

Genericity Theory from the Randomness Viewpoint

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Genericity and Randomness

Definition

A real $x \in 2^\omega$ is

- (i) *weakly n -generic* if $x \notin A$ for all Π_n^0 meager set A ;
- (ii) *weakly n -random* if $x \notin A$ for all Π_n^0 null set A .

Definition

A real $x \in 2^\omega$ is

- (i) *n -generic* if either $x \in A$ or there exists a finite string $\sigma \prec x$ so that $[\sigma] \cap A = \emptyset$ for all Σ_n^0 open set A ;
- (ii) *n -random* if $x \notin A$ for set A which is the intersection of a Σ_n^0 -test.

Effective Category and Measure Theory

Theorem

Fix a universal Σ_n^0 set $A \subseteq \omega \times 2^\omega$, then:

- 1 (Sacks) $\{(i, n) \mid \mu(A_i) > 2^{-n}\}$ is Σ_n^0 .
- 2 (Kechris) $\{i \mid A_i \text{ is not meager}\}$ is Σ_n^0 .

Theorem

For every Σ_n^0 set $A \subseteq 2^\omega$,

- 1 (Kurtz) there is a recursive sequence of Σ_{n-1}^0 closed sets $\{F_m\}_m$ so that $\bigcup_n F_m \subseteq A$ and $\mu(\bigcup_m F_m) = \mu(A)$.
- 2 (Forklore) A is comeager, then there is an recursive sequence of Σ_n^0 dense open sets $\{U_m\}_m$ so that $\bigcap_m U_m \subseteq A$.

Uniformizing the Notions

Corollary (Kurtz)

A real $x \in 2^\omega$ is

- 1 weakly n -generic iff $x \in U$ for every Σ_n^0 dense open set U .
- 2 weakly $n + 1$ -random iff $x \notin \bigcap_m U_m$ for all recursive sequence of Σ_n^0 open sets $\{U_m\}_m$ with $\lim_m \mu(U_m) = 0$.
- 3 n -random iff $x \notin \bigcap_m U_m$ for all recursive sequence of Σ_n^0 open sets $\{U_m\}_m$ with $\mu(U_m) \leq 2^{-m}$.

So weak $n + 1$ randomness \implies n -randomness \implies weak n -randomness and weak $n + 1$ -genericity \implies n -genericity \implies weak n -genericity

Definition

- A Turing degree is hyperimmune if it contains a function not dominated by any recursive functions. Otherwise, it is hyperimmune-free.
- A degree is recursively traceable if every function computed by it can be traced by a recursive function with identity bound.
- A Turing degree is *DNR* if it contains a function f so that $\forall n(f(n) \neq \Phi_n(n))$.
- A Turing degree is *PA* if it contains a real computing a completion of Peano's Axioms.

Characterizing Low Levels I

Theorem

- 1 (Forklore) Every weakly 1-generic real is weakly 1-random.
- 2
 - (Kurtz+Jockusch) x has a weakly 1-generic degree iff it has a hyperimmune degree, and every 1-generic real is REA.
 - (DNWY+Hirschfeldt, Miller) x has a weakly 2-random degree iff it has a 1-random degree and is incomparable with all of nonrecursive Δ_2^0 -degrees.
 - (Kurtz) Every 2-random real is REA.
- 3
 - (Forklore) Every 1-generic degree is GL_1 .
 - (Kautz) Every 2-random degree is GL_1 .
 - (Kucera) If $x \geq_T \emptyset'$, then it has a 1-random degree.

Characterizing Low Levels II

Theorem

- (Forklore) No 1-generic real has DNR-degree.
- (Forklore) Every 1-random degree is a DNR-degree.
- (Stephan) If x is 1-random, then x has a PA degree iff $x \geq_T \emptyset'$.
- (Yu) A real x is hyperimmune-free, then x is weakly 1-random iff x is weakly 2-random.

Question

Characterizing weakly 1-random degrees.

Kolmogorov Complexity vs Randomness

Theorem

- 1 (Schnorr) x is 1-random iff there is a constant c so that $\forall n(K(x \upharpoonright n) \geq n - c)$.
- 2 (Miller and Yu) x is 1-random iff for every computable function g with $\sum_n 2^{-g(n)} < \infty$, there is a constant c so that $\forall n(C(x \upharpoonright n) \geq n - g(n) - c)$.
- 3 (Miller and Yu) $x \oplus y$ is 1-random iff there is a constant so that $\forall n(K(x \upharpoonright n) + C(y \upharpoonright n) \geq 2n - c)$.
- 4 (NST+Miller) x is 2-random iff there is a constant c so that $\exists^\infty n(C(x \upharpoonright n) \geq n - c)$.

Kolmogorov Complexity vs Genericity

Theorem

- 1 (Nies) *There exists a K -trivial 1-generic real.*
- 2 (Forklore) *x is weakly 2-generic then x is K -“very low” and K -random infinitely often.*

Question

Finding out a complexity characterization of weak 2-randomness and genericity.

Lowness for Randomness and Genericity I

Definition

Given a notion G and its relativization G^x , a real x is low for G if $G = G^x$.

Theorem

- 1 (Stephan and Yu) A real is low for weakly 1-random then it must have a degree strictly between recursively traceability and hyperimmune-freeness.
- 2 (Stephan and Yu) A real is low for weakly 1-generic iff it hyperimmune-free and non-DNR.

Lowness for Randomness and Genericity II

Theorem

1

- (Hirschfeldt and Nies) A real x is low for 1-random iff it is there exists a constant c so that $\forall n(K(x \upharpoonright n) \leq K(n) + c)$.
- (Greenberg, Miller + Yu) A real is low for 1-generic iff x is recursive.

2

(DNWY+Miller + Nies) A real x is low for weakly 2-random iff it is there exists a constant c so that $\forall n(K(x \upharpoonright n) \leq K(n) + c)$.

Forcing vs Genericity and Randomness

Definition

Let (P_n, \leq) be a forcing notation where $P_n = \{A \subseteq 2^\omega \mid A \in \Pi_n^0 \wedge \mu(A) > 0\}$ and $\leq = \subseteq$. $A \Vdash \varphi$ if $\varphi(x)$ is true for all $x \in A$. x is Solovay n -generic if for every Π_n^0 -formula, there is a condition $x \in A$, A decides φ .

Theorem

- 1 (Jockusch) x is n -generic iff x forces all of Σ_n^0 sentences in the Cohen forcing sense.
- 2 (Kurtz) x is weakly n -random iff x is Solovay n -generic.

Van Lambalgen's Theorem

Theorem

- 1 (van Lambalgen) $x \oplus y$ is n -random iff x is n -random and y is n - x -random.
- 2 (Forklore) $x \oplus y$ is n -generic iff x is n -generic and y is n - x -generic.

Relativized Randomness

Theorem

- 1 (Miller and Yu) For any real z and 1-random reals $x \leq_T y$, if y is 1- z -random then x is 1- z -random.
- 2 (CDGHM) M-Y Theorem holds for n -genericity if $n \geq 2$, but fails for 1-genericity.

Higher Up

Definition

Given a class of sets of reals T ,

- 1 A real x is T -random if x is not in any null set in T .
- 2 A real x is T -generic if x is in every dense set in T where T is also a class of open sets.

Theorem

- 1 (Sacks+Hjorth, Nies+Chong, Nies, Yu) Π_1^1 -randomness \subset
 Π_1^1 -Martin-Löf randomness \subset Δ_1^1 -randomness
 $=$ Δ_1^1 -Martin-Löf randomness.
- 2 Π_1^1 -genericity $=$ Δ_1^1 -genericity.

Traceability

Definition

- (i) Let $h : \omega \rightarrow \omega$ be a nondecreasing unbounded function that is hyperarithmetical. A Π_1^1 -trace/ Δ_1^1 -trace with bound h is a uniformly Π_1^1 /uniformly Δ_1^1 sequence $(T_e)_{e \in \omega}$ such that $|T_e| \leq h(e)$ for each e .
- (ii) $A \subseteq \omega$ is Π_1^1 -traceable/ Δ_1^1 -traceable if there is $h \in \Delta_1^1$ such that, for each $f \leq_h A$, there is a Π_1^1 -trace/ Δ_1^1 -trace with bound h such that, for each e , $f(e) \in T_e$.

Proposition (Chong, Nies and Yu)

If x is Π_1^1 -traceable, then x is Δ_1^1 -traceable.

Lowness properties

Theorem

- 1 (Chong, Nies and Yu) *Lowness for Δ_1^1 randomness = Δ_1^1 -traceability.*
- 2 (Hjorth and Nies) *Lowness for Π_1^1 -Martin-Löf randomness = Hyperarithmetic.*
- 3 (Harrington, Nies and Slaman) *Lowness for Π_1^1 -randomness = Lowness for Δ_1^1 randomness + non-random-cuppable.*
- 4 (Yu) *Lowness for Δ_1^1 -genericity $\supseteq \Delta_1^1$ -traceability.*

Beyond Recursion Theory

Theorem

Assume PD if $n \geq 1$.

- (Kechris) There exists a largest Π_{2n+1}^1 and Σ_{2n}^1 null set.
- (Kechris) There exists a largest Π_{2n+1}^1 and Σ_{2n}^1 meager set.
- (Sacks+Tanaka+Kechris) Each non-null Π_{2n+1}^1 set contains a Δ_{2n}^1 real.
- (Hinman+Kechris) Each non-meager Π_{2n+1}^1 set contains a Δ_{2n}^1 real.

Some questions

Question

- 1 *How far can genericity and randomness theory go under PD?*
- 2 *Finding out an inner model to develop higher genericity and randomness theory.*

Thank you