A construction of an absolutely normal and continued fraction normal number

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Representation of real numbers in integer bases

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The expansion of a real number x in base b is a sequence $a_1a_2a_3\ldots$ of integers from $\{0,\ldots,b-1\}$ such that

$$x = \lfloor x \rfloor + \sum_{k\geq 1} \frac{a_k}{b^k} = \lfloor x \rfloor + 0.a_1a_2a_3\ldots$$

and the sequence $a_1a_2a_3\ldots$ does not end with a tail of $b-1$.

Normal numbers

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Theorem (Wall 1949)

A real x is normal to base b if and only if $(b^k x)_{k \geq 0}$ equidistributes modulo one for Lebesgue measure.

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- $0.123456789101112131\!4\ldots$ is normal to base 10 (Champernowne, 1933). It is unknown if it simply normal to bases that are not powers of 10.
- Stoneham number $\alpha_{2,3} = \sum_{k\geq 1} \frac{1}{3^k\ 2^{3^k}}$ is normal to base 2 but not simply normal to base 6 (Bailey, Borwein, 2012).

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Conjecture (Borel 1951)

All irrational algebraic numbers are absolutely normal.

Representation of real numbers by continued fractions

The continued fraction expansion of a positive real x is a sequence of positive integers $a_1, a_2, \ldots$ such that

$$x = \lfloor x \rfloor + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n + \cfrac{1}{\ddots}}}}}$$

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Examples,
$e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \ldots]$,$\Phi = [1; 1, 1, 1, 1, 1, 1, 1, 1, 1, \ldots]$.

The convergents $p_n(x)$ and $q_n(x)$

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And for $n \geq 1$,

$$p_n(x) = a_n p_{n-1}(x) + p_{n-2}(x),$$
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Then,

$$x = [a_1, \ldots, a_n] = \frac{p_n}{q_n}.$$

Gauss map and the Gauss measure

The Gauss map T is a function from real numbers in $[0, 1]$ to to real numbers in $[0, 1]$ defined by $T(0) = 0$ and $T(x) = 1/x - \lfloor 1/x \rfloor$.

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The map T has an invariant ergodic measure, the Gauss measure μ, which is absolutely continuous with respect to Lebesgue measure. For a Lebesgue measurable set A,

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Since Gauss measure is invariant under T, $\mu I_{v_1,\ldots,v_k}$ coincides with the measure of the set of numbers having $v_1, \ldots, v_k$ in some other position.

Continued fraction normal

Definition

A real number $x = [a_1, a_2, \dots]$ is continued fraction normal if the limit frequency of each possible block of integers $v_1, \dots, v_k$ coincides with the Gauss measure of the interval $I_{v_1,\dots,v_k}$, which is the interval formed by all the numbers whose continued fraction starts with $v_1, \dots, v_k$.

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$$\lim_{n\to\infty} \frac{1}{n} \#\Big\{ j : 1 \le j \le n, a_j = v_1, \ldots, a_{j+k-1} = v_k \Big\} = \mu I_{v_1,\ldots,v_k}.$$

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In other words, a real x is continued fraction normal if the forward orbit of x by T is equidistributed with respect to the Gauss measure.

Examples and counterexamples

Quadratic irrationals are not continued fraction normal

$$\sqrt{2} = 1.414\ldots = [1; 2, 2, 2, \ldots]$$
$$\sqrt{3} = 1.732\ldots = [1; 1, 2, 1, 2, 1, 2, \ldots]$$

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Constructions of continued fraction normal given by Postnikov and Pyatetskii-Shapiro, 1957 and Adler, Keane and Smorodinsky, 1981 and there are newer.

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Problem (Folklore; Queffelec 2006: Bugeaud 2012, Problem 10.49)

Give an example of an absolutely normal and continued fraction normal number.

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Scheerer (2017) gave an algorithm that yields one such number with doubly exponential computational complexity.

General construction of a computable real number

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This gives a construction of the unique computable real x in $\bigcap_{i \geq 1} I_i$.

Fundamental intervals b-ary

An interval I is b-ary for some integer base b if there is a block $d_1, \ldots, d_n$ of digits in $\{0, 1, \ldots, b-1\}$ such that I is the set of real numbers whose first n digits of their b-ary expansion are equal to $d_1, \ldots, d_n$.

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The set of b-ary intervals determined by n digits in base b is a partition of the unit interval in b^n many parts of equal length.

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An interval I is *cf*-ary if there is $[a_1, \ldots, a_n]$ such that the interval I is equal to the set of all the numbers whose first n digits of their continued fraction expansion are $a_1, \ldots, a_n$. Thus,

$$I_{a_1,\ldots,a_n} = ([a_1, \ldots, a_n], [a_1, \ldots, a_n + 1]), \text{ or}$$
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The set of *cf*-ary intervals determined by n digits also form a partition of the unit interval, but in infinite parts of different lengths.

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- choose digits without looking at the digits we put in previuos steps.
- choose enough many digits to make progress on normality (to avoid oscilations they should not be too many).

Two results on large deviations

1. Bernstein's inequality, 1920s, (or Hardy and Wright 1930s) to bound the measure of the sets of numbers whose expansion in a given integer base starts with k digits with too many or too few occurrences of some digit.
2. Kifer, Peres and Weiss, 2001, to bound the measure of the sets of numbers whose continued fractions start with k integers with too many or too few occurrences of some block integers.

t-bricks

Definition

For an integer $t \geq 2$, a t-brick is a t-uple $(\sigma_{cf}, \sigma_2, \ldots, \sigma_t)$ as follows

- the interval σ_{cf} is *cf*-ary;
- for every $d = 2, \ldots, t$, σ_d is d-ary interval or the union of two consecutive d-ary intervals of the same order;
- for every $d = 2, \ldots, t$, $\sigma_{cf} \subset \sigma_d$ and $|\sigma_{cf}|/|\sigma_d|$ is larger than $constant/d$;

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- Proved by Ibragimov, 1961

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The main difficulty is to control the length of these subintervals because, for any fixed positive integer n, there are intervals of the form $I_{a_1,\ldots,a_n}$ that are arbitrarily small.

We use that the distribution of the logarithm of the length of intervals of the form $I_{a_1,\ldots,a_n}$ is asymptotically Gaussian.

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Lemma (Morita 1994 (Theorem 8.1) Vallée 1997 (Théoreme 9))

There is K_0 and n_0 such that for every $n \geq n_0$,

$$\left| Pr\left[x \in (0,1) : -y \leq \frac{\log q_n(x) - nL}{\sigma\sqrt{n}} \leq y\right] - \frac{1}{\sqrt{2\pi}}\int_{-y}^{y} e^{-z^2/2}dz \right| < \frac{K_0}{\sqrt{n}},$$

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Problem

Give the values, or at least approximate, K_0 and n_0.

Optimal central limit theorem and explicit constants

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The expression for σ uses the dominant eigenvalue of L_2,

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where λ' and λ'' denote the derivative and second derivative of λ and

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Our use of σ occurs just in the next Lemma and we do not require its exact value; any upper bound suffices.

We control the length of *cf*-intervals

Lemma

There are positive constants K, c *and a positive integer* n_1 *such that for any* cf*-ary interval* I *and any integer* $n \geq n_1$*, the Lebesgue measure of the union of the* cf*-ary subintervals* J *of* I *of relative order* n *such that*

$$\frac{|I|}{4} e^{-2nL-2c} \leq |J| \leq 2|I| e^{-2nL+2c}$$

is greater than $K|I|/\sqrt{n}$*.*

Computational complexity

At step s

1. the choice of the t-brick $(\sigma_{cf}, \sigma_2, \ldots, \sigma_t)$ does not depend on the actual digits put at previous steps.
2. the relative order $n(s)$ of σ_{cf} is logaritmic in s. Similarly, for σ_d, $d = 2, \ldots t$.
3. the maximum integer t and maximum block size is sublogarithmic in s.
4. approximation to normality with tolerance $\varepsilon = 1/t$.
5. divide $\sigma_{cf}^{(s-1)}$ in $\lfloor 4\ e^{2n(s)L+2c} \rfloor + 1$ equal intervals I_{cf}.

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5. divide $\sigma_{cf}^{(s-1)}$ in $\lfloor 4\ e^{2n(s)L+2c} \rfloor + 1$ equal intervals I_{cf}. Notice that every interval contained in $\sigma_{cf}^{(s-1)}$ of length $\frac{1}{4} e^{-2n(s)\ L-2c} |\sigma_{cf}^{(s-1)}|$ will have an interior in one of these intervals I_{cf}. Check each endpoint !

Open problems

Problem

Give n_0 and K in Vallée's Central Limit theorem that establishes Gaussian distribution of $\log q_n$.

Open problems

In the ternary Cantor set with probability 1 a number is normal to base 2.
(Tool: measure whose Fourier transform on the fractal decays quickly)

Theorem (David Simmons and Barak Weiss 2016, Theorem 8.9)

In the ternary Cantor set with probability 1 a number is continued fraction normal.

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David Simmons and Barak Weiss, 2016

Random walks on homogeneous spaces and Diophantine approximation on fractals

http://www.math.tau.ac.il/~barakw/papers/master_for_arxiv.pdf

Problem

Give another proof of Simmons and Weiss's theorem.

Open problems

Problem

Normality together with pseudo-randomness.

Open problems

Problem

Normality together with pseudo-randomness.

The End