

# Sobre el Problema de Dar Ejemplos de Números Normales

Verónica Becher

Departamento de Computación, Facultad de Ciencias Exactas y Naturales  
Universidad de Buenos Aires & CONICET

IMAL, Abril de 2012

# Normal numbers

*Normality*, defined by Émile Borel in 1909, is a form of randomness:

## Definition (Borel 1909)

A real number is random to a given integer base if every block of digits of the same length occurs with the same limit frequency in the expansion of the number in that base.

For example, if a number is normal to base two, each of the digits '0' and '1' occur in the limit, half of the times; each of the blocks '00', '01', '10' and '11' occur one fourth of the times, and so on.

A real number that is normal to every integer base is called *absolutely normal*, or just *normal*.

## Counterexamples

Rationals are not absolutely normal: for every  $q \in \mathbb{Q}$  there is a base  $t$  such that the expansion of  $q$  ends with all zeros.

$0,1010010001000010000010000\dots$  not normal to base 2.

$0,1010101010101010101010101\dots$  not normal to base 2.

# A Fundamental Theorem of Borel

## Theorem (Borel 1909)

*Almost all real numbers are normal.*

That is, the set of normal numbers has Lebesgue measure 1.

Borel asked for an explicit example.

## Definition of Normality

Let  $b$  be an integer greater than 1, the *base*.

The digits in base  $b$  are  $0, \dots, b - 1$ .

A *block* in base  $b$  is a finite sequence of digits in base  $t$ .

### Definition (Borel 1909, 1922)

A real number  $r$  is *normal to base  $b$*  if for every block  $w$ ,

$$\lim_{k \rightarrow \infty} \frac{\text{occ}(r, b, w, k)}{k} = \frac{1}{b^{|w|}}$$

where  $\text{occ}(r, b, w, k)$  is the number of occurrences of the block  $w$  in the first  $k$  digits of the expansion of  $r$  in base  $b$ , allowing overlapping.

A real  $r$  is *normal* if it is normal to every base  $b \geq 2$ .

# Equivalent definitions of normality

## Definition (Borel 1909, the original)

A real  $r$  is *simply normal* to a given integer base  $b$  if each digit has the same asymptotic frequency  $1/b$  in the expansion of  $r$  in base  $b$ .

A real  $r$  is *normal to base  $b$*  if each  $r, br, b^2r..$  are simply normal to the bases  $b^n$ , for every  $n \geq 1$ .

A real  $r$  is *normal* if it is normal to every base  $b \geq 2$ .

Seemingly simpler:

## Definition (ver Kuipers, Niederreiter 2006)

A real  $r$  is normal to base  $b$  if each  $r, br, b^2r..$  are simply normal to the bases  $b^n$ , for every  $n \geq 1$ .

## Easier to conjecture than to prove

Are the fundamental constants normal to some base?

$$\pi, e? \quad \sqrt{2}?$$

### Borel's Conjecture (1950)

*Each real irrational algebraic number should be normal to every base.*

There are several examples constructed to be normal to one base.

Champernowne (1935): 0, 12345678910111213141516..

None of these have been proved normal to two independent bases.

No number has been proved normal to one base but not to another.

# Characterizations of Normality

Theorem (ver Kuipers and Niederreiter 2006)

*A real  $r$  is normal to a base  $b$  if, and only if, the sequence  $(xb^n)_{n \geq 1}$  is uniformly distributed modulo one.*

Theorem (Schnorr, Stimm 1971 + Dai et al 2004; Becher, Heiber 2012)

*A real is normal to base  $b$  if, and only if, its expansion in base  $b$  is incompressible by lossless finite-state compressors.*

Lossless finite-state compressors (David Huffman, 1959) are ordinary finite automata augmented with an output transition function such that the automata input-output behaviour is injective.



# Rephrasing Borel's conjecture

## Borel's Conjecture

*Each real irrational algebraic number should be normal to every integer base.*

## Borel's Conjecture

*Each real number whose expansion in some base is compressible by lossless finite-state compressors is either rational or transcendental.*

Proved to be true for *automatic numbers*.

Their compressibility follows from Bourke, Hitchcock and Vinodchandran 2005.

Automatic numbers are either rational or transcendental,  
Adamczewski and Bugeaud 2007

## The first two examples of numbers normal numbers

Henri Lebesgue and Waclaw Sierpiński, same journal issue, 1917.

Both gave a *partially constructive* proof of the measure of the set of normal numbers, and defined their respective examples as the limit of a set that includes all non-normal numbers—this limit point is outside the set—.

Their examples were not finitarily defined.

# Can a computer output an absolutely normal number?

Computability is the acceptable constructiveness since the 1930s.

The *computable real numbers* are those whose expansion in some integer base can be generated by a mechanical (finitary) method:

## Definition (Turing 1936)

A real number is computable if there is a machine that outputs each of the digits of its expansion in some base, one after the other.

## Are there computable normal numbers?

A measure-theoretic argument is not enough to see that the set of normal numbers and the set of computable numbers intersect.

The set of normal numbers in the unit interval has measure one.

But the computable numbers are just countable, hence they form a measure zero set.

## Indeed there are computable normal numbers!

Alan Turing's manuscript, presumably 1938 gave an algorithm that produces normal numbers.

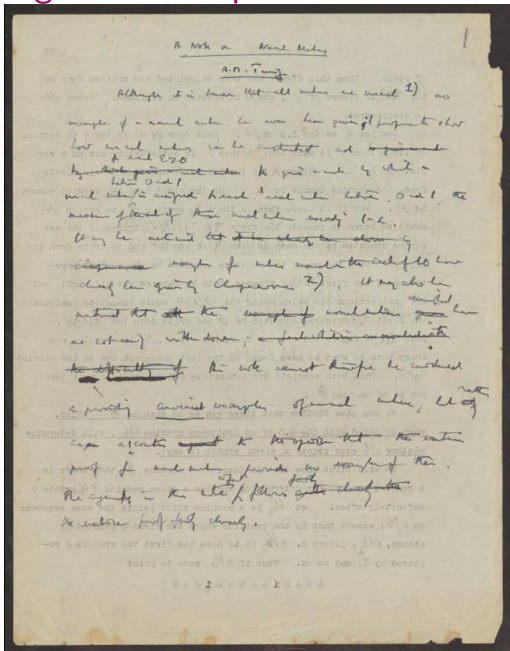
Only published in 1992 in his Collected Works, edited by Britton. The editorial notes say that the proof given by Turing is inadequate and speculate that the theorem could be false.

We reconstructed, corrected and completed Turing's manuscript (Becher, Figueira, Picchi, 2007; Becher 2012).

A computable reformulation of Sierpinski's construction (Becher, Figueira 2002).

These are the best know answer to date to Borel's question.

# Turing's Manuscript on Normal numbers 1/6



# Turing's Manuscript on Normal numbers 2/6

if you assume  $\epsilon$  is small in the scale of  $t$  be 2

4 for given  $\epsilon$  select  $t$  such that  $\epsilon < \frac{1}{2} \log t$

Definition  $N(\epsilon, t, \rho, A)$  is the number of  $n$  such that  $n \leq t$  and  $|S_n(\epsilon, t, \rho, A) - \rho| < \epsilon$

4  $\epsilon$  is small when  $\epsilon$  and  $t$  are of the same order of magnitude

$S(\epsilon, t, \rho, A)$  is the number of  $n$  such that  $n \leq t$  and  $|S_n(\epsilon, t, \rho, A) - \rho| < \epsilon$

$\rho$  is the normal number of  $\rho$  in the scale of  $t$

$\rho$  is a suitable normal number of  $t$  if

$R^t S(\epsilon, t, \rho, A) \rightarrow \rho$  as  $t \rightarrow \infty$  (1)

It is said that  $\rho$  is a normal number if  $R^t S(\epsilon, t, \rho, A) \rightarrow \rho$  as  $t \rightarrow \infty$

ad also  $\rho$  is a normal number if  $R^t S(\epsilon, t, \rho, A) \rightarrow \rho$  as  $t \rightarrow \infty$

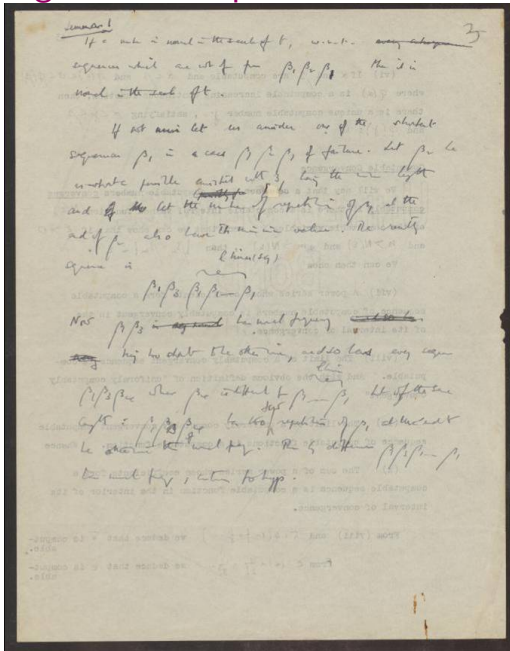
$N(\epsilon, t, \rho, A)$  is the number of  $n$  such that  $n \leq t$  and  $|S_n(\epsilon, t, \rho, A) - \rho| < \epsilon$

scale of  $t$  is that of  $\rho$  seen exactly a time

instead of  $\rho$  is a normal number of  $\rho$  in the scale of  $t$

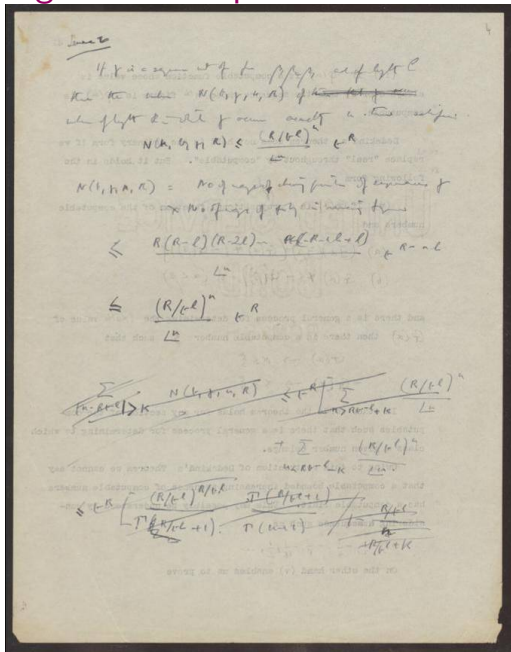
and  $\rho$  is a normal number of  $\rho$  in the scale of  $t$

# Turing's Manuscript on Normal numbers 3/6

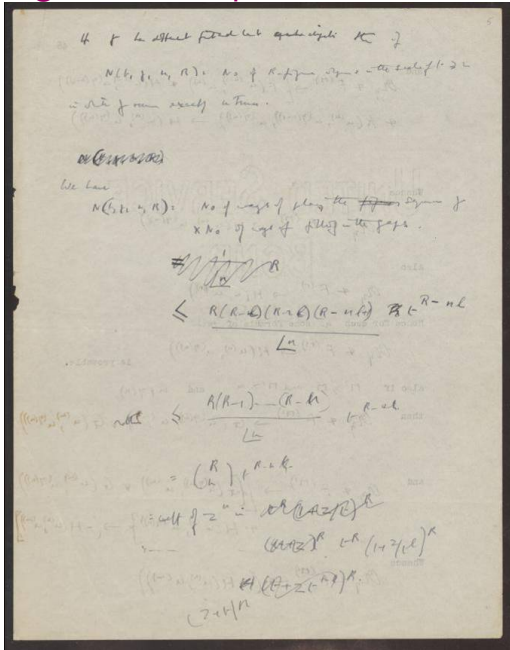




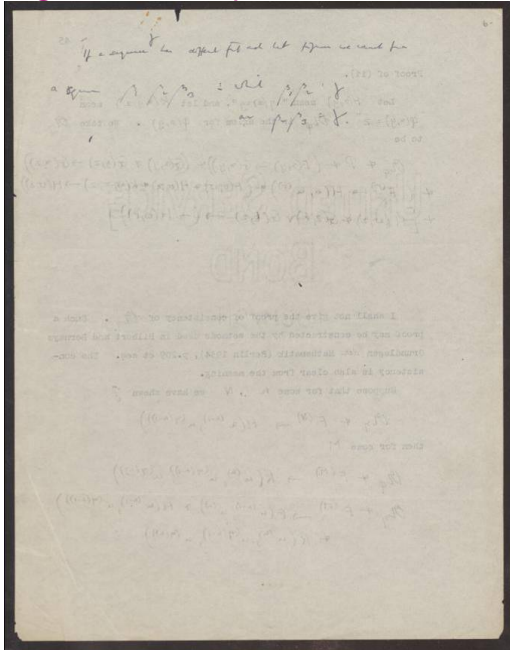
# Turing's Manuscript on Normal numbers 4/6



# Turing's Manuscript on Normal numbers 5/6



# Turing's Manuscript on Normal numbers 6/6



# Turing's Transcript 1/5

## A Note on Normal Numbers

Although it is known that almost all numbers are normal <sup>1)</sup> no example of a normal number has ever been given. I propose to shew how normal numbers may be constructed and to prove that almost all numbers are normal constructively

Consider the  $\mathbb{R}$ -figure integers in the scale of  $t$  ( $t \geq 2$ ). If  $\gamma$  is any sequence of figures in that scale we denote by  $N(t, \gamma, n, \mathbb{R})$  the number of these in which  $\gamma$  occurs exactly  $n$  times. Then it can be proved without difficulty that

$$\frac{\sum_{n=0}^{\mathbb{R}} n N(t, \gamma, n, \mathbb{R})}{\sum_{n=0}^{\mathbb{R}} N(t, \gamma, n, \mathbb{R})} = \frac{\mathbb{R} - r + 1}{\mathbb{R}} t^{-r}$$

where  $l(\gamma) = r$  is the length of the sequence  $\gamma$ ; it is also possible <sup>2)</sup> to prove that

$$\sum_{|n - \mathbb{R}t^{-r}| > k} N(t, \gamma, n, \mathbb{R}) < 2t^{\mathbb{R}} e^{-k^2 \frac{t^r}{4\mathbb{R}}} \quad \text{provided} \quad \frac{k t^r}{\mathbb{R}} < .3 \quad (1)$$

Let  $\alpha$  be a real number and  $S(\alpha, t, \gamma, \mathbb{R})$  the number of occurrences of  $\gamma$  in the first  $\mathbb{R}$  figures after the decimal point in the expression of  $\alpha$  in the scale of  $t$ .  $\alpha$  is said to be normal if

$$\mathbb{R}^{-2} S(\alpha, t, \gamma, \mathbb{R}) \rightarrow t^{-r} \quad \text{as } \mathbb{R} \rightarrow \infty \text{ for each } \gamma, t$$

where  $r = l(\gamma)$ .

Now consider sums of a finite number of open intervals with rational end points. These can be enumerated constructively. We take a particular constructive enumeration: let  $\bar{K}_n$  be the  $n$ th

# Turing's Transcript 2/5

set of intervals in the enumeration. Then we have

Theorem 1

We can find a constructive<sup>3)</sup> function  $c(K, n)$  of two integral variables, such that

$$\text{and } K_{c(K, n+1)} \subseteq K_{c(K, n)}$$

$$\text{and } \mu K_{c(K, n)} > 1 - \frac{1}{K} \quad \text{for each } K, n$$

and  $K_{(K)} = \prod_{k=1}^{\infty} K_{c(K, k)}$  consists entirely of normal numbers for each  $K$ .

$$0 < \alpha < 1$$

Let  $B(\Delta, \gamma, t, R)$  be the set of numbers  $\alpha, \lambda$  for which

$$\left| S(\alpha, t, \gamma, R) - R t^{-\alpha} \right| < \frac{R}{\Delta t^{\gamma}} \quad \left( K = \frac{R}{\Delta t^{\gamma}} \right) \quad (2)$$

then by (1)

$$\mu B(\Delta, \gamma, t, R) > 1 - 2e^{-R \frac{t^{\gamma}}{4 \Delta^2}} \quad \text{if } \frac{R t^{\gamma}}{4 \Delta^2} > 3$$

Let  $H(\Delta, T, L, R)$  be the set of those  $\alpha$  for which (2) holds whenever  $2 \leq t \leq T$  and  $\ell(t) \leq L$  i.e.

$$H(\Delta, T, L, R) = \prod_{t=2}^T \prod_{\ell(t) \leq L} B(\Delta, \gamma, t, R)$$

The number of factors in the product is at most  $T^{L+1}$  so that

$$\mu H(\Delta, T, L, R) > 1 - T^{L+1} e^{-\frac{R T^{\gamma}}{4 \Delta^2}}$$

Let

$$A_k = A \left( \left[ k^{1/a} \right], \left[ e^{\sqrt{\log k}} \right], \left[ \sqrt{\log k} - 1 \right], k \right)$$

$$A_k = A \left( k, \left[ e^{\sqrt{\log k}} \right], \left[ \sqrt{\log k} - 1 \right], k^k \right)$$

# Turing's Transcript 3/5

then if  $k > 1000$  we shall have  $\mu \beta_k > 1 - k e^{-\frac{1}{2} k^{1/2}} > 1 - \frac{1}{k(k-1)}$ .  
 $C(H, \mu)$  ( $H > 1000$ ) is to be defined as follows

$C(H, 0)$  is  $(0, 1)$

$C(H, \mu+1)$  is the intersection of an interval  $(\beta_k, 1)$ , ( $0 \leq \beta_k < 1$ ) with  $\beta_{H+H+1}$  and  $C(H, \mu)$ ,  $\beta_k$  being so chosen that the measure of  $C(H, \mu+1)$  is  $1 - \frac{1}{k} + \frac{1}{k+H+1}$ . This is possible since the measure of  $C(H, \mu)$  is  $1 - \frac{1}{k} + \frac{1}{k+H}$  and that of  $\beta_{H+H+1}$  is at least  $1 - \frac{1}{(H+1)(H+2)}$ ; consequently the measure of  $C(H, \mu) \cap \beta_{H+H+1}$  is at least  $1 - \frac{1}{k} + \frac{1}{k+H+1}$ .

If  $H < 1000$  we define  $C(H, \mu)$  to be  $C(1000, \mu) \cdot C(H, \mu)$  is a finite sum of intervals for each  $H, \mu$ . When we remove the boundary points we obtain a set of form  $\bar{K}_{C(H, \mu)}$  of measure  $1 - \frac{1}{k} + \frac{1}{k+H}$  ( $H > 1000$ ). The intervals of which  $\bar{K}_{C(H, \mu)}$  is composed may be found by a mechanical process and so the function  $C(H, \mu)$  is

constructive. The set  $\bar{K}_{(H)}$  =  $\prod_{\mu=1}^{\infty} \bar{K}_{C(H, \mu)}$  consists of normal numbers, for if  $\alpha \in \bar{K}_{(H)}$  then  $\alpha \in \bar{K}_{k, \mu}$   $\forall k > 1000, \mu > 1000$ . If  $\gamma$  is a sequence of length  $r$  in the scale of  $t$  and if  $\beta_k$  be such that  $[\epsilon \sqrt{\log k}] > t$  and  $[\sqrt{\log k}] > r+1$  then for  $k > k_0$

$|S(\beta, t, \gamma, k) - k t^{-r}| < k [k^{-1/2}]^{-1}$  when  $\beta$  is in  $\beta_k$   
 (by the definition of  $\beta_k$ ). Hence  $k^{-1} S(\alpha, t, \gamma, k) \rightarrow t^{-r}$   
 as  $k$  tends to infinity, i.e.  $\alpha$  is normal.

## Theorem 2

There is a rule whereby given an integer  $H$  and an infinite sequence of figures 0 and 1 (the  $p$ th figure in the sequence being  $\alpha^p(\gamma)$ ) we can find a normal number  $\alpha(H, \beta)$  in the interval  $(0, 1)$  and in such a way that for fixed  $H$  these numbers form a set of measure at least  $1 - \frac{2}{H}$ , and so that the first  $n$  figures of  $\alpha^S$  determine  $\alpha(H, \beta)$  to within  $2^{-n}$ .

# Turing's Transcript 4/5

With each integer  $n$  we associate an interval of the form

$\left(\frac{m_n}{2^n}, \frac{m_n+1}{2^n}\right)$  whose intersection with  $\widehat{N}$  is of positive measure. and given  $m_n$  we obtain  $m_{n+1}$  as follows. Put

$$m_n \quad \widehat{N} \cap (H, n) \cap \left(\frac{m_n}{2^n}, \frac{2m_n+1}{2^{n+1}}\right) = a_{n, m_n}$$

$$m_n \quad E \cap (K, n) \cap \left(\frac{2m_n+1}{2^{n+1}}, \frac{m_n+1}{2^n}\right) = b_{n, m_n}$$

and let  $r_n$  be the smallest  $m$  for which either  $a_{n, m} < K^{-1} 2^{-2n}$  or  $b_{n, m} < K^{-1} 2^{-2n}$  or both  $a_{n, m} > \frac{1}{K(K+n+2)}$  and  $b_{n, m} > \frac{1}{K(K+n+2)}$

There exists such an  $r_n$  for  $a_{n, m}$  and  $b_{n, m}$  decrease either to 0

or to some positive number. In the case where  $a_{n, r_n} < K^{-1} 2^{-2n}$  we

put  $m_{n+1} = 2m_n + 1$  : if  $a_{n, r_n} > K^{-1} 2^{-2n}$  but  $b_{n, r_n} < K^{-1} 2^{-2n}$

we put  $m_{n+1} = 2m_n$ , and in the third case we put  $m_{n+1} = 2m_n$

or  $m_{n+1} = 2m_n + 1$  according as  $\gamma(n) = 0$  or 1. For each  $n$  the

interval  $\left(\frac{m_n}{2^n}, \frac{m_n+1}{2^{n+1}}\right)$  includes normal numbers in positive measure.

The intersection of these intervals contains only one number

which must be normal.

Now consider the set  $A(K, n)$  consisting of all possible intervals

$\left(\frac{m_n}{2^n}, \frac{m_n+1}{2^{n+1}}\right)$  i.e. the sum of all these intervals as we allow the first  $n$  figures of  $\gamma^g$  to run through all possibilities. Then

$$m_n \quad K(K) \cap A(K, n+1) = m_n \quad E(K) \cap A(K, n)$$

$$= \sum_{m=0}^{2^{n+1}} m_n \quad K(K) \cap (A(K, n) - A(K, n+1)) \cap \left(\frac{m}{2^n}, \frac{m+1}{2^n}\right)$$

$$\text{But} \quad m_n (A(K, n) - A(K, n+1)) \cap \left(\frac{m}{2^n}, \frac{m+1}{2^n}\right) < 2^{-2n} K^{-1}$$

# Turing's Transcript 5/5

so that

$$m_{(K)}^{-1} A(K, n+2) > m_{(K)}^{-1} A(K, n) - 2^{-n-2} K^{-2} > m_{(K)}^{-1} - K^{-1} > 1 - 2/K$$

The set of all possible numbers  $x(K, n)$  is therefore of measure at least  $1 - 2/K$ .

By taking particular sequences  $x^c$  (e.g.  $x^c(n) = 0$  for all  $n$ ) we obtain particular normal numbers.





# Turing's Transcript

## A Note on Normal Numbers

Although it is known that almost all numbers are normal <sup>1)</sup> no example of a normal number has ever been given. I propose to shew how normal numbers may be constructed and to prove that almost all numbers are normal constructively

Consider the  $R$ -figure integers in the scale of  $t$  ( $t \geq 2$ ). If  $\gamma$  is any sequence of figures in that scale we denote by  $N(t, \gamma, n, R)$  the number of these in which  $\gamma$  occurs exactly  $n$  times. Then it can be proved without difficulty that

$$\frac{\sum_{n=1}^R n N(t, \gamma, n, R)}{\sum_{n=1}^R N(t, \gamma, n, R)} = \frac{R-r+1}{R} t^{-r}$$

where  $l(\gamma) = r$  is the length of the sequence  $\gamma$ : it is also possible <sup>2)</sup> to prove that

$$\sum_{|n - Rt^{-r}| > k} N(t, \gamma, n, R) < 2t^R e^{-k^2 t^r / 4R} \quad \text{provided } \frac{kt^r}{R} < .3 \quad (1)$$

Let  $\alpha$  be a real number and  $S(\alpha, t, \gamma, R)$  the number of occurrences

# Turing's manuscript

## A Note on Normal Numbers

Although it is known that almost all numbers are normal <sup>1)</sup> no example of a normal number has ever been given . I propose to shew how normal numbers may be constructed and to prove that almost all numbers are normal constructively

# Turing's Theorem 1

## Theorem 1

We can find a constructive<sup>3)</sup> function  $c(K, n)$  of two integral variables, such that

$$K_{c(K, n+1)} \subseteq K_{c(K, n)}$$

and  $n \cdot K_{c(K, n)} > 1 - \frac{1}{K}$  for each  $K, n$

and  $K_{(K)} = \bigcap_{n=1}^{\infty} K_{c(K, n)}$  consists entirely of normal numbers for each  $K$ .

# Turing's Theorem 1

It is a computable version of Borel's fundamental theorem that establishes that almost all real numbers are normal.

## Turing's Theorem 1

*There is a computable construction in terms of intervals with rational endpoints in  $[0, 1]$  to show that almost all real numbers are normal.*

## Turing's Theorem 1

*There is a computable function  $c(k, n)$  of two integer variables with values consisting of finite sets of pairs of rational numbers such that, for each  $k$  and  $n$ ,*

*if  $E_{c(k,n)} = (a_1, b_1) \cup (a_2, b_2) \cup \dots (a_m, b_m)$  denotes the finite union of the intervals whose rational endpoints are the pairs given by  $c(k, n)$ , then*

*$E_{c(k,n)}$  is included in  $E_{c(k,n-1)}$  and the measure of  $E_{c(k,n)}$  is greater than  $1 - 1/k$ .*

*And for each  $k$ ,*

*$E(k) = \bigcap_n E_{c(k,n)}$  has measure  $1 - 1/k$  and consists entirely of normal numbers.*

## Turing's Theorem 2

### Theorem 2

There is a rule whereby given an integer  $K$  and an infinite sequence of figures 0 and 1 ( the  $p$  th figure in the sequence being  $v^p$  ) we can find a normal number  $\alpha(K, v)$  in the interval  $(0,1)$  and in such a way that for fixed  $K$  these numbers form a set of measure at least  $1 - 2/K$ , and so that the first  $n$  figures of  $v$  determine  $\alpha(K, v)$  to within  $2^{-n}$ .

## Turing's Theorem 2

*There is an algorithm to produce normal numbers.*

The proof uses Theorem 1.

## Turing's Theorem 2

*There is an algorithm that, given an integer  $k$  and an infinite sequence  $\nu$  of zeros and ones, produces a normal number  $\alpha(k, \nu)$  in the unit interval, expressed in base two, such that in order to write down the first  $n$  digits of  $\alpha(k, \nu)$  the algorithm requires at most the first  $n$  digits of  $\nu$ .*

*For a fixed  $k$  these numbers  $\alpha(k, \nu)$  form a set of measure at least  $1 - 2/k$ .*

## Turing's Theorem 2

Turing gives an algorithm to output the expansion of a normal number in base two.

The algorithm is a computable functional: it receives an integer value that acts as a parameter to control measure, and an infinite sequence  $\nu$  in base two to be used as an oracle to possibly determine some digits of the output sequence.

When  $\nu$  is a computable sequence (Turing puts all zeros), the algorithm yields a computable normal number.

With this result Turing is the first one to prove the existence of computable normal numbers.

The algorithm can be adapted to intercalate the bits of the input sequence  $\nu$  at fixed positions of the output sequence. Thus, one obtains non-computable normal numbers in each Turing degree.



## Computational complexity of Turing's algorithm

A literal reading of Turing's algorithm yields that at most *simple-exponentially* many operations suffice.

Our reconstruction worsens this amount to *double-exponentially* many, due to a modification we had to introduce in one expression that Turing wrote without a proof.

A theorem of Strauss (2003) asserts that normal numbers computable in simple exponential time do exist, but this existential result yields no specific instances.

## Turing says his normal numbers are not convenient

In his manuscript, Turing alerts the reader that the provided examples of normal numbers are not convenient and he explicitly says that one would like that the expansion of such numbers be actually exhibited.

From his wording we suppose that he was aware of the problem that the  $n$ -th digit in the expansion of a number output by his algorithm is defined by exponentially many operations in  $n$ .

# Turing's Manuscript on Normal numbers

A Note on Normal Numbers

A. N. Turing

Although it is known that all numbers are normal <sup>1)</sup>, no

example of a normal number has been given. It is proposed to show  
how normal numbers can be constructed and approximated  
by ~~the~~  $\epsilon$ -normal numbers.  $\epsilon$ -normal numbers are those for which  
the number of occurrences of any given string of length  $k$  is  
approximately  $\frac{1}{26^k}$  of the total number of strings of length  $k$ .

It may be noticed that the above definition is only a  
rough approximation of the true normality property. It is  
clear that a normal number must satisfy the above conditions  
for all strings of length  $k$ .

It may be noticed that the above definition is only a

rough approximation of the true normality property. It is  
clear that a normal number must satisfy the above conditions  
for all strings of length  $k$ .

It may be noticed that the above definition is only a  
rough approximation of the true normality property. It is  
clear that a normal number must satisfy the above conditions  
for all strings of length  $k$ .

It may be noticed that the above definition is only a  
rough approximation of the true normality property. It is  
clear that a normal number must satisfy the above conditions  
for all strings of length  $k$ .

It may be noticed that the above definition is only a  
rough approximation of the true normality property. It is  
clear that a normal number must satisfy the above conditions  
for all strings of length  $k$ .

## Over 100 years

- 1909 Borel asks for examples of normal numbers.
- 1917 Independent non-finitarily based constructions by Lebesgue Sierpiński, hence a priori non computable.
- 1937 Unpublished Turing's algorithm for absolutely normal numbers, answers the then outstanding question.
- 1962 Construction by Schmidt. It is unknown whether it is computable.
- 1975 Random reals ( Martin-Löf/Chaitin) are normal but non computable.
- 1979 Construction by M.B.Levin. It is unknown whether is computable.
- 1997 Martin Strauss gives proves the existences of normal numbers computable in exponential time. No instances are known yet.
- 2002 We gave a computable reformulation of Sierpinski's construction. Double exponential time complexity.
- 2007 We reconstructed Turing's algorithm. Double exponential time complexity.

## Borel's question is still unresolved

The few known instances are not completely satisfactory.

It is desirable to show that a *known* irrational number is normal, or, at least, to exhibit the number explicitly.

We would like an example with a simple mathematical definition and such that, in addition of normality, extra properties are proved.

We would also like that the number be easily computable.