Normal numbers with digit dependencies

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Expansion of a real number in an integer base

For a real number x, its fractional expansion in an integer base $b \geq 2$ is a sequence of integers $a_1, a_2 \ldots$, where $0 \leq a_j < b$ for every j, such that

$$x = \lfloor x \rfloor + \sum_{j=1}^{\infty} a_j b^{-j} = \lfloor x \rfloor + 0.a_1 a_2 a_3 \ldots$$

We require that $a_j < b - 1$ infinitely often to ensure that every number has a unique representation.

Borel normal numbers

A real number x is simply normal to base b if every digit in $\{0, \ldots, b-1\}$ occurs in the b-ary expansion of x with the same asymptotic frequency (that is, with frequency $1/b$).

A real number x is normal to base b if it is simply normal to all the bases $b, b^2, b^3, \ldots$.

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Émile Borel proved that almost all numbers with respect to Lebesgue measure are normal to all integer bases.

The question

How much digit dependence can be allowed so that, still, almost all real numbers are normal?

First result

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We prove that, for any fixed integer base b, almost all real numbers whose base b-expansion is such that for every sufficiently large n at least slightly more than $\log\log n$ consecutive digits with indices starting at position n are independent, are normal to base b.

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Independence of just $\log\log n$ consecutive digits is not sufficient.

First result

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. Let integer $b \geq 2$. Let $X_1, X_2, \ldots$ be a sequence of random variables (that is, measurable functions) from Ω into $\{0, \ldots, b-1\}$.

Theorem 1

Assume that for every n the random variable X_n is uniformly distributed on $\{0, \ldots, b-1\}$. Assume that there exists a function $g : \mathbb{N} \mapsto \mathbb{R}$ unbounded and monotonically increasing such that for all sufficiently large n the random variables

$$X_n, \; X_{n+1}, \; \ldots, \; X_{n+\lceil g(n) \log \log n \rceil}$$

are mutually independent. Let x be the real number whose base-b expansion is given by $x = 0.X_1 X_2 \ldots$. Then $\mathbb{P}$-almost surely x is normal to base b.

First result

Theorem 1, continued

On the other hand, for every base b and every positive constant K there is an example where for every $n \geq 1$ the random variable X_n is uniformly distributed on $\{0, \ldots, b-1\}$ and where for all sufficiently large n the random variables

$$X_n,\ X_{n+1},\ \ldots,\ X_{n+\lceil K \log\log n \rceil}$$

are mutually independent but $\mathbb{P}$-almost surely the number $x = 0.X_1 X_2 \ldots$ fails to be simply normal.

Toeplitz sequences (Jacobs and Keane 1969)

Fix an integer $b \geq 2$, let $A = \{0, \ldots, b-1\}$ and let A^ω be the set of all infinite sequences of symbols from A.

For $P = \{2\}$, T_P is the set of all sequences $t_1 t_2 \ldots$ such that, for every n,

$$t_n = t_{2n}$$

Thus,

$$\begin{aligned} t_1 &= t_2 &&= t_{2^2} &&= \ldots \\ t_3 &= t_{2\ 3} &&= t_{2^2\ 3} &&= \ldots \\ t_5 &= t_{2\ 5} &&= t_{2^2\ 5} &&= \ldots \end{aligned}$$

...

$$t_j = t_{2\ j} = t_{2^2\ j} = \ldots$$

for every j that is not a multiple of 2.

Toeplitz sequences

For a positive integer r and a set $P = \{p_1, \ldots, p_r\}$ of r prime numbers, let T_P be the set of all Toeplitz sequences, that is, the set of all sequences $t_1 t_2 t_3 \cdots$ in A^ω such that for every $n \geq 1$ and for every $i = 1, \ldots, r$,

$$t_n = t_{np_i}.$$

Toeplitz transform τ_P

Let $P = \{p_1, \dots, p_r\}$ a set of r primes. Let $j_1, j_2, j_3, \dots$ be the enumeration in increasing order of all positive integers that are not divisible by any of the primes $p_1, \dots, p_r$.

The Toeplitz transform $\tau_P : A^\omega \to T_P$ is defined as

$$\tau_P(a_1 a_2 a_3 \dots) = t_1 t_2 t_3 \dots$$

where $t_n = a_k$ when n has the decomposition $n = j_k p_1^{e_1} \cdots p_r^{e_r}$.

Since elements of A^ω can be identified with real numbers in $[0, 1]$ via their expansion, the transform τ_P induces a transform $[0, 1] \mapsto T_P$, which we denote by τ_P as well.

Uniform probability measure μ on T_P

Let λ be the uniform probability measure λ on A^ω (the infinite product measure generated by the uniform measure on $\{0, \ldots, b-1\}$).

We endow T_P with a probability measure μ, which is the forward-push by τ_P of the uniorm measure λ.

For any measurable set $X \subseteq T_P$, $\mu(X) = \lambda(\tau_P^{-1}(X))$.

By identifying infinite sequences with real numbers, the measure μ on A^ω also induces a measure on $[0, 1]$, which we denote by μ as well.

Independence

The Toeplitz transform τ_P also induces a function $\delta : \mathbb{N} \mapsto \mathbb{N}$ where

$$t_1 t_2 t_3 \cdots = \tau_P(a_1 a_2 a_3 \cdots) = a_{\delta(1)} a_{\delta(2)} a_{\delta(3)} \cdots .$$

The n-th symbol $t_n(x)$ of $\tau_P(x)$, $t_n(x)$, is a random variable on the space $([0,1], \mathcal{B}(0,1), \lambda)$. That is, it is a measurable function $[0,1] \mapsto \{0, \ldots, b-1\}$.

Since $t_n(x) = a_{\delta(n)}$ for all n, it is easy to see that two random variables t_m and t_n are independent (with respect to both measures λ and μ) if and only if $\delta(m) \neq \delta(n)$, that is, if they do not originate in the same digit of x by means of the Toeplitz transform.

Second result

We show that almost all Toeplitz sequences are normal to a given base.

Theorem 2

Let $b \geq 2$ be an integer, and let P be a finite set of primes. Let μ be the uniform probability measure on the set T_P. Then, μ-almost all elements of T_P are the expansion in base b of a normal number.

About the proof of Theorem 2

We prove Theorem 2 by showing that it is a consequence of Theorem 1, together with number-theoretic results of Tijdeman 1973.

Theorem (Becher, Carton and Heiber 2016)

We construct a normal sequence in T_P for $P = \{2\}$.

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Problem

Construct a normal sequence in T_P for $P = \{2, 3\}$.

Third main result

Fix an integer base b. Almost all real numbers whose base-b expansion, $t_1t_2\ldots$, is such that for every n, t_n is equal to t_{2n} are normal to every integer base.

Theorem 3

Let $b \geq 2$ be an integer, let $P = \{2\}$ and let μ be the uniform probability measure on T_P. Then, μ-almost all elements of T_P are the expansion in base b of an absolutely normal number.

About the proof of Theorem 3

To prove Theorem 3 we adapt the work of Cassels 1959 and Schmidt 1961.
Our argument is also based on giving upper bounds for certain Riesz products.

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To prove Theorem 3 we adapt the work of Cassels 1959 and Schmidt 1961. Our argument is also based on giving upper bounds for certain Riesz products.

Cassels worked on a Cantor-type set of real numbers whose ternary expansion avoids the digit 2 (and which therefore cannot be normal to base 3), and he established certain regularity properties of the uniform measure supported on this fractal set. In contrast, we deal with the measure μ which is the uniform measure on the set of real numbers which respect the digit dependencies.

A moral

Imposing digit dependencies does not destroy the fact that almost all numbers are normal.

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