Normality together with other properties

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Normal numbers

A *base* is an integer b greater than or equal to 2.

Definition (Borel, 1909)

A real x is simply normal to base b if in the expansion of x in base b, each digit $0, \dots b-1$ occurs with limiting frequency equal to $1/b$.

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A real x is absolutely normal if x is normal to every base.

Theorem (Wall 1949)

A real x is normal to base b if and only if $(b^k x)_{k \geq 0}$ equidistributes modulo one for Lebesgue measure.

Examples and counterexamples

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  It is unknown if it simply normal to bases that are not powers of 10.

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- $0.123456789101112131415\ldots$ is normal to base 10 (Champernowne, 1933).
  It is unknown if it simply normal to bases that are not powers of 10.
- Stoneham number $\alpha_{2,3} = \sum_{k \geq 1} \frac{1}{3^k \ 2^{3^k}}$ is normal to base 2 but not simply normal to base 6 (Bailey, Borwein, 2012).

Absolutely normal numbers

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He asked for one example.

Absolutely normal, non-computable constructions

Bulletin de la Société Mathématique de France (1917) 45:127–132; 132–144

DÉMONSTRATION ÉLÉMENTAIRE DU THÉORÈME DE M. BOREL SUR LES NOMBRES ABSOLUMENT NORMAUX ET DÉTERMINATION EFFECTIVE D'UN TEL NOMBRE;

PAR M. W. SIERPINSKI.

On appelle, d'après M. Borel, *simplement normal* par rapport à la base q (1) tout nombre réel x dont la partie fractionnaire

(1) E. BOREL, *Leçons sur la théorie des fonctions*, p. 197, Paris, 1914.

SUR CERTAINES DÉMONSTRATIONS D'EXISTENCE;

PAR M. H. LEBESGUE.

Dans une lettre, adressée à M. Borel, et qui accompagnait l'envoi de l'article précédent, M. Sierpinski se demandait si cet article devait être publié, s'il ne ferait pas double emploi avec une démonstration que j'avais indiquée à M. Borel et que celui-ci a signalée dans la deuxième édition de ses *Leçons sur la théorie des fonctions* (p. 198).

Absolutely normal numbers

Conjecture (Borel 1950)

Irrational algebraic numbers are absolutely normal.

Computable absolutely normal

Definition (Turing 1936)

A real number x is computable is there is a program that produces the expansion of x in some base.

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Theorem (Turing 1937?)

There is a computable absolutely normal number.

To produce the n-th binary digit, Turing's algorithm performs a number of operations that is exponential in n. This is called exponential time complexity.

Corrected and completed in Becher, Figueira and Picchi, 2007.

General construction of a computable real number

Consider a computable sequence of intervals $I_1, I_2, I_3 \ldots$ with rational endpoints (left endpoint increasing, right endpoint decreasing), nested, lengths go to 0.

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Turing uses dyadic intervals. To select $I_1, I_2, I_3 \ldots$ his strategy is to "follow the measure". The computed number is the trace of left/right choices.

Computable Absolutely normal numbers

Theorem (Lutz, Mayordomo 2013; Figueira, Nies 2013; Becher, Heiber, Slaman 2013)

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Output of algorithm Becher, Heiber and Slaman, 2013 programmed by Martin Epszteyn.

$0.4031290542003809132371428380827059102765116777624189775110896366\ldots$

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Lutz and Mayordomo (2016) gave an algorithm with nearly linear time.

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Theorem (Cassels 1959; Schmidt 1961/1962; Becher, Slaman 2013)

For any subset S of the multiplicative dependence classes, there is a real x which is normal to the bases in S and not simply normal to the bases in the complement of S. Furthermore, the real x is computable from S.

Simple normality to different bases

Theorem (Becher, Bugeaud, Slaman 2013)

Let f be any function from the multiplicative dependence classes to their subsets such that

- *for each b, if $b^{km} \in f(b)$ then $b^k \in f(b)$*
- *if $f(b)$ is infinite then $f(b) = \{b^k : k \geq 1\}$.*

Then, there is a real x which is simply normal to exactly the bases specified by f. Furthermore, the real x is computable from the function f.

The theorem gives a complete characterization (necessary and sufficient conditions).

Normality together with other properties

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Belief

Typical elements of well-structured sets, with respect to appropriate measures, are absolutely normal, unless the set displays an obvious obstruction.

For non-zero dimensional sets, Hochman and Shmerkin (2015) give geometrical conditions on a measure μ so that μ-almost all numbers are normal to a given base.

Measures whose Fourier transform decays quickly

Weyl's criterion: x is normal to base b if and only if for every non-zero integer t,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} e^{2\pi i t b^k x} = 0.$$

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Lemma (application of Davenport, Erdős, LeVeque's Theorem, 1963)

Let μ be a measure whose Fourier transform decays quickly, let I be an interval and let b a base. If for every non-zero integer t,

$$\sum_{n\geq 1} \frac{1}{n} \int_I \left| \frac{1}{n} \sum_{k=0}^{n-1} e^{2\pi i t b^k x} \right|^2 d\mu(x) \ < \infty$$

then for μ-almost all x in interval I are normal to base b.

Irrationality exponent

Definition (Liouville 1855)

The irrationality exponent of a real number x, is the supremum of the set of real numbers z for which

$$0 < \left| x - \frac{p}{q} \right| < \frac{1}{q^z}$$

is satisfied by an infinite number of integer pairs (p, q) with $q > 0$.

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- Every real greater than 2 is the irrationality exponent of some real.
- Irrational algebraic numbers have irrationality exponent equal to 2
- Rational numbers have irrationality exponent equal to 1.

Jarník's fractal

Fix a real a greater than 2. Jarník gave a Cantor-like construction of a set in $[0,1]$.
Let $(m_k)_{k\geq 1}$ be an appropriate increasing sequence of positive integers.
For each $k \geq 1$,

$$E(k) = \dot{\bigcup_{\substack{q \text{ prime} \\ m_k < q < 2m_k}}} \left\{ x \in \left(\frac{1}{q^a}, 1 - \frac{1}{q^a} \right) : \exists p \in \mathbb{N}, \left| \frac{p}{q} - x \right| < \frac{1}{q^a} \right\}$$

$E(k)$ has about $\dfrac{m_k^2}{\log m_k}$ disjoint intervals, each of length at least $\dfrac{2}{(2m_k)^a}$.

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Jarník's's fractal for the real a is

$$J = \bigcap_{k \geq 1} E(k).$$

Absolutely normal Liouville numbers

Kaufman (1981) defined for each a greater than 2, a measure on Jarník's fractal for a whose Fourier transform decays quickly.

Bluhm (2000) defined a measure such that it is supported by the Liouville numbers and its Fourier transform decays quickly.

Theorem (Bugeaud 2002)

There is an absolutely normal Liouville number.

Theorem (Becher, Heiber, Slaman 2014)

There is a computable absolutely normal Liouville number.

Absolutely normal and continued fraction normal

For a real number x in the unit interval, the continued fraction expansion of x is a sequence of positive integers $a_1, a_2, \ldots$, such that

$$x = \lfloor x \rfloor + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n + \cfrac{1}{\ddots}}}}}$$

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An application of Birkhoffs Ergodic Theorem yields that almost all reals (in the sense of Lebesgue measure) are continued fraction normal.

Absolutely normal and continued fraction normal

Theorem

There is an algorithm that computes a number that is absolutely normal and continued fraction normal.

Exponential time algorithm, Scheerer (2016);
Polynomial time $O(n^4)$, Becher and Yuhjtman (2017).

Pisot absolutely normal

Theorem (Madritsch, Scheerer, Tichy, 2016)

There is a a polynomial algorithm that computes number that is normal to all Pisot bases.

The discrepancy estimate of normal numbers

For a sequence of real numbers in the unit interval $(x_j)_{j\geq 1}$,

$$D_N((x_j)_{j\geq 1}) = \sup_{0\leq u<v\leq 1} \left| \frac{\#\{j : 1 \leq j \leq N \text{ and } u \leq x_j < v\}}{N} - (v-u) \right|.$$

Schmidt (2974) showed that for every $(x_j)_{j\geq 1}$ there are infinitely many N such that

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Now, a real number x is normal to base b exactly when, writing $\{x\} = x - \lfloor x \rfloor$,

$$\lim_{N\to\infty} D_N(\{b^j x\}_{j\geq 0}) = 0,$$

For just one base Levin 1999 constructed a real x such that

$$D_N(\{b^j x\}_{j\geq 0}) \text{ is } O\left(\frac{\log^2(N)}{N}\right).$$

The discrepancy estimate of normal numbers

Gál and Gál (1964) there is C such that for almost all reals x,

$$\limsup_{N\to\infty} \frac{D_N(\{2^j x\}_{j\geq 0})\sqrt{N}}{\sqrt{\log\log N}} < C.$$

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Philipp (1975), for every integer b, for almost all reals x,

$$\limsup_{N\to\infty} \frac{D_N(\{b_j^j x\}_{j\geq 1})\sqrt{N}}{\sqrt{\log\log N}} < 166 + 664/(\sqrt{b}-1).$$

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Fukuyama (2008) for any real $\theta > 1$, for almost all reals x,

$$\limsup_{N\to\infty} \frac{D_N(\{\theta^j x\}_{j\geq 0})\sqrt{N}}{\sqrt{\log\log N}} = C'_\theta.$$

For instance, in case θ is an integer greater than or equal to 2,

$$C'_\theta = \begin{cases} \sqrt{84}/9, & \text{if } \theta = 2 \\ \sqrt{2(\theta+1)/(\theta-1)}/2, & \text{if } \theta \text{ is odd} \\ \sqrt{2(\theta+1)\theta(\theta-2)/(\theta-1)^3}/2, & \text{if } \theta \geq 4 \text{ is even.} \end{cases}$$

The discrepancy estimate of normal numbers

Theorem (Becher, Scheerer and Slaman 2017)

There is an algorithm that computes a real x such that for every integer $b \geq 2$,

$$\limsup_{N \to \infty} \frac{D_N(\{b^j x\}_{j \geq 0})\sqrt{N}}{\sqrt{\log \log N}} < 3C_b,$$

where

$C_b = 166 + 664/(\sqrt{b} - 1)$ *is Philipp's constant.*

The algorithm computes the first n digits of the expansion of x in base 2 after performing triply-exponential in n mathematical operations.

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This improves Levin (1979) where $D_N(\{b^j x\}_{j \geq 1})$ is $O\left(\frac{(\log N)^3}{N}\right)$ (see Alvarez and Becher 2017).

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- Problem: consider discrepancy $D_N(\{b^k x\}_{k\geq 0})$.
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- Some constructions for normality together with null properties.
  Tool : measure whose Fourier transform decays quickly, effectivized.

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The End

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