

# Effectively Closed Sets of Measures and Randomness

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# Motivation

## Hausdorff measures and probability measures

- ▶ **Hausdorff measures** are an indispensable tool in fractal geometry: **self-similar sets, rectifiability, dimension concepts**.
- ▶ As measures, they are rather unpleasant to deal with: in general not  $\sigma$ -finite, no integration theory, etc.
- ▶ Consequently, the study of sets of finite Hausdorff  $s$ -measure is very complicated.
- ▶ It is possible to “approximate” Hausdorff measures by probability measures and make use their “good behavior” .
- ▶ **Question:** Can the theory of effective dimension, especially the connections to randomness and Kolmogorov complexity, contribute to this?

# Motivation

The basic paradigm

random reals + Turing reductions = existence of measures

# Measures on Cantor Space

## Outer measures from premeasures

Approximate sets from outside by open sets and weigh with a general measure function.

- ▶ A **premeasure** is a function  $\rho : 2^{<\omega} \rightarrow \mathbb{R}_0^+ \cup \{\infty\}$ .
- ▶ One can obtain an **outer measure**  $\mu_\rho$  from  $\rho$  by letting

$$\mu_\rho(X) = \inf_{C \subseteq 2^{<\omega}} \left\{ \sum_{\sigma \in C} \rho(\sigma) : \bigcup_{\sigma \in C} N_\sigma \supseteq X \right\},$$

where  $N_\sigma$  is the **basic open set** induced by  $\sigma$ .  
(Set  $\mu_\rho(\emptyset) = 0$ .)

The resulting  $\mu = \mu_\rho$  is a countably subadditive, monotone set function, an **outer measure**.

# Measures on Cantor Space

## Types of measures

**Probability measures:** based on a premeasure  $\rho$  which satisfies

- ▶  $\rho(\emptyset) = 1$  and
- ▶  $\rho(\sigma) = \rho(\sigma \frown 0) + \rho(\sigma \frown 1)$ .

For probability measures it holds that  $\mu_\rho(N_\sigma) = \rho(\sigma)$ .

**Hausdorff measures:** based on a premeasure  $\rho$  which satisfies

- ▶ If  $|\sigma| = |\tau|$ , then  $\rho(\sigma) = \rho(\tau)$ .
- ▶  $\rho(n)$  is nonincreasing.
- ▶  $\rho(n) \rightarrow 0$  as  $n \rightarrow \infty$ .
- ▶ For example:  $\rho(\sigma) = 2^{-|\sigma|^s}$ ,  $s \geq 0$ .

# Measures on Cantor Space

## Nullsets

The way we constructed outer measures,  $\mu(A) = 0$  is equivalent to the existence of a sequence  $(W_n)_{n \in \omega}$ ,  $W_n \subseteq 2^{<\omega}$ , such that for all  $n$ ,

$$A \subseteq \bigcup_{\sigma \in W_n} N_\sigma \quad \text{and} \quad \sum_{\sigma \in W_n} \rho(\sigma) \leq 2^{-n}.$$

Thus,

every nullset is contained in a  $G_\delta$  nullset.

# Randomness for Outer Measures

## Effective $G_\delta$ sets

By requiring that the covering nullset is **effectively**  $G_\delta$ , we obtain a notion of **effective nullsets**.

### Definition

- ▶ A **test relative to**  $z \in 2^\omega$  is a set  $W \subseteq \mathbb{N} \times 2^{<\omega}$  which is c.e. in  $z$ .
- ▶ A real  $x$  **passes** a test  $W$  if  $x \notin \bigcap_n N(W_n)$ , where  $W_n = \{\sigma : (n, \sigma) \in W\}$ .

Hence a real passes a test  $W$  if it is not in the  $G_\delta$ -set represented by  $W$ .

# Randomness for Outer Measures

## Martin-Löf tests

To test for randomness, we want to ensure that  $W$  actually describes a nullset.

### Definition

Suppose  $\mu$  is a measure on  $2^\omega$ . A test  $W$  is **correct for  $\mu$**  if for all  $n$ ,

$$\sum_{\sigma \in W_n} \mu(N_\sigma) \leq 2^{-n}.$$

Any test which is correct for  $\mu$  will be called a **test for  $\mu$** .

# Randomness for Outer Measures

## Representation of measures

An effective test for randomness should have **access to the measure** it is testing for.

- ▶ Therefore, **represent** it by an infinite binary sequence.
- ▶ Outer measures are determined by the underlying premeasure  $\rho$ . It seems reasonable to represent these values via **approximation by rational intervals**.

### Definition

Given a premeasure  $\rho$ , define its **rational representation**  $r_\rho$  by letting, for all  $\sigma \in 2^{<\omega}$ ,  $q_1, q_2 \in \mathbb{Q}$ ,

$$\langle \sigma, q_1, q_2 \rangle \in r_\rho \Leftrightarrow q_1 < \rho(\sigma) < q_2.$$

# Randomness for Outer Measures

## Tests for Arbitrary Measures

### Definition

Suppose  $\rho$  is a premeasure on  $2^\omega$  and  $z \in 2^\omega$ . A real is  $\mu_\rho$ -**z-random** if it passes all  $r_\rho \oplus z$ -tests which are correct for  $\mu_\rho$ .

Hence, a real  $x$  is random with respect to an arbitrary measure  $\mu_\rho$  if and only if it passes all tests which are enumerable in the representation  $r_\rho$  of the underlying premeasure  $\rho$ .

# Topology for Probability Measures

## The weak\*-topology

If  $\mu_\rho$  is a probability measure, the representation  $r_\rho$  can be interpreted topologically, by means of the **weak\*-topology** of Banach spaces.

- ▶ Denote by  $\mathcal{P}$  the set of all probability measures on  $2^\omega$ . For this section, we identify measures and their underlying premeasures.
- ▶ The **Riesz representation theorem** lets us identify measures with **linear functionals on the space of continuous functions** on  $2^\omega$ , by means of **integration**.
- ▶ The **weak\*-topology** on  $\mathcal{P}$  is the topology generated by the mappings  $f \mapsto \int f d\mu$ .

# Topology for Probability Measures

## A compatible metric

To generate the weak topology of  $\mathcal{P}$ , it suffices to consider a dense set of continuous functions on  $2^\omega$ .

- ▶ A **countable** dense set is given by the set of continuous functions on  $2^\omega$  that take only **finitely many, rational values**.
- ▶ Denote this set by  $D(2^\omega) = \{f_n\}_{n \in \omega}$ .

The mapping  $\mu \mapsto (\int f_n \mu / \|f_n\|_\infty)_{n \in \omega}$  embeds  $\mathcal{P}$  into  $[-1, 1]^\omega$ .

- ▶ We can pull back the product metric on  $[-1, 1]^\omega$  to  $\mathcal{P}$  to obtain a compatible metric

$$d(\mu, \nu) = \sum_{n=0}^{\infty} 2^{-n-1} \frac{|\int f_n d\mu - \int f_n d\nu|}{\|f_n\|_\infty}.$$

# Topology for Probability Measures

An effective dense subset

With the weak topology,  $\mathcal{P}$  becomes a **compact Polish space**.

A **countable dense subset** of  $\mathcal{P}$  is given as follows:

- ▶ Let  $Q$  be the set of all reals of the form  $\sigma \frown 0^\omega$ .
- ▶ Given  $\bar{q} = (q_1, \dots, q_n) \in Q^{<\omega}$  and non-negative rational numbers  $\alpha_1, \dots, \alpha_n$ , let

$$\delta_{\bar{q}} = \sum_{k=1}^n \alpha_k \delta_{q_k},$$

where  $\delta_x$  denotes the **Dirac point measure** for  $x$ .

# Topology for Probability Measures

## Effective representations

We want to exploit the topological structure of  $\mathcal{P}$  to prove results about algorithmic randomness.

- ▶ One can show that sets of the form

$$\{\mu \in \mathcal{P} : q_1 < \mu(\sigma) < q_2\}, \quad \sigma \in 2^{<\omega}, q_1, q_2 \in \mathbb{Q}$$

form a **subbasis** of the weak topology.

- ▶ Hence, the rational representation  $r_\mu$  indicates to which basic open sets  $\mu$  belongs.
- ▶ However, **not every real is a rational representation** of some probability measure.
- ▶ Moreover, the set of all  $x \in 2^\omega$  such that  $x = r_\mu$  for some  $\mu \in \mathcal{P}$  is **not**  $\Pi_1^0$ , so it does not effectively reflect the topological properties of  $\mathcal{P}$ .

# Topology for Probability Measures

## Effective representations

**Alternative:** Use the recursive dense subset  $\{\delta_{\bar{q}}\}$  and the effectiveness of the metric  $d$  between measures of the form  $\delta_{\bar{q}}$  to represent measures.

### Theorem

*There is a recursive surjection*

$$\pi: 2^\omega \rightarrow \mathcal{P}$$

*and a  $\Pi_1^0$  subset  $P$  of  $2^\omega$  such that  $\pi|_P$  is one-to-one and  $\pi(P) = \mathcal{P}$ .*

- ▶ The argument – as an effective version of a classical theorem of descriptive set theory – is applicable in much greater generality, essentially to any Polish space which allows for a recursive presentation (see **Moschovakis'** book)

# Effectively Closed Sets of Measures

## Uniform tests for randomness

Levin (1973) was the first to use  $\Pi_1^0$  classes of measures in algorithmic randomness.

### Observation

*Given a test  $W$ , the set of probability measures that are correct for  $W$  is  $\Pi_1^0$ .*

Levin was interested in devising **uniform tests for randomness**.

- ▶ A uniform test tests randomness for a whole class of measures, not only a single one.
- ▶ By the observation above, uniform tests can only exist for effectively closed sets of measures.

# Effectively Closed Sets of Measures

Uniform tests for randomness

## Theorem (Levin, 1973)

*Given a  $\Pi_1^0$  class  $S$  of probability measures, there exists a test  $\mathcal{U}$  such that for any  $x$  that passes  $\mathcal{U}$  there exists a measure  $\mu \in S$  such that  $x$  passes any  $\mu$ -test.*

Note that this is a kind of **lowness property**.

# Hausdorff Measures

## Outer measures from premeasures – Method II

Let  $\rho(\sigma) = 2^{-|\sigma|s}$ . In general,  $\mu_\rho$  is **not a Borel measure**.

- ▶ For example,  $\mu_\rho$  is not additive on cylinders.

Therefore, one refines the transition from a premeasure to an outer measure.

- ▶ Given  $\delta > 0$ , define the set function

$$\mathcal{H}_\delta^h(A) = \inf \left\{ \sum_{i=0}^{\infty} \rho_h(N_{\sigma_i}) : A \subseteq \bigcup_i N(\sigma_i), 2^{-|\sigma_i|} < \delta \right\}.$$

- ▶ Let  $\mathcal{H}^h(A) = \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^h(A)$ .

# Hausdorff Measures

## Difficulties of Hausdorff measures

The  $s$ -dimensional Hausdorff measure  $\mathcal{H}^s$  is a Borel measure.

- ▶ For  $s = 1$ ,  $\mathcal{H}^1$  is the same as Lebesgue measure on  $2^\omega$ .
- ▶ For  $s < 1$ , all basic open sets have infinite  $\mathcal{H}^s$ -measure. In particular, not all compact subsets of  $2^\omega$  have finite  $\mathcal{H}^s$  measure.

This makes the study of non-integral Hausdorff measures rather complicated.

- ▶ In particular, if  $\dim_{\mathcal{H}} A = s$  and  $\mathcal{H}^s(A) = \infty$ .
- ▶ Recall:  $\dim_{\mathcal{H}} A = \inf\{s : \mathcal{H}^s(A) = 0\}$ .

# Mass Distributions

Approximating Hausdorff measure by probability measures

**Idea:** If a set  $A$  supports a probability measure that is “close” to uniform, then its Hausdorff dimension is close to 1.

- ▶ Recall: The **support** of a measure  $\mu$ ,  $\text{supp}(\mu)$ , is the smallest closed set  $F$  such that  $\mu(2^\omega \setminus F) = 0$ .
- ▶  $A$  **supports** a measure  $\mu$  if  $\text{supp}(\mu) \subseteq A$ .

## Mass Distribution Principle

If  $A$  supports a probability measure  $\mu$  such that for almost all  $\sigma$ ,

$$\mu(\sigma) \leq c 2^{-|\sigma|s},$$

then  $\mathcal{H}^s(A) \geq 1/c$ .

# Mass Distributions and Hausdorff Measures

## Frostman's Lemma

A fundamental result due to **Frostman** (1935) asserts that the converse holds, too, as long as  $A$  is not too complex.

### Theorem

*If  $A$  is analytic and  $\dim_{\mathcal{H}} A > s > 0$ , then there exists a probability measure  $\mu$  such that  $\text{supp}(\mu) \subseteq A$  and for some  $c > 0$ ,*

$$\mu(\sigma) \leq c 2^{-|\sigma|s}$$

Frostman's Lemma is an important ingredient in the proof that every analytic set of infinite  $\mathcal{H}^s$ -measure has a subset of finite  $\mathcal{H}^s$ -measure.

# Effective Dimension and Continuous Randomness

Making reals of positive dimension random

We first show that every real of positive effective dimension is random with respect to a continuous probability measure.

- ▶ The theorem is an **effective version of Frostman's Lemma**.

## Theorem

*If  $\dim_{\mathbb{H}}^1 x > s > 0$ , then there exists a probability measure  $\mu$  such that  $x$  is  $\mu$ -random and for all  $\sigma$ ,*

$$\mu(\sigma) \leq c2^{-|\sigma|s}$$

# Effective Dimension and Continuous Randomness

## Transforming Randomness

By the **Kucera-Gacs Theorem**, there exists a  $\lambda$ -random real  $y$  such that  $y \geq_{\text{wtt}} x$  via some reduction  $\Phi$ .

- ▶ We will use  $y$  and the reduction to **transform randomness**.
- ▶ If  $\nu$  is a probability measure and  $f : 2^\omega \rightarrow 2^\omega$  is continuous, then the **image measure**  $\nu_f$ , defined by  $\nu_f(\sigma) = \nu(f^{-1}[\mathbf{N}_\sigma])$ , is also a probability measure.
- ▶ If  $f$  is effective (i.e. **truth-table**), then  $f$  transforms a computable probability measure into a computable probability measure.
- ▶ **Conservation of randomness**: If  $z$  is  $\nu$ -random and  $f$  is truth-table, then  $f(z)$  is  $\nu_f$ -random.

# Effective Dimension and Continuous Randomness

## Transforming Randomness

**Problem:** The Kucera-Gacs result holds only for a **wtt**-reduction.

- ▶ **Nota bene:** It can be easily seen that it cannot hold for truth-table since there are reals which are not random for any computable probability measure.

Partial reductions yield **semimeasures**.

- ▶ A **(continuous) semimeasure** is a function  $M : 2^{<\omega} \rightarrow [0, 1]$  such that  $M(\emptyset) \leq 1$  and  $M(\sigma) \geq M(\sigma \frown 0) + M(\sigma \frown 1)$ .

# Effective Dimension and Continuous Randomness

## Completing semimeasures

We want to define  $\mu(\sigma)$ ,  $\sigma \in 2^{<\omega}$ . We have to satisfy two requirements:

1. The measure  $\mu$  will **dominate** an image measure induced by  $\Phi$ . This will ensure that any Martin-Löf random sequence is mapped by  $\Phi$  to a  $\mu$ -random sequence.
2. The measure must respect the upper bound.

To meet these requirements, we restrict the values of  $\mu$  in the following way:

$$\lambda(\Phi^{-1}(\sigma)) \leq \mu(\sigma) \leq c2^{-|\sigma|s}. \quad (*)$$

This singles out **suitable completions** of the semimeasure induced by  $\Phi$ .

# Effective Dimension and Continuous Randomness

## Completing semimeasures

What is  $c$ ?

- ▶ Make use of the **semimeasure characterization** of effective Hausdorff measure:

$$x \text{ not effectively } \mathcal{H}^s\text{-null} \Rightarrow (\exists c_0)(\forall n) \widetilde{M}(x \upharpoonright_n) \leq c_0 2^{-ns},$$

where  $\widetilde{M}$  is an optimal enumerable continuous semimeasure.

- ▶ Choose  $c > c_0$ .

It can be shown that

$$\mathcal{M} := \{\mu : \mu \text{ satisfies } (*)\}$$

is a non-empty  $\Pi_1^0$  subset of  $\mathcal{P}$ .

# Effective Dimension and Continuous Randomness

A lowness property for  $\Pi_1^0$  classes

Note that if  $(V_n)$  were a  $\mu$ -test covering  $x$ , then  $\Phi^{-1}(V_n)$  would be a  $\lambda$ -test relative to  $\mu$  covering  $y$ .

- ▶ So, what we need to show is that  $y$  is  $\lambda$ -random relative to  $\mu$  for some  $\mu \in \mathcal{M}$ .

The following result ensures the existence of such a  $\mu$ . (Downey, Hirschfeldt, Miller, and Nies; Reimann and Slaman)

## Theorem

*If  $B \subseteq 2^\omega$  is nonempty and  $\Pi_1^0$ , then, for every  $y$  which is  $\lambda$ -random there is  $z \in B$  such that  $y$  is  $\lambda$ -random relative to  $z$ .*

The proof is essentially a compactness argument.

# Obtaining the Mass Distribution

## Compact subsets

Frostman's Lemma yields a mass distribution such that  $\text{supp}(\mu) \subseteq A$ .

- ▶ The base case is that  $A$  is closed.
- ▶ The proof for Borel sets uses clever approximations in measure.

If  $A$  is  $\Pi_1^0$ , then it is  $\Pi_1^0(z)$  relative to some  $z$ .

- ▶ **Relativize** the argument and add the  $\Pi_1^0$  conditions for  $A$  to (\*) determining the set of suitable measures  $M$ .

# Information Theoretic and Classical Methods

## A comparison

There are essentially two known proofs of Frostman's Lemma:

- ▶ By means of a **direct construction**, using the **compactness** of  $\mathcal{P}$ .
- ▶ Using the **Hahn-Banach theorem**, completing a functional defined on the subspace of constant functions constructed via **weighted Hausdorff measures**.

The second method works in arbitrary **compact metric spaces**.

- ▶ The information theoretic method can also be applied to arbitrary compact effective metric spaces, using **Gacs'** framework of randomness.

It seems that essentially the extension from subspaces in the Hahn-Banach theorem is replaced by a lowness property of  $\Pi_1^0$  classes.