

Bounded Genericity

Ludwig Staiger

Lehrstuhl für Theoretische Informatik
Institut für Informatik
Martin-Luther-Universität Halle-Wittenberg

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- 1 Notation and Preliminary Definitions
- 2 Structure Functions of Languages
- 3 Dimensions of ω -languages

Notation: Strings and Languages

- Finite Alphabet $X = \{0, \dots, r - 1\}$, cardinality $|X| = r \geq 2$
- Finite strings (words): $w = x_1 \cdots x_l \in X^*$, $x_i \in X$, length $|w| = l$
- Languages: $W \subseteq X^*$
- Infinite strings (ω -words): $\xi = x_1 \cdots x_l \cdots \in X^\omega$
- Prefixes of infinite strings: $\xi[0..n] \in X^*$, $|\xi[0..n]| = n$
- Prefix relation: $w \sqsubseteq \eta$, for $\eta \in X^* \cup X^\omega$
- Language of prefixes: $\mathbf{pref}(B) := \{w : \exists \eta (\eta \in B \wedge w \sqsubseteq \eta)\}$

X^ω as CANTOR space

Metric: $\rho(\eta, \xi) := \inf \{r^{-|w|} : w \sqsubset \eta \wedge w \sqsubset \xi\}$

Balls in (X^ω, ρ) : $w \cdot X^\omega = \{\eta : w \sqsubset \eta\}$

Diameter: $\text{diam } w \cdot X^\omega = r^{-|w|}$

Open sets: $W \cdot X^\omega = \bigcup_{w \in W} w \cdot X^\omega$

Closure: $\mathcal{C}(F) = \{\xi : \mathbf{pref}(\xi) \subseteq \mathbf{pref}(F)\}$

Property

$F \subseteq X^\omega$ is closed if and only if $\mathbf{pref}(\xi) \subseteq \mathbf{pref}(F)$ implies $\xi \in F$.

Extension functions (Ambos-Spies & Busse '03)

Extension function: $f : X^* \rightarrow X^*$

k -bounded extension function: $f : X^* \rightarrow X^k$

$f : X^* \rightarrow X^*$ **meets** $\xi \in X^\omega$: $\exists w (w \sqsubset \xi \wedge w \cdot f(w) \sqsubset \xi)$

$$M_f := \bigcup_{w \in X^*} w \cdot f(w) \cdot X^\omega,$$

$$F_f := X^\omega \setminus M_f$$

$$M_{\mathcal{F}} := \bigcap_{f \in \mathcal{F}} M_f,$$

$$F_{\mathcal{F}} := X^\omega \setminus M_{\mathcal{F}} = \bigcup_{f \in \mathcal{F}} F_f$$

ξ is $(f_n)_{n \in \mathbb{N}}$ -**generic**: $\xi \in \bigcap_{n \in \mathbb{N}} M_{f_n} = X^\omega \setminus \bigcup_{n \in \mathbb{N}} F_{f_n}$

k -bounded automatic genericity

$f : X^* \rightarrow X^k$ is defined by a finite automaton $\mathcal{A} = (X, Q, q_0, \delta)$ and a function $g : Q \rightarrow X^k$ via

$$f(w) := g(\delta(q_0, w)).$$

Lemma

If f is a k -bounded automatic extension function then F_f is definable by a finite automaton.

Theorem (Ambos-Spies & Busse '03)

Let $\{f_n^k : n \in \mathbb{N}\}$ be the family of all k -bounded automatic extension functions. Then

$$\bigcup_{n \in \mathbb{N}} F_{f_n^k} \subset \bigcup_{n \in \mathbb{N}} F_{f_n^{k+1}}.$$

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Special case: Forbidden subwords

For some $v \in X^*$, $f \equiv v$ is a constant function, that is,

$$E(\neg v) := F_f = X^\omega \setminus X^* \cdot v \cdot X^\omega.$$

Definition (Overlap-free words)

A word $u \in X^*$ is **overlap-free** if and only if no non-empty proper prefix of u is a suffix of u .

A hierarchy result by VOLKMANN

Theorem (Volkmann '53)

1. **length k :** Let $v \in X^k$ be overlap-free, and $u \in X^k$ neither overlap-free nor of the form a^k , $a \in X$. Then

$$\dim_{\mathbb{H}} E(\neg v) < \dim_{\mathbb{H}} E(\neg u) < \dim_{\mathbb{H}} E(\neg 0^k).$$

2. **length k and $k + 1$:** Let $u \in X^{k+1}$. If $|X| \geq 3$ or u is not overlap-free then

$$\dim_{\mathbb{H}} \bigcup_{|v|=k} E(\neg v) < \dim_{\mathbb{H}} E(\neg u).$$

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Structure function

Definition (Structure Function of $W \subseteq X^*$ and $F \subseteq X^\omega$)

$$s_W(n) := |\{w : w \in W \wedge |w| = n\}| = |W \cap X^n|$$

$$s_F := s_{\text{pref}(F)}$$

Special case: $E(\neg 0^{k-1}1)$ and $E(\neg 0^k)$

$$\begin{aligned} \underline{E}_k &:= E(\neg 0^{k-1}1) = X^\omega \setminus X^* \cdot \mathbf{0}^{k-1}\mathbf{1} \cdot X^\omega \\ E_k &:= E(\neg 0^k) = X^\omega \setminus X^* \cdot \mathbf{0}^k \cdot X^\omega \end{aligned}$$

Their structure functions satisfy

$$\begin{aligned} \underline{s}_k(n) &= s_k(n) = r^n, & \text{if } 0 \leq n < k \text{ and} \\ \underline{s}_k(k) &= s_k(k) = r^k - 1, & \end{aligned} \quad (1)$$

and the recurrences

$$\begin{aligned} \underline{s}_k(n) &= r \cdot \underline{s}_k(n-1) - \underline{s}_k(n-k), & \text{if } n > k \\ s_k(n) &= (r-1) \cdot \sum_{i=1}^k s_k(n-i), & \text{if } n > k. \end{aligned} \quad (2)$$

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Asymptotic behaviour of \underline{s}_k and s_k

Let $\underline{\lambda}_k$ and λ_k be the maximum roots of the polynomials

$$\underline{p}_k(z) = z^k - r \cdot z^{k-1} + 1 \text{ and}$$

$$p_k(z) = z^k - \sum_{i=0}^{k-1} (r-1) \cdot z^i, \text{ respectively.}$$

Remark. $\underline{\lambda}_k < \lambda_k$, and if $r > 2$ then $\lambda_k < \lambda_{k+1}$.

$$\underline{s}_k(n) \approx \underline{\lambda}_k^n \quad \text{and} \quad s_k(n) \approx \lambda_k^n$$

Overlapping

Definition

Let $f : X^* \rightarrow X^k$ be an extension function.

- ① We call f **overlap-free** : \iff
 $\forall w \forall v (v \sqsubset w \sqsubset v \cdot f(v) \rightarrow v \cdot f(v) \not\sqsubseteq w \cdot f(w))$.
- ② We call f **fully overlapping** : \iff
 $\forall w \forall v (v \sqsubset w \sqsubset v \cdot f(v) \rightarrow v \cdot f(v) \sqsubseteq w \cdot f(w))$.

Lemma

Let $f : X^* \rightarrow X^k$ be an extension function.

- ① There is an overlap-free extension function $g : X^* \rightarrow X^k$ such that $F_g \subseteq F_f$.
- ② There is a fully overlapping extension function $h : X^* \rightarrow X^k$ such that $F_h \supseteq F_f$.

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Let $f : X^* \rightarrow X^k$ be an extension function.

- ① *There is an overlap-free extension function $g : X^* \rightarrow X^k$ such that $F_g \subseteq F_f$.*
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Minimal and maximal elements

$$\mathcal{E}_k := \{F_f : f : X^* \rightarrow X^k\}$$

Lemma

If F_f is a minimal (maximal) w.r.t. set inclusion element in \mathcal{E}_k then there is an overlap-free extension function $g : X^* \rightarrow X^k$ (a fully overlapping extension function $h : X^* \rightarrow X^k$) such that $F_f = F_g$ ($F_f = F_h$).

Theorem

- 1 If $f : X^* \rightarrow X^k$ then $\underline{s}_k \leq s_{F_f} \leq s_k$.
- 2 If $g : X^* \rightarrow X^k$ is overlap-free then $s_{F_g} = \underline{s}_k$ and F_g is minimal in \mathcal{E}_k .
- 3 If $h : X^* \rightarrow X^k$ is fully overlapping then $s_{F_h} = s_k$ and F_h is maximal in \mathcal{E}_k .

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- 3 If $h : X^* \rightarrow X^k$ is fully overlapping then $s_{F_h} = s_k$ and F_h is maximal in \mathcal{E}_k .

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Minkowski or box-counting dimensions

$$\underline{\dim}_{\mathbf{B}} F := \liminf_{n \rightarrow \infty} \frac{1}{n} \cdot \log_{|\mathcal{X}|}(s_F(n) + 1)$$

$$\overline{\dim}_{\mathbf{B}} F := \limsup_{n \rightarrow \infty} \frac{1}{n} \cdot \log_{|\mathcal{X}|}(s_F(n) + 1)$$

Property

$$\underline{\dim}_{\mathbf{B}} F = \underline{\dim}_{\mathbf{B}} \mathcal{C}(F)$$

$$\overline{\dim}_{\mathbf{B}} F = \overline{\dim}_{\mathbf{B}} \mathcal{C}(F)$$

$$E \subseteq F \rightarrow \underline{\dim}_{\mathbf{B}} E \leq \underline{\dim}_{\mathbf{B}} F$$

$$\overline{\dim}_{\mathbf{B}} E \cup F = \max \{ \overline{\dim}_{\mathbf{B}} E, \overline{\dim}_{\mathbf{B}} F \}$$

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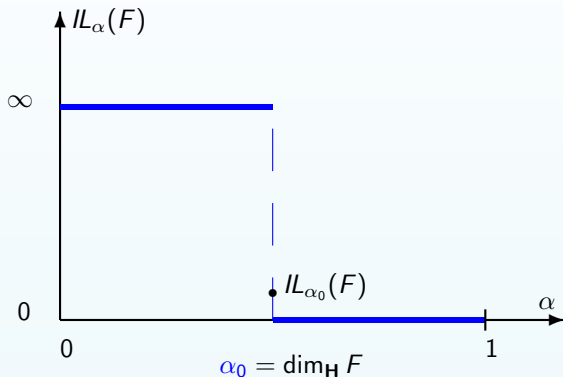
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Hausdorff measure

$$IL_\alpha(F) := \lim_{l \rightarrow \infty} \inf \left\{ \sum_{w \in W} r^{-\alpha \cdot |w|} : F \subseteq \bigcup_{w \in W} w \cdot X^\omega \wedge \forall w (w \in W \rightarrow |w| \geq l) \right\}$$



Hausdorff dimension

$$\dim_{\mathbf{H}} F = \sup \{ \alpha : IL_{\alpha}(F) = \infty \} = \inf \{ \alpha : IL_{\alpha}(F) = 0 \}$$

Proposition

- 1 $\dim_{\mathbf{H}} \bigcup_{i \in \mathbb{N}} F_i = \sup \{ \dim_{\mathbf{H}} F_i : i \in \mathbb{N} \}$
- 2 $\dim_{\mathbf{H}} w \cdot F = \dim_{\mathbf{H}} F$
- 3 $\dim_{\mathbf{H}} F \leq \underline{\dim}_{\mathbf{B}} F \leq \overline{\dim}_{\mathbf{B}} F$

Relations between the dimensions

Definition ("Subtree" rooted at $w \in X^*$)

$$F/w := \{\xi : w \cdot \xi \in F\}$$

Theorem (St.'89)

Let $\log_r \gamma(n) = o(n)$ for some $\gamma : \mathbb{N} \rightarrow \mathbb{N}$. If
 $s_{F/w}(n) \leq \gamma(|w|) \cdot s_F(n)$ for all $w \in X^*$ and all $n \in \mathbb{N}$ then

$$\dim_{\mathbb{H}} F = \overline{\dim}_{\mathbb{B}} F.$$

Observation

If $f : X^* \rightarrow X^k$ is overlapfree or fully overlapping then
 $s_{F_f/w}(n) \leq s_{F_f}(n)$, for all $w \in X^*$ and all $n \in \mathbb{N}$.

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 $s_{F_f/w}(n) \leq s_{F_f}(n)$, for all $w \in X^*$ and all $n \in \mathbb{N}$.

The dimensions of F_f

Corollary

If $f : X^* \rightarrow X^k$ is overlapfree or fully overlapping then

$$\dim_{\mathbf{H}} F_f = \overline{\dim}_{\mathbf{B}} F_f.$$

Lemma (Mauldin & Williams '88, Bandt '89, St.'89)

If $f : X^* \rightarrow X^k$ is automatic then

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General hierarchy result I

Theorem

If f is a k -bounded extension function and f' is a $(k + 1)$ -bounded extension function over X then $F_f \not\subseteq F_{f'}$ and

$$\log_{|X|} \underline{\lambda}_k \leq \dim_H F_f \leq \log_{|X|} \lambda_k \leq$$

$$\log_{|X|} \underline{\lambda}_{k+1} \leq \dim_H F_{f'} \leq \log_{|X|} \lambda_{k+1}.$$

Equality of Hausdorff dimension $\dim_H F_f = \dim_H F_{f'}$ can hold only if $|X| = 2$, $\dim_H F_f = \log_2 \lambda_k$ and $\dim_H F_{f'} = \log_2 \underline{\lambda}_{k+1}$.

General hierarchy result II

Theorem

Let, for $k \in \mathbb{N}$, $\{f_n^k : n \in \mathbb{N}\}$ be a countable family of k -bounded extension functions containing all constant functions. Then

$$\bigcup_{n \in \mathbb{N}} F_{f_n^k} \not\subseteq \bigcup_{n \in \mathbb{N}} F_{f_n^{k+1}}.$$

Corollary

Let $\{f_n^k : n \in \mathbb{N}\}$ be the family of all k -bounded computable (recursive) extension functions. Then

$$\bigcup_{n \in \mathbb{N}} F_{f_n^k} \subset \bigcup_{n \in \mathbb{N}} F_{f_n^{k+1}}.$$

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