On normal numbers

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Février 2, 2015

Normal numbers

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Definition

A base is an integer greater than or equal to 2. For a real number x, the expansion of x in base b is a sequence $(a_k)_{k\geq 1}$ of integers a_k from $\{0,1,\ldots,b-1\}$ such that

$$x = \lfloor x \rfloor + \sum_{k\geq 1} \frac{a_k}{b^k} = \lfloor x \rfloor + 0.a_1a_2a_3\ldots$$

where infinitely many of the a_k are not equal to $b-1$.

Normal numbers

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A real number x is normal to base b if x is simply normal to every base b^k, for every positive integer k.

Normal numbers

A real x is normal to base b if its expansion in base b, $0.x_1x_2x_3\ldots$, satisfies for every digit d in $\{0,..,b-1\}$,

$$\lim_{n\to\infty}\frac{\#\{j : 1\le j\le n,\ x_j=d\}}{n}=\frac{1}{b}.$$

Normal numbers

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$$\lim_{n\to\infty} \frac{\#\{j : 1 \le j \le n,\ x_j = d\}}{n} = \frac{1}{b}.$$

Equivalently, for every digit d in $\{0,..,b-1\}$,

$$\forall \varepsilon > 0\ \exists m\ \forall n \ge m\ \left|\frac{\#\{j : 1 \le j \le n, x_j = d\}}{n} - \frac{1}{b}\right| < \varepsilon.$$

Then, a real number x is normal to base b if $\forall\varepsilon\exists m\forall n\ \varphi(x,b,n,\varepsilon)$.
This is a formula with one free real variable x, one free integer variable b and quantification only on integers, of the form $\forall\exists\forall$.

Normal numbers

Theorem (Borel 1922, Niven and Zuckerman 1951)

A real number x is normal to base b if, for every $k \geq 1$, every block of k digits occurs in the expansion of x in base b with limiting frequency $1/b^k$.

Normality as uniform distribution modulo one

Theorem (Wall 1949)

A real x is normal to base b if and only if $(b^k x)_{k\geq 0}$ equidistributes modulo one for Lebesgue measure.

Not normal

$0.01\ 002\ 0003\ 00004\ 000005\ 0000006\ 00000007\ 000000008\ldots$
is not simply normal to base 10.

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is simply normal to base 10, but not simply normal to base 100.

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Rational numbers are normal to no base.

Normal to a given base

Theorem (Champernowne, 1933)

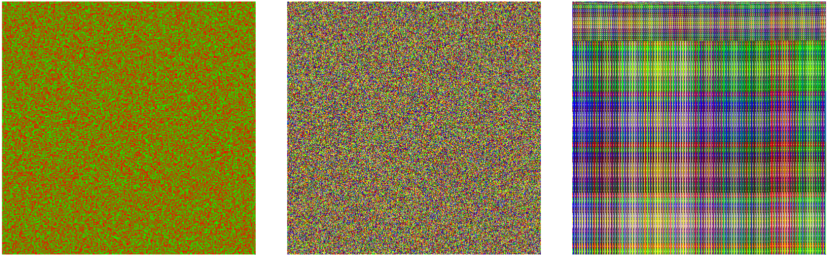
$0.12345678910111213141516171819202122232425\ldots$ *is normal to base* 10.

Normal to a given base

Theorem (Champernowne, 1933)

$0.12345678910111213141516171819202122232425\ldots$ *is normal to base* 10.

It is unknown if it is normal to bases that are not powers of 10.



base 2 base 6 base 10

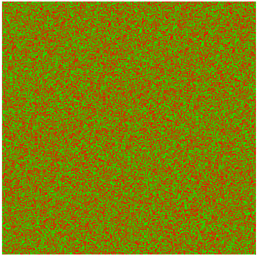
Plots of the first 250000 digits of Champernowne's number.

Normal to one base, but not to another

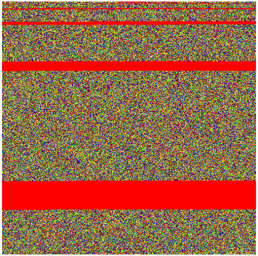
Bailey and Borwein (2012) proved that the Stoneham number $\alpha_{2,3}$,

$$\alpha_{2,3} = \sum_{k\geq 1} \frac{1}{3^k \; 2^{3^k}}$$

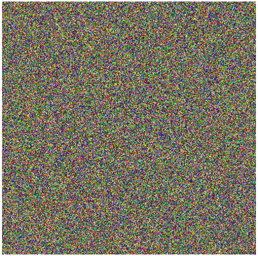
is normal to base 2 but not simply normal to base 6.



base 2



base 6



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Plots of the first 250000 digits of Stoneham number $\alpha_{2,3}$.

Normal to all bases

A real number x is absolutely normal if x is normal to all bases.

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Theorem (Borel 1909)

The set of absolutely normal numbers in $[0, 1]$ *has Lebesgue measure* 1.

Problem (Borel 1909)

Give one example.

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Are the usual mathematical constants, such as π, e, or $\sqrt{2}$, absolutely normal? Or at least simply normal to some base?

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Conjecture (Borel 1950)

Irrational algebraic numbers are absolutely normal.

Normal to all bases

Bulletin de la Société Mathématique de France (1917) 45:127–132; 132–144

DÉMONSTRATION ÉLÉMENTAIRE DU THÉORÈME DE M. BOREL SUR LES NOMBRES ABSOLUMENT NORMAUX ET DÉTERMINATION EFFECTIVE D'UN TEL NOMBRE;

PAR M. W. SIERPINSKI.

On appelle, d'après M. Borel, *simplement normal* par rapport à la base q [1] tout nombre réel x dont la partie fractionnaire

---

[1] E. BOREL, *Leçons sur la théorie des fonctions*, p. 197, Paris, 1914.

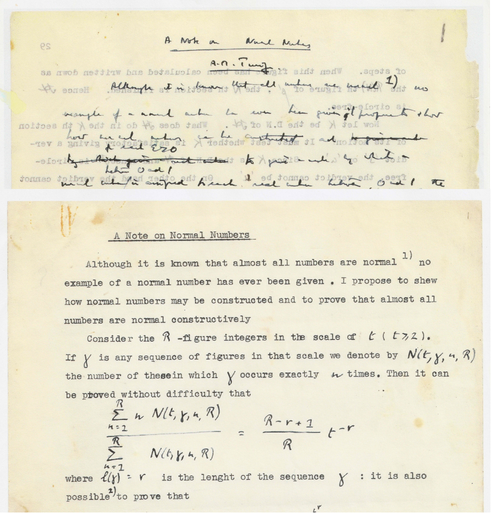
SUR CERTAINES DÉMONSTRATIONS D'EXISTENCE;

PAR M. H. LEBESGUE.

Dans une lettre, adressée à M. Borel, et qui accompagnait l'envoi de l'article précédent, M. Sierpinski se demandait si cet article devait être publié, s'il ne ferait pas double emploi avec une démonstration que j'avais indiquée à M. Borel et que celui-ci a signalée dans la deuxième édition de ses *Leçons sur la théorie des fonctions* (p. 198).

Normal to all bases

Turing, A. M. A Note on Normal Numbers. Collected Works of Alan M. Turing, Pure Mathematics, 117-119. Notes of editor J.L. Britton, 263-265. North Holland, 1992.



A Note on Normal Numbers

Although it is known that almost all numbers are normal [1] no example of a normal number has ever been given . I propose to shew how normal numbers may be constructed and to prove that almost all numbers are normal constructively

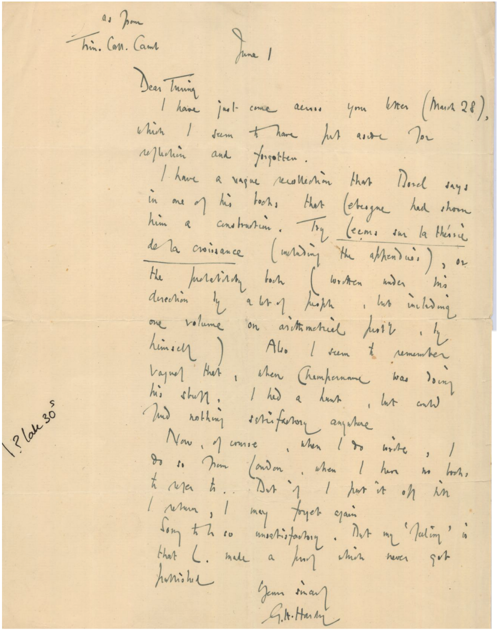
Consider the R -figure integers in the scale of t ($t \geq 2$). If γ is any sequence of figures in that scale we denote by $N(t,\gamma,n,R)$ the number of these in which γ occurs exactly n times. Then it can be proved without difficulty that

$$\frac{\sum_{n=1}^{R} n\, N(t,\gamma,n,R)}{\sum_{n=1}^{R} N(t,\gamma,n,R)} = \frac{R-r+1}{R}\, t^{-r}$$

where $\ell(\gamma) = r$ is the lenght of the sequence γ : it is also possible[2] to prove that

Corrected and completed in Becher, Figueira and Picchi, 2007.

Letter exchange between Turing and Hardy (AMT/D/5)



as from
Trin. Coll. Camb

June 1

Dear Turing

I have just come across your letter (March 28), which I seem to have put aside for reflection and forgotten.

I have a vague recollection that Borel says in one of his books that Lebesgue had shown him a construction. Try Leçons sur la théorie de la croissance (including the appendices), or the productivity book (written under his direction by a lot of people, but including one volume on arithmetical [illegible], by himself) Also I seem to remember vaguely that, when Champernowne was doing his stuff, I had a hunt, but could find nothing satisfactory anywhere.

Now, of course, when I do write, I do so from London, where I have no books to refer to. But if I put it off till I return, I may forget again.

Sorry to be so unsatisfactory. But my 'feeling' is that L. made a proof which never got published.

Yours sincerely
G.H. Hardy

1? Plate 30s

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Normal to all bases (or a prescribed set of bases)

Recall that a real number is computable if there is an algorithm that computes its expansion in some base (given n the algorithm computes the n-th digit).

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2015 Computability and Diophantine approximations (prescribed irrationality exponents). Becher, Bugeaud and Slaman (in progress).

Normal to all bases

Theorem (Becher, Heiber and Slaman, 2013)

There is an algorithm that computes an absolutely normal number with just above quadratic time-complexity.

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There is an algorithm that computes an absolutely normal number with just above quadratic time-complexity.

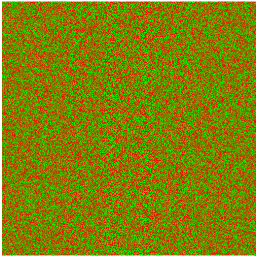
That is,

For any computable non-decreasing unbounded function f, there is an algorithm that outputs the first n digits in the expansion of a real number in base 2 after performing $O(f(n)\, n^2)$ elementary operations.

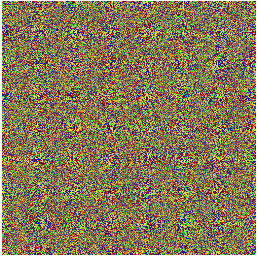
Normal to all bases

Output of algorithm Becher, Heiber and Slaman, 2013 programmed by Martin Epszteyn.

$0.4031290542003809132371428380827059102765116777624189775110896366\ldots$



base 2



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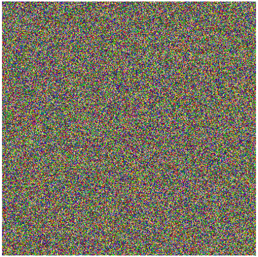

base10

Plots of the first 250000 digits of the output of our algorithm.

Open question

Question

Is there an absolutely normal number computable in polynomial time having a nearly optimal rate of convergence to normality?

Constructions of normal numbers

Concatenation works if we consider just one base.
For two bases, concatenation in general fails.

For example,

	base 10	*base* 3
$x =$	$(0.25)_{10} =$	$(0.020202020202\ldots)_3\ldots$
$y =$	$(0.0017)_{10} =$	$(0.000001020110110010 2\ldots)_3$
$x + y =$	$(0.2517)_{10} =$	$(0.0202101110122\ldots)_3$

Simple normality to different bases

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Because $8 = 2^3$ and the digits in $\{0, .., 7\}$ correspond to the blocks in base 2:

$$000 \quad 001 \quad 010 \quad 011 \quad 100 \quad 101 \; 110 \quad 111$$

Half of the digits are equal to 0.

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Observation

Simple normality to b^k implies simple normality to b^ℓ, for each ℓ that divides k.

Simple normality to infinitely many powers of b implies normality to b (Long 1957)

Multiplicative independence

Definition

Two positive integers are multiplicatively dependent if one is a rational power of the other. Then, 2 and 8 are dependent, but 2 and 6 are independent.

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The positive integers that are not perfect powers, $2, 3, 5, 6, 7, 10, 11, \ldots$, are pairwise multiplicatively independent.

Normality to different bases

Theorem (Maxfield 1953)

Let b and b' multiplicatively dependent. For any real number x, x is normal to base b if and only if x is normal to base b'.

Normality to different bases

Theorem (Cassels, 1959)

Almost every real number in the middle third Cantor set is normal to every base which is not a power of 3.

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Theorem (Schmidt 1961/1962)

For any given set S of bases closed under multiplicative dependence, there are real numbers normal to every base in S and not normal to any base in its complement. Furthermore, there is a real x computable from S.

Becher and Slaman 2014 refuted simple normality, and answered to Brown, Moran and Pearce.

An old silent question on simple normality

Made explicit by Yann Bugeaud in 2013:

> *What are the necessary and sufficient conditions on a set of bases so that there is a real number which is simply normal exactly to the bases in such a set?*

Necessary and sufficient conditions for simple normality

Theorem (Becher, Bugeaud and Slaman, 2015)

Let f be any function from the set of integers that are not perfect powers to sets of integers such that for each b,

- *if for some k, b^k is in $f(b)$ then, for every ℓ that divides k, b^ℓ is in $f(b)$;*
- *if $f(b)$ is infinite then $f(b) = \{b^k : k \geq 1\}$.*

Then, there is a real x simply normal to exactly the bases specified by f.

Moreover, the set of real numbers that satisfy this condition has full Hausdorff dimension.

Also, the real x is computable from the function f.

Arithmetical independence

Theorem (informal statement) (Becher and Slaman 2014)

The set of bases to which a real number can be normal is not tied to any arithmetical properties other than multiplicative dependence.

In other words,

The set of bases for which a real x is normal can coincide with any arithmetical property on the set of integers that are not perfect powers, definable by a Π^0_3 formula relative to x.

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Formally, this is a problem in descriptive set theory. Achim Ditzen conjectured it in 1994, after a question of A. Kechris. We confirmed it.

Borel hierarchy for subsets of the real numbers

Recall Borel hierarchy for subsets of the real numbers is the stratification of the σ-algebra generated by the open sets with the usual interval topology.

- A set is $\mathbf{\Sigma}^0_1$ iff it is open.
- A set is $\mathbf{\Pi}^0_1$ iff it is closed.
- A set is $\mathbf{\Sigma}^0_{n+1}$ iff it is countable union of $\mathbf{\Pi}^0_n$ sets.
- A set is $\mathbf{\Pi}^0_{n+1}$ iff it is a countable intersection of $\mathbf{\Sigma}^0_n$ sets.

A set is is hard for a Borel class if every set in the class is reducible to it by a continuous map.

A set is complete for a class if it is hard for this class and belongs to the class.

Effective Borel hierarchy for subsets of the real numbers

Arithmetic hierarchy of formulas in the language of second-order arithmetic:

- formulas involve only quantification over integers.
- atomic formulas assert algebraic identities between integers or membership of real numbers in intervals with rational endpoints.
- a formula is Π^0_0 and Σ^0_0 if all its quantifiers are bounded.
- a formula is Σ^0_{n+1} if it has the form $\exists x\, \theta$ where θ is Π^0_n.
- a formula is Π^0_{n+1} if it has the form $\forall x\, \theta$ where θ is Σ^0_n.

A set of real numbers is Σ^0_n (respectively Π^0_n) in the effective Borel hierarchy if membership in that set is definable by a formula Σ^0_n (respectively Π^0_n).

Effective reductions are computable maps.

Effective case implies the general case

- Every Σ^0_n set is $\mathbf{\Sigma}^0_n$ and every Π^0_n set is $\mathbf{\Pi}^0_n$.
- For every $\mathbf{\Sigma}^0_n$ set A there is a Σ^0_n formula and a real parameter such that membership in A is defined by that Σ^0_n formula relative to that real parameter.

Since computable maps are continuous, proofs of hardness in the effective hierarchy yield proofs of hardness in general by relativization.

Normal numbers

A real x is normal to base b if its expansion in base b, $0.x_1x_2x_3\ldots$, satisfies

$$\text{for every digit } d \text{ in } \{0, ...b1\},\ \lim_{n\to\infty} \frac{\#\{j : x_j = d\}}{n} = \frac{1}{b},$$

Equivalently,

$$\forall d \in \{0, .., b-1\}\ \forall \varepsilon > 0 \exists m\ \forall n \geq m\ \left| \frac{\#\{j : x_j = d\}}{n} - \frac{1}{b} \right| \leq \varepsilon.$$

A real number x is normal to base b if $\forall \epsilon \exists m \forall n\ \varphi(x, b, n, \varepsilon)$ where φ has one free real variable x, one free integer variable b and quantification only on integers.

Normality in the effective Borel Hierarchy

In the effective Borel hierarchy for susbets of real numbers,

- The set of normal numbers to a fixed base b is Π^0_3.
- The set of normal numbers to all bases is also Π^0_3.
- The set of numbers normal to some base is Σ^0_4.

Normal to all bases is complete at the third level

Asked first by Kechris 1994.

Theorem (Ki and Linton 1994)

The set of real numbers that are normal to any fixed base is Π^0_3-complete.

Theorem (Becher, Heiber, Slaman 2014)

The set of real numbers that are absolutely normal is Π^0_3-complete.

Arithmetical independence

Theorem (Becher and Slaman 2014)

The set of real numbers that are normal to some base is Σ^0_4*-complete in the effective Borel Hierachy on subsets of real numbers.*

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The proof shows that the discrepancy functions (speed of convergence to normality) on multiplicatively independent bases are pairwise independent.

Research line

Little is knwon about the interplay between combinatorial, recursion-theoretic and number-theoretic properties of the expansions of real numbers.

These investigations on normal numbers aim to make progress in this direction.

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The End

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New normal numbers

Belief

If we consider appropriate measures, most elements of well structured sets are absolutely normal, unless the sets have evident obstacles.

Normality and Weyl's criterion

A number x is normal to base b if $(b^k x)_{k \geq 0}$ is uniformly distributed modulo one.

A sequence of real numbers is uniformly distributed if and only if
for every Riemann-integrable (complex-valued) function f,
$\int_0^1 f(x)dx$ is the limit of the average values of f on the sequence.

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By Weyl's criterion, a real number x is normal to base b if and only if

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} e^{2\pi i t b^k x} = 0.$$

Appropriate measures for normality

Lemma (direct application of Davenport, Erdős, LeVeque's Theorem 1963)

Let μ be a measure, I an interval and b a base. If for every non-zero integer t,

$$\sum_{n \geq 1} \frac{1}{n} \int_I \left| \frac{1}{n} \sum_{k=0}^{n-1} e^{2\pi i t b^k x} \right|^2 d\mu(x) \ < \infty$$

then for μ-almost all x in interval I are normal to base b.

Normal numbers and Diophantine approximations

Definition (Liouville 1855)

The irrationality exponent of a real number x, is the supremum of the set of real numbers z for which the inequality $0 < \left| x - \frac{p}{q} \right| < \frac{1}{q^z}$ is satisfied by an infinite number of integer pairs (p, q) with positive q.

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- Almost all real numbers have irrationality exponent equal to 2.
- Every real greater than 2 is the irrationality exponent of some real.
- Irrational algebraic numbers have irrationality exponent equal to 2 (Thue - Siegel - Roth theorem 1955).
- Rational numbers have irrationality exponent equal to 1.

Fractals, measures and approximations

- Jarník (1929) and Besicovich (1934) defined a fractal for real numbers with irrationality exponent equal to a given a greater than or equal to 2.
- Kaufman (1981) defined for each real number a greater than 2, a measure on Jarník 's fractal whose Fourier transform decays quickly.
- Bluhm (2000) defined a measure supported by the Liouville numbers, whose Fourier transform decays quickly.
- We adapted the measures, tailored for effective approximations.

Normal Liouville numbers

Theorem (Bugeaud 2002)

There is an absolutely normal Liouville number.

Theorem (Becher, Heiber and Slaman 2015)

There is a computable absolutely normal Liouville number.

Simple normality and irrationality exponents

Theorem (Becher, Bugeaud and Slaman)

Let S be a set of bases satisfying the conditions for simple normality.

- *There is a Liouville number x simply normal to exactly the bases in S.*
- *For every a greater than or equal to 2 there is a real x with irrationality exponent equal to a and simply normal to exactly the bases in S.*

Furthermore, x is computable from S and, for non- Liouville, also from a.

Normality and finite automata

Fix an alphabet A. Consider finite automata that input and output infinite sequences of symbols from A.

Definition

A sequence $a_1 a_2 a_3 \ldots$ is compressible by an input-output finite automata if

$$\liminf_{k \to \infty} \frac{\#\text{output symbols after reading } a_1 \ldots a_k}{k} < 1$$

Normality and finite automata

Theorem (Schnorr and Stimm 1971 + Dai, Lathrop, Lutz and Mayordomo 2004)

A real is normal to base b if, and only if, its expansion in base b is incompressible by injective input-output finite automata.

A direct proof of the above theorem Becher and Heiber, 2012.

Normality and finite automata

Theorem (Becher, Carton, Heiber 2013)

Non-deterministic bounded-to-one input-output finite automata, even if augmented with a fixed number of counters, can not compress expansions of normal numbers.

Theorem (Boasson 2012)

Non-deterministic pushdown input-output finite automata can compress expansions of normal numbers.

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Problem

Can deterministic pushdown input-output finite automata compress expansions of normal numbers?

Normality preservation and finite automata

Let $a_1 a_2 a_3 \cdots$ be an infinite sequence. Consider the infinite sequence obtained by selection of some elements

$a_1 \ a_2 \ a_3 \ a_4 \ a_5 \ a_6 \ a_7 \ a_8 \ a_9 \ \ldots$

Normality preservation and finite automata

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Theorem (Agafonov 1968)

Prefix selection by a regular set of finite sequences preserves normality.

Theorem (Becher, Carton and Heiber 2013)

Suffix selection by a regular set of infinite sequences preserves normality.

Normality preservation and finite automata

Theorem (Becher, Carton and Heiber 2013)

Two sided selectors do not preserve normality.

Theorem (Merkle and Reimann 2006)

Neither deterministic one-counter sets nor linear sets preserve normality (these are the sets recognized by pushdown finite automata with a unary stack and by one-turn pushdown finite automata, respectively)

Problem

What is the least powerful selection that does not preserve normality?

Normality and finite automata

Definition

A *transducer* is a tuple $T = \langle Q, A, B, \delta, q_0 \rangle$, where

- Q is a finite set of states,
- A and B are the input and output alphabets, respectively,
- $\delta : Q \times A \to B^* \times Q$ is the transition function,
- $q_0 \in Q$ is the starting state.

If $\delta(p, a) = \langle v, q \rangle$ we write $p \xrightarrow{a|v} q$.
An *infinite run* is $p_0 \xrightarrow{a_1|v_1} p_1 \xrightarrow{a_2|v_2} p_2 \xrightarrow{a_3|v_3} p_3 \cdots$ is accepting if $p_0 = q_0$.

This is the Büchi acceptance condition where all states are accepting.

Normality and finite automata

Definition

A sequence $x = a_1 a_2 a_3 \cdots$ is *compressible* by a transducer if and only if its accepting run $q_0 \xrightarrow{a_1|v_1} q_1 \xrightarrow{a_2|v_2} q_2 \xrightarrow{a_3|v_3} q_3 \cdots$ satisfies

$$\liminf_{n\to\infty} \frac{|v_1 v_2 \cdots v_n|}{n} \frac{\log |B|}{\log |A|} < 1.$$

Normality and finite automata

Let $L \subseteq A^*$. The infinite word obtained by *prefix-selection* by L is $a_{p(1)}a_{p(2)}\cdots$, where $p(j)$ is the j-th in sorted $\{i \ : \ a_1a_2\cdots a_{i-1} \in L\}$.

Let $X \subseteq A^\omega$. The infinite word obtained by *suffix-selection* by X is $a_{s(1)}a_{s(2)}\cdots$, where $s(j)$ is the j-th in sorted $\{i : a_{i+1}a_{i+2}\cdots \in X\}$.