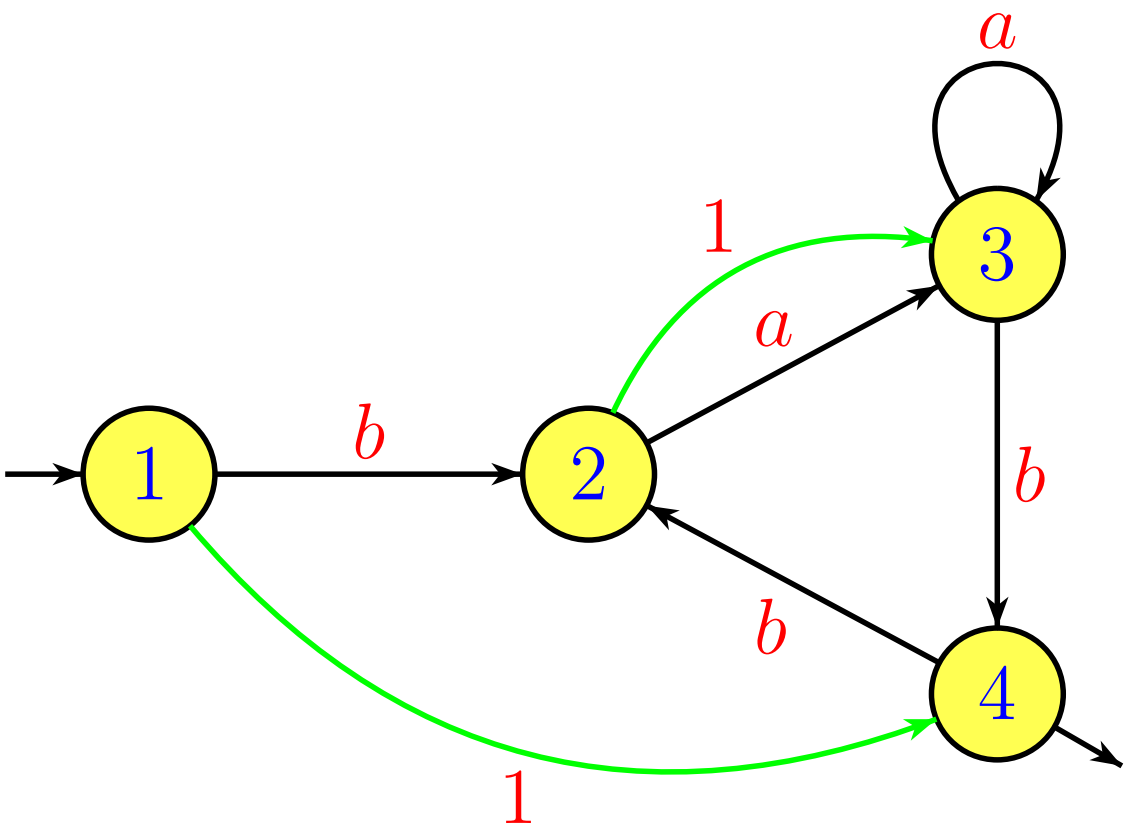

Now, there are a path from 1 to 4 and a loop around 4, both labeled by bb. The closure of L is $(ba^*b)^*$.

Part V

Some consequences

Some consequences in finite semigroup theory

- The variety of finite monoids generated by the monoids of partial injective maps on a finite set is the variety of monoids whose idempotent commute [Margolis-Pin 87, Ash 87].

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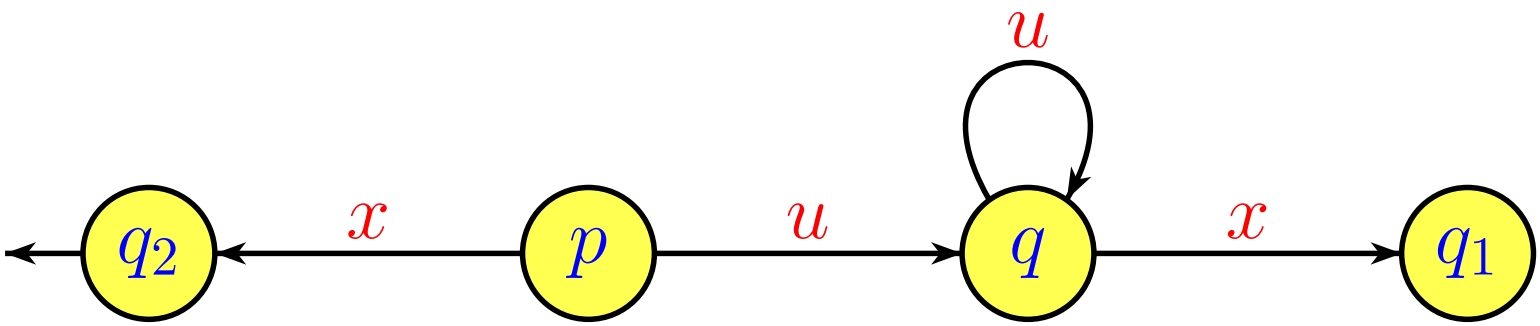
- The variety of finite monoids generated by the monoids of partial injective maps on a finite set is the variety of monoids whose idempotent commute [Margolis-Pin 87, Ash 87].
- The Mal'cev product of a decidable variety of finite monoids with the variety of finite groups is still a decidable variety. (In general, Mal'cev products do not preserve decidability) [Ash 91].

Some consequences in automata theory

- One can decide whether a given regular language can be accepted by a reversible automaton [Pin 87] .
- A regular language is open in the pro-group topology iff its minimal automaton does not contain the following pattern, where u and x are words, and $q_1 \neq q_2$.



Part VI

Conclusion and open problems

Open problems

One can define pro-p-group, pro-group pro-nilpotent, pro-solvable topologies, etc.

Algorithms are known to compute the closure of a regular language for the pro-group, the pro-p and the pro-nilpotent topology. It is still an open problem for the pro-solvable topology!

The problem amounts to deciding whether a partial group automaton can be completed, by adding an arbitray number of states and transitions, into a solvable group automaton.

Beyond groups

One can build profinite topologies for other varieties of finite monoids. The free monoid A^* is then equipped with a structure of metric space, whose completion is a compact monoid.

These objects are still mysterious and are the key to the solution of numerous problems of the theory of automata and the topic of active research [Almeida, Auinger, Steinberg, Weil, etc.]

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