Normal numbers with digit dependencies

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Expansion of a real number in an integer base

For a real number x, its fractional expansion in an integer base $b \geqslant 2$ is a sequence of integers $a_1, a_2 \ldots$, where $0 \leqslant a_n < b$ for every n, such that

$$x - \lfloor x \rfloor = \sum_{j \geqslant 1} a_n / b^n = 0.a_1 a_2 a_3 \ldots$$

We require that $a_n < b - 1$ infinitely often to ensure that every number has a unique representation.

Borel normal numbers

Let integer $b \geqslant 2$. A real number x is simply normal to base b if every digit in $\{0, \ldots, b-1\}$ occurs in the base-b expansion of x with the same asymptotic frequency (that is, with frequency $1/b$).

A real number x is normal to base b if it is simply normal to all the bases $b, b^2, b^3, \ldots$.

A real number x is aboslutely normal if it is normal to all integer bases.

Examples and counterexamples of Borel normal numbers

$0.01010101010\ldots$ is simply normal to base 2 but not to 2^2 nor 2^3, etc.

Each number in Cantor middle third set is not simply normal to base 3.

Champernowne's number in base 10

$$0.12345679101112131415161718192021$$

is normal to base 10, but is not known whether it is normal to the multiplicatively independent bases.

Stoneham number $\alpha_{2,3} = \sum_{n \geqslant 1} \frac{1}{3^n \ 2^{3^n}}$ is normal to base 2 but not simply normal to base 6 (Bailey and Borwein, 2012).

Borel normal numbers

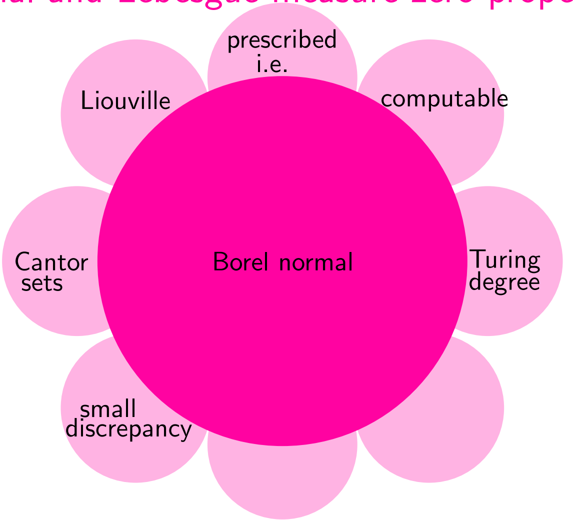
Borel (1909) proved that almost all numbers, with respect to Lebesgue measure, are absolutely normal.

Borel normality and other properties of full measure

- continued fraction normal
- x and $1/x$ absolutely normal
- Poisson generic in base b (subset of normal in base b)

...

Borel normal and Lebesgue measure zero properties



Turing(1937), Cassels (1959), Schmidt (1961/1962), Bugeaud (2002), Levin (1999)

Conjecture (Borel 1951)

Algebraic numbers are absolutely normal.

Question

How many consecutive digits have to be independent, so that almost all numbers are Borel normal?

Answers today

For each position n, slightly more than $\log \log n$ (Theorem 1).

Metric theorems for Borel normal Toeplitz numbers (Theorems 2 and 5).

Examples of Borel (simply) normal Toeplitz numbers (Theorems 3 and 4).

Theorem 1 (Aistleitner, Becher and Carton 2019)

Let integer $b \geqslant 2$. Let $X_1, X_2, \ldots$ be a sequence of random variables from a given probability space $(\Omega, \mathcal{F}, P)$ into $\{0, .., b-1\}$.

Assume that for every n, X_n is uniformly distributed on $\{0, .., b-1\}$. Suppose there is a function $g: \mathbb{N} \mapsto \mathbb{R}$ unbounded and monotonically increasing such that for all sufficiently large n the random variables

$$X_n,\ X_{n+1},\ \ldots,\ X_{n+\lceil g(n) \log\log n \rceil}$$

are mutually independent. Then, P-almost surely $x = 0.X_1X_2\ldots$ is normal to base b.

Theorem 1, continued

On the other hand, for every integer $b \geqslant 2$ and every positive K there is an example where $X_1, X_2, \ldots$ are uniformly distributed on $\{0, \ldots, b-1\}$ and for all sufficiently large n the random variables

$$X_n,\ X_{n+1},\ \ldots,\ X_{n+\lceil K \log\log n \rceil}$$

are mutually independent but P-almost surely the number $x = 0.X_1 X_2 \ldots$ is not simply normal to base b.

Proof of Theorem 1, simple normality to base b

Fix base b. Fix ε.

Partition $\mathbb{N}$ in $N_1, N_2, \ldots$ such that each $|N_j|$ grows exponentially in j (N_j goes from $(1+\varepsilon)^{j-1}$ to $(1+\varepsilon)^j$).

N_1 | N_2 | N_3 | N_4 | ...

Let j be large enough.

Partition N_j in $S_1, \ldots S_r$, each $|S| = \lceil (\log j)/\varepsilon^2 \rceil$.

$N_j =$ [S_1 | S_2 | S_3 | ... | S_r]

Variables with indices in each S are independent because

$$|S| > \frac{\log \log n}{\varepsilon^2}, \quad \text{for } n \in N_j,$$

while by the assumption independence holds for random variables whose indices are within distance $g(n) \log \log n$ of each other with $g(n) \to \infty$.

Proof of Theorem 1, simple normality to base b

Fix a digit d.

By Hoeffding's inequality, for each S,

$$P\Big(\Big|\frac{1}{|S|}\sum_{n\in S}\mathbf{1}(X_n=d)-\frac{1}{b}\Big|>\varepsilon\Big)\leqslant 2e^{-2\varepsilon^2|S|}\leqslant\frac{2}{j^2}.$$

Proof of Theorem 1, simple normality to base b

Let D_S be the random variable for $\dfrac{1}{|S|}\sum_{n\in S}\mathbf{1}(X_n = d) - \dfrac{1}{b}$,

obtain (in some steps)

$$P\Big(\sum_{S\in\{S_1,\ldots,S_r\}} |D_S| > 2\varepsilon r\Big) \leqslant \frac{2}{\varepsilon j^2}.$$

These exceptional probabilities form a convergent series summing over j.

Thus, P-almost surely $\quad \Big|\dfrac{1}{|N_j|}|\{n \in N_j : X_n = d\}| - \dfrac{1}{b}\Big| \leqslant 2\varepsilon,$

By Borel Cantelli lemma, $\quad \Big|\dfrac{1}{N}|\{n : 1 \leqslant n \leqslant N, X_n = d\}| - \dfrac{1}{b}\Big| \leqslant 4\varepsilon.$

Proof of Theorem 1, normality to base b

The same argument yields simple normality to $b^2, b^3, b^4, \ldots$.
For b^2 we have

$$(0.X_1X_2X_3X_4\ldots)_b = (0.Y_1Y_2\ldots)_{b^2}$$

where, for each $n \geqslant 1$,

$$Y_n = bX_{2n-1} + X_{2n}.$$

Mutual independence of

$$X_{2n-1},\ X_{2n},\ \ldots,\ X_{2n-1+\lceil g(2n-1)\log\log(2n-1)\rceil}$$

implies there is a monotonous increasing function $\hat{g}$ such that for all sufficiently large n,

$$Y_n,\ Y_{n+1},\ \ldots,\ Y_{n+\lceil \hat{g}(n)\log\log n\rceil}$$

are mutually independent. □

Toeplitz numbers (Jacobs and Keane 1969)

Let integer $b \geqslant 2$. Let $\mathbb{P}$ denote the set of prime numbers and let $\mathcal{P} \subseteq \mathbb{P}$.

The set of Toeplitz numbers $\mathcal{T}_{b,\mathcal{P}}$ is the set of all real numbers $\xi \in [0,1)$ whose base-b expansion $\xi = \sum_{n \geqslant 1} a_n / b^n$ satisfies

$$a_n = a_{np} \qquad (n \geqslant 1,\, p \in \mathcal{P}).$$

For example, $0.a_1 a_2 a_3 \ldots$ is a Toeplitz number for $\mathcal{P} = \{2, 3\}$ if, for every $n \geqslant 1$, we have

$$a_n = a_{2n} = a_{3n}.$$

Then, $a_1, a_5, a_7, a_{11}, \ldots$ are independent while $a_2, a_3, a_4, a_6, \ldots$ are completely determined by earlier digits.

Uniform measure on $\mathcal{T}_{b,\mathcal{P}}$

Let $\mathcal{P}$ be a set of primes included in $\mathbb{P}$.
Let $j_1, j_2, \ldots$ be the enumeration in increasing order of all positive integers that are not divisible by any of the primes in $\mathcal{P}$.

The Toeplitz transform $\tau_{b,\mathcal{P}} : [0,1) \to \mathcal{T}_{b,\mathcal{P}}$ is defined as

$$\tau_{b,\mathcal{P}}(0.a_1 a_2 a_3 \ldots) := 0.t_1 t_2 t_3 \ldots$$

such that when $n = j_k p_1^{e_1} \cdots p_r^{e_r} \quad (p_1, \cdots, p_r \in \mathcal{P})$,

$$t_n = a_k.$$

We endow $\mathcal{T}_{b,\mathcal{P}}$ with a probability measure μ, which is the forward-push by $\tau_{b,\mathcal{P}}$ of the Lebesgue measure λ. For any measurable set $X \subseteq \mathcal{T}_{b,\mathcal{P}}$,

$$\mu(X) = \lambda(\tau_{b,\mathcal{P}}^{-1}(X)).$$

Theorem 2 (Aistleitner, Becher and Carton 2019)

Let integer $b \geqslant 2$, let finite $\mathcal{P} \subset \mathbb{P}$ and let μ be the uniform probability measure on $\mathcal{T}_{b,\mathcal{P}}$. Then, μ-almost all elements of $\mathcal{T}_{b,\mathcal{P}}$ are normal to base b.

For $\mathcal{P} = \{2\}$ was obtained by Alexander Shen (2016), and by Lingmin Liao and Michal Rams (2021).

Yann Bugeaud (personal communication 2017) observed the theorem holds for infinite $\mathcal{P} \subset \mathbb{P}$ (it is possible that there is some publication!).

Proof of Theorem 2

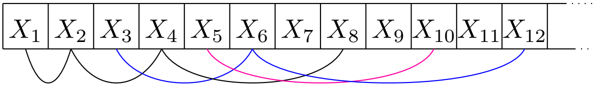
The Toeplitz transform $\tau_{b,\mathcal{P}}$ also induces a function $\delta : \mathbb{N} \mapsto \mathbb{N}$ where

$$\tau_{b,\mathcal{P}}(0.a_1a_2a_3\cdots) = 0.t_1t_2t_3\cdots = 0.a_{\delta(1)}a_{\delta(2)}a_{\delta(3)}\cdots .$$

For each n, $t_n(x)$, is a random variable on space $([0,1), \mathcal{B}(0,1), \lambda)$.

Since $t_n(x) = a_{\delta(n)}(x)$ for all n, t_m and t_n are independent, with respect to both measures λ and μ, if and only if $\delta(m) \neq \delta(n)$.

For $\mathcal{P} = \{2\}$



$1 = \delta(1) = \delta(2) = \delta(4) = \delta(8) = \ldots$
$3 = \delta(3) = \delta(6) = \delta(12) = \ldots$
$5 = \delta(5) = \delta(10) = \ldots$

Proof of Theorem 2

Let $\mathcal{P} \subset \mathbb{P}$, $\mathcal{P} = \{p_1, \ldots, p_r\}$ be a finite set of r primes.

Define $n \sim n'$ whenever there are exponents $e_1, \ldots e_r, e'_1, \ldots e'_r$ and a positive integer k such that

k is coprime with each $p \in \mathcal{P}$,

$n = kp_1^{e_1} \ldots p_r^{e_r}$ and $n' = kp_1^{e'_1} \ldots p_r^{e'_r}$.

Lemma (follows from Tijdeman 1973)

There is n_0 such that if $n' \sim n$ and $n' > n > n_0$, then $n' - n > 2\sqrt{n}$.

Since $n \sim n'$ holds exacly when $\delta(n) = \delta(n')$, and given that $\lfloor 2\sqrt{n} \rfloor \gg g(n) \log\log(n)$, we have

$$\delta(n), \delta(n+1), \ldots, \delta(n + \lfloor 2\sqrt{n} \rfloor)$$

are pairwise different.

Thus, $a_{\delta(n)}, a_{\delta(n+1)}, \ldots, a_{\delta(n+\lfloor 2\sqrt{n} \rfloor)}$ are mutually independent. □

Example of a simply number in $\mathcal{T}_{b,\mathcal{P}}$

Let $\mathcal{P} \subset \mathbb{P}$. Define $\Omega_{\mathcal{P}}(n) : \mathbb{N} \to \mathbb{N}$, the sum of the exponents in the factorization of n of those prime factors that are *not* in $\mathcal{P}$.

For example, for $\mathcal{P} = \{2, 3\}$,
$\Omega_{\mathcal{P}}(2) = \Omega_{\mathcal{P}}(3) = \Omega_{\mathcal{P}}(6) = \Omega_{\mathcal{P}}(8) = 0$
$\Omega_{\mathcal{P}}(5) = \Omega_{\mathcal{P}}(10) = 1$
$\Omega_{\mathcal{P}}(35) = 2$

Given $\mathcal{P} \subset \mathbb{P}$ and integer $b \geqslant 2$, the number

$$\xi_{\mathcal{P}} := \sum_{n \geqslant 1} t_n / b^n$$

where

$$t_n := (\Omega_{\mathcal{P}}(n) \mod b).$$

Clearly $\xi_{\mathcal{P}} \in \mathcal{T}_{b,\mathcal{P}}$.

Theorem 3 (Becher, Marchionna and Tenenbaum 2023)

Let integer $b \geqslant 2$ and $\mathcal{P} \subset \mathbb{P}$. The number $\xi_{\mathcal{P}}$ is simply normal to base b if, and only if, $\sum_{p \in (\mathbb{P} \setminus \mathcal{P})} 1/p = \infty$. Moreover, defining for $k = 0, \ldots, (b-1)$

$$\varepsilon_{N,k} := \left| \frac{1}{N} |\{n : 1 \leqslant n \leqslant N, (\Omega_{\mathcal{P}}(n) \mod b) = k\}| - \frac{1}{b} \right|$$

we have

$$\varepsilon_{N,k} \ll \frac{1}{b} \mathrm{e}^{-E(N)/180b^2}, \text{ where } E(N) := \sum_{p \leqslant N,\, p \in (\mathbb{P} \setminus \mathcal{P})} 1/p \quad (N \geqslant 1)$$

Proof of Theorem 3

Lemma

Let $\mathcal{P} \subset \mathbb{P}$ and let b be an integer $\geqslant 2$. The number $\xi_{\mathcal{P}}$ is simply normal to base b if, and only if,

$$\frac{1}{N} \sum_{1 \leqslant n \leqslant N} \mathrm{e}\big(a\Omega_{\mathcal{P}}(n)/b\big) = o(1) \quad (a = 1, 2, \ldots b-1,\ N \to \infty).$$

with usual notation $\mathrm{e}(u) := \mathrm{e}^{2\pi i u}$ $(u \in \mathbb{R})$.

Proof of Theorem 3

Lemma

Let $\mathcal{P} \subset \mathbb{P}$ and let b be an integer $\geqslant 2$. The number $\xi_{\mathcal{P}}$ is simply normal to the base b if, and only if,

$$\frac{1}{N} \sum_{1 \leqslant n \leqslant N} \mathrm{e}\big(a\Omega_{\mathcal{P}}(n)/b\big) = o(1) \quad (a = 1, 2, \ldots b-1,\ N \to \infty).$$

with usual notation $\mathrm{e}(u) := \mathrm{e}^{2\pi i u}$ $(u \in \mathbb{R})$.

Proof.

The necessity of the criterion is clear. We show the sufficiency. Define

$$b_{k,N} := \frac{1}{N}\big|\{1 \leqslant n \leqslant N : (\Omega_{\mathcal{P}}(n) \mod b) = k\}\big| \quad (0 \leqslant k < b,\ N \geqslant 1).$$

Then,

$$b_{k,N} = \frac{1}{bN} \sum_{0 \leqslant a < b} \mathrm{e}(-ak/b) \sum_{1 \leqslant n \leqslant N} \mathrm{e}\big(a\Omega_{\mathcal{P}}(n)/b\big) = \frac{1}{b} + o(1)$$

because by hypothesis all inner sums with $a \neq 0$ contribute $o(N)$. □

Proof of Theorem 3

Define

$$S(N;a/b) := \sum_{1\leqslant n\leqslant N} \mathrm{e}\big(a\Omega_{\mathcal{P}}(n)/b\big) \quad (a\in\mathbb{Z},\ b\geqslant 2,\ N\geqslant 1).$$

Ramanujan J.
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Corrig. **51**, n° 1 (2020), 243-244.

Moyennes effectives de fonctions multiplicatives complexes*

Gérald Tenenbaum

Abstract. We establish effective mean-value estimates for a wide class of multiplicative arithmetic functions, thereby providing (essentially optimal) quantitative versions of Wirsing's classical estimates and extending those of Halász. Several applications are derived, including: estimates for the difference of mean-values of so-called pretentious functions, local laws for the distribution of prime factors in an arbitrary set, and weighted distribution of additive functions.

Proof of Theorem 3

Notice $\{a \in \mathbb{Z} : |a| \leqslant \frac{1}{2}b\}$ describes a complete set of residues $\pmod b$.

Whenever a and b are coprime, $b \geqslant 2$ and $|a| \leqslant b/2$, apply Tenenbaum's effective mean-value estimates for a arithmetic multiplicative functions (quantitative versions of Wirsing's classical estimates):

$$S(N; a/b) \ll N e^{-a^2 E(N)/(180 b^2)}.$$

So, if $\sum_{p \in (\mathbb{P} \setminus \mathcal{P})} 1/p = \infty$ holds, $S(N, a/b) = o(N)$ as $N \to +\infty$ and $\xi_{\mathcal{P}}$ is simply normal to the base b.

Proof of Theorem 3

If, on the contrary, $\sum_{p\in(\mathbb{P}\setminus\mathcal{P})} 1/p < \infty$ we need to prove $S(N,a/b) \gg N$.

Use $\sum_{p\in(\mathbb{P}\setminus\mathcal{P}),\ p\leqslant N} \frac{\log p}{p} \ll \eta_N \log N$, for some $\eta_N \to 0$.

A possible choice is $\eta_N := \min_{1\leqslant z\leqslant N} \left(\frac{\log z}{\log N} + \sum_{p\in(\mathbb{P}\setminus\mathcal{P}),\, p>z} \frac{1}{p} \right)$.

Apply Tenenbaum's effective version of a result of Delange,

$$S(N;a/b) = \frac{N}{\log N} \left(\prod_p \sum_{p^\nu \leqslant N} \frac{e(\nu a \Omega_{\mathcal{P}}(p)/b)}{p^\nu} + O\left(\eta_N^{1/8} \mathrm{e}^{E(N)} + \frac{\mathrm{e}^{E(N)}}{\log^{1/12} N} \right) \right)$$

Show

$$\log N \ll \prod_p \sum_{p^\nu \leqslant N} \frac{e(\nu a \Omega_{\mathcal{P}}(p)/b)}{p^\nu}$$

and conclude $S(N,a/b) \gg N$, $\square$

Example of a normal number in $\mathcal{T}_{b,\mathcal{P}}$ for singleton $\mathcal{P}$

Theorem 4 (Becher, Carton and Heiber 2018)

We construct a number in $\mathcal{T}_{b,\mathcal{P}}$ for $b = 2$ and $\mathcal{P} = \{2\}$, normal to base 2.

Proof of Theorem 4

Fix alphabet of two symbols. We construct a sequence x such that $x = even(x)$.
A word x is ℓ-perfect if each of the 2^ℓ many words of length ℓ occurs aligned in x the same number of times .

The construction consists in concatenating perfect sequences $s_1, s_2, \ldots$ such that $|s_{i+1}| = 2|s_i|$, $s_i = even(s_{i+1})$ and each s_i is ℓ_i-perfect for ℓ_i a power of 2.

Start with $s_1 = 01$, $s_2 := 1001$ and $\ell_2 = 1$. For $i > 2$,

If $|s_i| = \ell_i 2^{2\ell_i}$ and s_i is ℓ_i-perfect then construct s_{i+1} by transforming the n-th occurrence of u into $w = v \vee u$, where v is the n-th word of length ℓ_i in lexico order. Then s_i is $2\ell_i$-perfect, because all words of length $2\ell_i$ occur once in s_{i+1}. Set $\ell_{i+1} := 2\ell_i$.

If $|s_i| = m2^{2\ell_i}$, with m a multiple of ℓ_i but $m \neq \ell_i 2^{\ell_i}$, and s_i is ℓ_i-perfect then construct s_{i+1} as before, but now with multiplicity m. Notice that s_{i+1} is ℓ_i-perfect, each word of length ℓ_i occurs twice the number of times it occurred before. Set $\ell_{i+1} := \ell_i$.

A metric theorem in $\mathcal{T}_{b,\mathcal{P}}$, $\mathcal{P} = \{2\}$, for absolute normality

Theorem 5 (Aistleitner, Becher and Carton 2019)

Let integer $b \geqslant 2$, $\mathcal{P} = \{2\}$ and μ be the uniform probability measure on $\mathcal{T}_{b,\mathcal{P}}$. Then, μ-almost all elements of $\mathcal{T}_{b,\mathcal{P}}$ are absolutely normal.

Two positive integers are multiplicatively dependent if one is a rational power of the other.

In case b and r are multiplicatively dependent, Theorem 5 follows immediately from Theorem 2 because normality to base b is equivalent to normality to any multiplicatively dependent base r.

Weyl's criterion

Again we write $e(u)$ to denote $\mathrm{e}^{2\pi iu}$.

A sequence $x_1, x_2, \ldots$ of real numbers is equidistributed modulo 1 if and only if for all non-zero integers h,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} e(hx_n) = 0.$$

A number x is Borel normal to integer base $b \geqslant 2$ exactly when $(b^n x)_{n \geqslant 0}$ is equidistributed modulo 1 which holds exactly when for all non zero integers h,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N} e(hb^n x) = 0.$$

Proof of Theorem 5

We adapt the work of Cassels 1959 and Schmidt 1961/1962. Our argument is also based on giving upper bounds for certain Riesz products.

Cassels worked on a Cantor-type set of real numbers whose ternary expansion avoids the digit 2 (hence not normal to base 3) and he established regularity properties of the uniform measure supported on this Cantor-type set.

In contrast, we deal with the measure μ which is the uniform measure on the set of real numbers which respect the digit dependencies.

Proof of Theorem 5

To prove μ-almost all $x \in \mathcal{T}_{b,\mathcal{P}}$ are normal to base r use Weyl criterion.

1 Define initial segments of subexponential growth M_k for $k = 1, 2, 3\dots$ Fix a positive h. Define sets

$$Bad_k = \Big\{ x \in \mathcal{T}_{b,\mathcal{P}} : \frac{1}{M_k - M_{k-1}} \sum_{n=M_{k-1}}^{M_k} e(r^n hx) > 1/k. \Big\}$$

2 Prove $\mu(Bad_k)$ is small enough to convergent series summing over k

Give upper bound of mean value of $\Big| \frac{1}{M_k - M_{k-1}} \sum_{j=M_{k-1}}^{M_k} e(r^n hx) \Big|^2$.

Using Chebishev inequality give an upper bound for $\mu(Bad_k)$.

3 Apply Borel Cantelli, obtain μ-almost all $x \in \mathcal{T}_{\mathcal{P}}$ outside $\bigcup_k Bad_k$.

4 For any N there is k such that $N - M_k = o(N)$.Then,μ-almost all x

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} e(r^n hx) = 0.$$

5 Countably many h and $r \geqslant 2$ multiplicatively independent to b. □

Proof of Theorem 5

Lemma

Let $r \geqslant 2$ be multiplicatively independent to b.
Then for all integers $h \geqslant 1$ there exist constants $c > 0$ and $k_0 > 0$, depending only on b, r and h such that for all positive integers k, m satisfying $k_0 \leqslant k + 1 + 2\log_r b \leqslant m$,

$$\int_0^1 \left| \sum_{j=m+1}^{m+k} e(r^j hx) \right|^2 d\mu(x) \leqslant k^{2-c}$$

Proof of Theorem 5

Lemma (adapted from Schmidt's Hilfssatz 5, 1961)

Let r and b be multiplicatively independent. There is a constant $c > 0$, depending only on r and b, such that for all positive integers K and L with $L \geqslant b^K$,

$$\sum_{n=0}^{N-1} \prod_{\substack{k=K+1 \\ k \text{ odd}}}^{\infty} \left(\frac{1}{b} + \frac{b-1}{b} \left| \cos\left(\pi r^n L b^{-k} \right) \right| \right) \leqslant 2N^{1-c}.$$

The proof of Schmidt's *Hilfssatz*, uses that the function $|\cos(\pi x)|$ is periodic, the fact that $|\cos(\pi x)| \leqslant 1$ and that $|\cos(\pi/b^2)| < 1$.

All these properties also hold for the function $\dfrac{1}{b} + \dfrac{b-1}{b} \left|\cos(\pi x)\right|$.

□

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