Effectively Closed Sets of Measures and Randomness

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Outline

Motivation

Hausdorff measures and probability measures

- Hausdorff measures are an indispensable tool in fractal geometry: self-similar sets, rectifiability, dimension concepts.
- As measures, they are rather unpleasant to deal with: in general not σ-finite, no integration theory, etc.
- Consequently, the study of sets of finite Hausdorff s-measure is very complicated.
- It is possible to "approximate" Hausdorff measures by probability measures and make use their "good behavior".
- Question: Can the theory of effective dimension, especially the connections to randomness and Kolmogorov complexity, contribute to this?

Motivation

The basic paradigm

random reals $+$ Turing reductions $=$ existence of measures

Measures on Cantor Space

Outer measures from premeasures

Approximate sets from outside by open sets and weigh with a general measure function.

- A premeasure is a function $\rho : 2^{<\omega} \to \mathbb{R}_0^+ \cup \{\infty\}$.
- One can obtain an outer measure μ_ρ from ρ by letting

$$\mu_\rho(X) = \inf_{C \subseteq 2^{<\omega}} \left\{ \sum_{\sigma \in C} \rho(\sigma) : \bigcup_{\sigma \in C} N_\sigma \supseteq X \right\},$$

  where N_σ is the basic open set induced by σ.
  (Set $\mu_\rho(\emptyset) = 0$.)

The resulting $\mu = \mu_\rho$ is a countably subadditive, monotone set function, an outer measure.

Measures on Cantor Space

Types of measures

Probability measures: based on a premeasure ρ which satisfies

- $\rho(\emptyset) = 1$ and
- $\rho(\sigma) = \rho(\sigma^\frown 0) + \rho(\sigma^\frown 1)$.

For probability measures it holds that $\mu_\rho(N_\sigma) = \rho(\sigma)$.

Hausdorff measures: based on a premeasure ρ which satisfies

- If $|\sigma| = |\tau|$, then $\rho(\sigma) = \rho(\tau)$.
- $\rho(n)$ is nonincreasing.
- $\rho(n) \to 0$ as $n \to \infty$.
- For example: $\rho(\sigma) = 2^{-|\sigma|s}$, $s \geqslant 0$.

The way we constructed outer measures, $\mu(A) = 0$ is equivalent to the existence of a sequence $(W_n)_{n\in\omega}$, $W_n \subseteq 2^{<\omega}$, such that for all n,

$$A \subseteq \bigcup_{\sigma\in W_n} N_\sigma \quad \text{and} \quad \sum_{\sigma\in W_n} \rho(\sigma) \leqslant 2^{-n}.$$

Thus,

every nullset is contained in a G_δ nullset.

By requiring that the covering nullset is effectively G_δ, we obtain a notion of effective nullsets.

Definition

- A test relative to $z \in 2^\omega$ is a set $W \subseteq \mathbb{N} \times 2^{<\omega}$ which is c.e. in z.
- A real x passes a test W if $x \notin \bigcap_n N(W_n)$, where $W_n = \{\sigma : (n, \sigma) \in W\}$.

Hence a real passes a test W if it is not in the G_δ-set represented by W.

To test for randomness, we want to ensure that W actually describes a nullset.

Definition

Suppose μ is a measure on 2^ω. A test W is correct for μ if for all n,

$$\sum_{\sigma \in W_n} \mu(N_\sigma) \leqslant 2^{-n}.$$

Any test which is correct for μ will be called a test for μ.

An effective test for randomness should have access to the measure it is testing for.

- Therefore, represent it by an infinite binary sequence.
- Outer measures are determined by the underlying premeasure ρ. It seems reasonable to represent these values via approximation by rational intervals.

Definition

Given a premeasure ρ, define its rational representation r_ρ by letting, for all $\sigma \in 2^{<\omega}$, $q_1, q_2 \in \mathbb{Q}$,

$$\langle \sigma, q_1, q_2 \rangle \in r_\rho \;\Leftrightarrow\; q_1 < \rho(\sigma) < q_2.$$

Definition

Suppose ρ is a premeasure on 2^ω and $z \in 2^\omega$. A real is μ_ρ-z-random if it passes all $r_\rho \oplus z$-tests which are correct for μ_ρ.

Hence, a real x is random with respect to an arbitrary measure μ_ρ if and only if it passes all tests which are enumerable in the representation r_ρ of the underlying premeasure ρ.

Topology for Probability Measures

The weak*-topology

If μ_ρ is a probability measure, the representation r_ρ can be interpreted topologically, by means of the weak*-topology of Banach spaces.

- Denote by $\mathcal{P}$ the set of all probability measures on 2^ω. For this section, we identify measures and their underlying premeasures.
- The Riesz representation theorem lets us identify measures with linear functionals on the space of continuous functions on 2^ω, by means of integration.
- The weak*-topology on $\mathcal{P}$ is the topology generated by the mappings $f \mapsto \int f \, d\mu$.

To generate the weak topology of $\mathcal{P}$, it suffices to consider a dense set of continuous functions on 2^ω.

- A countable dense set is given by the set of continuous functions on 2^ω that take only finitely many, rational values.
- Denote this set by $D(2^\omega) = \{f_n\}_{n\in\omega}$.

The mapping $\mu \mapsto (\int f_n\mu / \|f_n\|_\infty)_{n\in\omega}$ embeds $\mathcal{P}$ into $[-1,1]^\omega$.

- We can pull back the product metric on $[-1,1]^\omega$ to $\mathcal{P}$ to obtain a compatible metric

$$d(\mu,\nu) = \sum_{n=0}^{\infty} 2^{-n-1} \frac{|\int f_n\, d\mu - \int f_n \nu|}{\|f_n\|_\infty}.$$

With the weak topology, $\mathcal{P}$ becomes a compact Polish space.

A countable dense subset of $\mathcal{P}$ is given as follows:

- Let Q be the set of all reals of the form $\sigma^\frown 0^\omega$.
- Given $\bar{q} = (q_1, \ldots, q_n) \in Q^{<\omega}$ and non-negative rational numbers $\alpha_1, \ldots, \alpha_n$, let

$$\delta_{\bar{q}} = \sum_{k=1}^{n} \alpha_k \delta_{q_k},$$

where δ_x denotes the Dirac point measure for x.

We want to exploit the topological structure of $\mathcal{P}$ to prove results about algorithmic randomness.

- One can show that sets of the form

$$\{\mu \in \mathcal{P} : \; q_1 < \mu(\sigma) < q_2\}, \quad \sigma \in 2^{<\omega}, q_1, q_2 \in \mathbb{Q}$$

  form a subbasis of the weak topology.
- Hence, the rational representation r_μ indicates to which basic open sets μ belongs.
- However, not every real is a rational representation of some probability measure.
- Moreover, the set of all $x \in 2^\omega$ such that $x = r_\mu$ for some $\mu \in \mathcal{P}$ is not Π^0_1, so it does not effectively reflect the topological properties of $\mathcal{P}$.

Alternative: Use the recursive dense subset $\{\delta_{\bar{q}}\}$ and the effectiveness of the metric d between measures of the form $\delta_{\bar{q}}$ to represent measures.

Theorem

There is a recursive surjection

$$\pi : 2^\omega \to \mathcal{P}$$

and a Π^0_1 *subset* P *of* 2^ω *such that* $\pi\restriction_P$ *is one-to-one and* $\pi(P) = \mathcal{P}$.

- The argument – as an effective version of a classical theorem of descriptive set theory – is applicable in much greater generality, essentially to any Polish space which allows for a recursive presentation (see Moschovakis' book)

Levin (1973) was the first to use Π^0_1 classes of measures in algorithmic randomness.

Observation

Given a test W, the set of probability measures that are correct for W is Π^0_1.

Levin was interested in devising uniform tests for randomness.

- A uniform test tests randomness for a whole class of measures, not only a single one.
- By the observation above, uniform tests can only exist for effectively closed sets of measures.

Theorem (Levin, 1973)

Given a Π^0_1 class S of probability measures, there exists a test U such that for any x that passes U there exists a measure $\mu \in S$ such that x passes any μ-test.

Note that this is a kind of lowness property.

Hausdorff Measures

Outer measures from premeasures – Method II

Let $\rho(\sigma) = 2^{-|\sigma|s}$. In general, μ_ρ is not a Borel measure.

- For example, μ_ρ is not additive on cylinders.

Therefore, one refines the transition from a premeasure to an outer measure.

- Given $\delta > 0$, define the set function

$$\mathcal{H}^h_\delta(A) = \inf\left\{\sum_{i=0}^{\infty} \rho_h(N_\sigma) : A \subseteq \bigcup_i N(\sigma_i),\ 2^{-|\sigma_i|} < \delta\right\}.$$

- Let $\mathcal{H}^h(A) = \lim_{\delta\to 0} \mathcal{H}^h_\delta(A)$.

The s-dimensional Hausdorff measure $\mathcal{H}^s$ is a Borel measure.

- For $s = 1$, $\mathcal{H}^1$ is the same as Lebesgue measure on 2^ω.
- For $s < 1$, all basic open sets have infinite $\mathcal{H}^s$-measure. In particular, not all compact subsets of 2^ω have finite $\mathcal{H}^s$ measure.

This makes the study of non-integral Hausdorff measures rather complicated.

- In particular, if $\dim_H A = s$ and $\mathcal{H}^s(A) = \infty$.
- Recall: $\dim_H A = \inf\{s : \mathcal{H}^s(A) = 0\}$.

Mass Distributions

Approximating Hausdorff measure by probability measures

Idea: If a set A supports a probability measure that is "close" to uniform, then its Hausdorff dimension is close to 1.

- Recall: The support of a measure μ, $\mathrm{supp}(\mu)$, is the smallest closed set F such that $\mu(2^\omega \setminus F) = 0$.
- A supports a measure μ if $\mathrm{supp}(\mu) \subseteq A$.

Mass Distribution Principle

If A supports a probability measure μ such that for almost all σ,

$$\mu(\sigma) \leqslant c2^{-|\sigma|s},$$

then $\mathcal{H}^s(A) \geqslant 1/c$.

A fundamental result due to Frostman (1935) asserts that the converse holds, too, as long as A is not too complex.

Theorem

If A is analytic and $\dim_H A > s > 0$, then there exists a probability measure μ such that $\mathrm{supp}(\mu) \subseteq A$ and for some $c > 0$,

$$\mu(\sigma) \leqslant c 2^{-|\sigma| s}$$

Frostman's Lemma is an important ingredient in the proof that every analytic set of inifinite $\mathcal{H}^s$-measure has a subset of finite $\mathcal{H}^s$-measure.

We first show that every real of positive effective dimension is random with respect to a continuous probability measure.

- The theorem is an effective version of Frostman's Lemma.

Theorem

If $\dim_H^1 x > s > 0$, then there exists a probability measure μ such that x is μ-random and for all σ,

$$\mu(\sigma) \leqslant c2^{-|\sigma|s}$$

By the Kucera-Gacs Theorem, there exists a λ-random real y such that $y \geqslant_{\mathrm{wtt}} x$ via some reduction Φ.

- We will use y and the reduction to transform randomness.
- If ν is a probability measure and $f : 2^\omega \to 2^\omega$ is continuous, then the image measure ν_f, defined by $\nu_f(\sigma) = \nu(f^{-1}[N_\sigma])$, is also a probability measure.
- If f is effective (i.e. truth-table), then f transforms a computable probability measure into a computable probability measure.
- Conservation of randomness: If z is ν-random and f is truth-table, then $f(z)$ is ν_f-random.

Problem: The Kucera-Gacs result holds only for a wtt-reduction.

- Nota bene: It can be easily seen that it cannot hold for truth-table since there are reals which are not random for any computable probability measure.

Partial reductions yield semimeasures.

- A (continuous) semimeasure is a function $M : 2^{<\omega} \to [0,1]$ such that $M(\emptyset) \leqslant 1$ and $M(\sigma) \geqslant M(\sigma^\frown 0) + M(\sigma^\frown 1)$.

We want to define $\mu(\sigma)$, $\sigma \in 2^{<\omega}$. We have to satisfy two requirements:

1. The measure μ will dominate an image measure induced by Φ. This will ensure that any Martin-Löf random sequence is mapped by Φ to a μ-random sequence.
2. The measure must respect the upper bound.

To meet these requirements, we restrict the values of μ in the following way:

$$\lambda(\Phi^{-1}(\sigma)) \leqslant \mu(\sigma) \leqslant c2^{-|\sigma|s}. \qquad (*)$$

This singles out suitable completions of the semimeasure induced by Φ.

What is c?

- Make use of the semimeasure characterization of effective Hausdorff measure:

$$x \text{ not effectively } \mathcal{H}^s\text{-null} \;\Rightarrow\; (\exists c_0)(\forall n)\, \widetilde{M}(x\restriction_n) \leqslant c_0 2^{-ns},$$

  where $\widetilde{M}$ is an optimal enumerable continuous semimeasure.
- Choose $c > c_0$.

It can be shown that

$$M := \{\mu : \ \mu \text{ satisfies } (*)\}$$

is a non-empty Π^0_1 subset of $\mathcal{P}$.

Note that if (V_n) were a μ-test covering x, then $\Phi^{-1}(V_n)$ would be a λ-test relative to μ covering y.

- So, what we need to show is that y is λ-random relative to μ for some $\mu \in M$.

The following result ensures the existence of such a μ. (Downey, Hirschfeldt, Miller, and Nies; Reimann and Slaman)

Theorem

If $B \subseteq 2^\omega$ is nonempty and Π^0_1, then, for every y which is λ-random there is $z \in B$ such that y is λ-random relative to z.

The proof is essentially a compactness argument.

Obtaining the Mass Distribution

Compact subsets

Frostman's Lemma yields a mass distribution such that $\text{supp}(\mu) \subseteq A$.

- The base case is that A is closed.
- The proof for Borel sets uses clever approximations in measure.

If A is $\boldsymbol{\Pi}^0_1$, then it is $\Pi^0_1(z)$ relative to some z.

- Relativize the argument and add the Π^0_1 conditions for A to $(*)$ determining the set of suitable measures M.

Information Theoretic and Classical Methods

A comparison

There are essentially two known proofs of Frostman's Lemma:

- By means of a direct construction, using the compactness of $\mathcal{P}$.
- Using the Hahn-Banach theorem, completing a functional defined on the subspace of constant functions constructed via weighted Hausdorff measures.

The second method works in arbitrary compact metric spaces.

- The information theoretic method can also be applied to arbitrary compact effective metric spaces, using Gacs' framework of randomness.

It seems that essentially the extension from subspaces in the Hahn-Banach theorem is replaced by a lowness property of Π_1^0 classes.