Characterization and recognition of Helly circular-arc clique-perfect graphs

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Abstract

A clique-transversal of a graph $G$ is a subset of vertices that meets all the cliques of $G$. A clique-independent set is a collection of pairwise vertex-disjoint cliques. A graph $G$ is clique-perfect if the sizes of a minimum clique-transversal and a maximum clique-independent set are equal for every induced subgraph of $G$. The list of minimal forbidden induced subgraphs for the class of clique-perfect graphs is not known. Another open question concerning clique-perfect graphs is the complexity of the recognition problem. In this work we characterize clique-perfect graphs by a restricted list of minimal forbidden induced subgraphs when the graph is a Helly circular-arc graph. This characterization leads to a polynomial time recognition algorithm for clique-perfect graphs inside this class of graphs.

Keywords: Clique-perfect graphs, Helly circular-arc graphs, K-perfect graphs, perfect graphs.
1 Introduction

Let $G$ be a simple finite undirected graph, with vertex set $V(G)$ and edge set $E(G)$. Denote by $\overline{G}$, the complement of $G$.

A family of sets $S$ is said to satisfy the Helly property if every subfamily of it, consisting of pairwise intersecting sets, has a common element. A circular-arc graph is the intersection graph of arcs of a circle. A Helly circular-arc (HCA) graph is a circular-arc graph admitting a model whose arcs satisfy the Helly property.

A clique is a complete subgraph maximal under inclusion. A graph is clique-Helly (CH) if its cliques satisfy the Helly property, and it is hereditary clique-Helly (HCH) if $H$ is clique-Helly for every induced subgraph $H$ of $G$.

A graph $G$ is perfect when the chromatic number equals the clique number for every induced subgraph of $G$. It has been proved recently that perfect graphs can be characterized by two families of minimal forbidden induced subgraphs \cite{4} and recognized in polynomial time \cite{3}. The clique graph $K(G)$ of $G$ is the intersection graph of the cliques of $G$. A graph $G$ is $K$-perfect if $K(G)$ is perfect.

A clique-transversal of a graph $G$ is a subset of vertices that meets all the cliques of $G$. A clique-independent set is a collection of pairwise vertex-disjoint cliques. The clique-transversal number and clique-independence number of $G$, denoted by $\tau_c(G)$ and $\alpha_c(G)$, are the sizes of a minimum clique-transversal and a maximum clique-independent set of $G$, respectively. It is easy to see that $\tau_c(G) \geq \alpha_c(G)$ for any graph $G$. A graph $G$ is clique-perfect if $\tau_c(H) = \alpha_c(H)$ for every induced subgraph $H$ of $G$. Clique-perfect graphs have been implicitly studied in a lot of works, but the terminology “clique-perfect” has been introduced in \cite{8}. The list of minimal forbidden induced subgraphs for the class of clique-perfect graphs is not known. Another open question concerning clique-perfect graphs is the complexity of the recognition problem.

There are some partial results in this direction. In \cite{9}, clique-perfect graphs are characterized by minimal forbidden subgraphs for the class of chordal

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graphs. In [10], minimal graphs $G$ with $\alpha_c(G) = 1$ and $\tau_c(G) > 1$ are explicitly described. In [1], clique-perfect graphs are characterized by minimal forbidden subgraphs for the classes of line graphs and hereditary clique-Helly claw-free graphs, and by forbidden subgraphs for the class of diamond-free graphs.

In this work, we give a characterization of clique-perfect graphs for the whole class of Helly circular-arc graphs by minimal forbidden subgraphs.

2 Main results

Let $G$ be a graph and $C$ be a cycle of $G$ not necessarily induced. An edge of $C$ is non proper if it forms a triangle with some vertex of $C$. An $r$-generalized sun, $r \geq 3$, is a graph $G$ whose vertex set can be partitioned into two sets: a cycle $C$ of $r$ vertices, with all its non proper edges $\{e_j\}_{j \in J}$ ($J$ is permitted be an empty set) and a stable set $U = \{u_j\}_{j \in J}$, such that for each $j \in J$, $u_j$ is adjacent only to the endpoints of $e_j$. An $r$-generalized sun is said to be odd if $r$ is odd. Odd generalized suns are not clique-perfect [2], but, unfortunately, they are not necessary minimal (with respect to taking induced subgraphs). However, the odd generalized suns involved in the characterization of HCA clique-perfect graphs by forbidden subgraphs can be described as a union of some families which are minimally clique-imperfect.

A hole is a chordless cycle of length $n \geq 4$, and it is denoted by $C_n$. A hole $C_n$ is said to be odd if $n$ is odd. Clearly odd holes are odd generalized suns.

The graphs $S^1_k$, $S^2_k$, $S^3_k$ and $S^4_k$ in Figure 1, where $k \geq 2$ and the length of the induced path depicted as a dotted line is $2k - 3$, are minimally clique-imperfect. In particular, $S^3_k$ and $S^4_k$ are $2k + 1$-generalized suns.

**Theorem 2.1** Let $G$ be a HCA graph. Then $G$ is clique-perfect if and only if $G$ does not contain any of the graphs of Figure 1 as an induced subgraph.

![Fig. 1. Minimal forbidden subgraphs for clique-perfect graphs inside the class of HCA graphs. Dotted lines replace any induced path of odd length at least 1.](image)

Moreover, we prove that a HCA graph which does not contain any of the graphs of Figure 1 as an induced subgraph is K-perfect. In general, clique-
perfect graphs are not necessarily K-perfect, and conversely. But, if a hereditary graph class is $HCH$ and K-perfect, then it is clique-perfect. We use that in the proof of Theorem 2.1, and handle separately the case of $HCA \setminus HCH$.

Helly circular-arc graphs can be recognized in polynomial time [7] and, given a Helly model of a $HCA$ graph $G$, both parameters $\tau_c(G)$ and $\alpha_c(G)$ can be computed in linear time [5,6]. However, clearly it is not straightforward from these properties the existence of a polynomial time recognition algorithm for clique-perfect $HCA$ graphs. The characterization in Theorem 2.1 leads to such an algorithm, which is strongly based on the recognition of perfect graphs [3].

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References