On normal numbers

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Workshop on Logic Language and Information, July 2014

Normal numbers

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Definition

- A base is an integer b greater than or equal to 2.
- For a real number x, the expansion of x in base b is a sequence $(a_k)_{k\geq 1}$ of integers a_k from $\{0, 1, \ldots, b-1\}$ such that

$$x = \lfloor x \rfloor + \sum_{k\geq 1} \frac{a_k}{b^k} = \lfloor x \rfloor + 0.a_1 a_2 a_3 \ldots$$

  where infinitely many of the a_k are not equal to $b-1$.

Normal numbers

Definition (Borel, 1909)

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Normal numbers

Theorem (Borel 1922, Niven and Zuckerman 1951)

A real number x is normal to base b if, for every $k \geq 1$, every block of k digits occurs in the expansion of x in base b with the limiting frequency $1/b^k$.

Normal to no base

$0.01\ 002\ 0003\ 00004\ 000005\ 0000006\ 00000007\ 000000008\ldots$
is not simply normal to base 10.

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is simply normal to base 10, but not simply normal to base 100.

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$0.0123456789\ \ 0123456789\ \ 0123456789\ \ 0123456789\ \ 0123456789\ldots$
is simply normal to base 10, but not simply normal to base 100.

Rational numbers are normal to no base.

Normal to some base

Theorem (Champernowne, 1933)

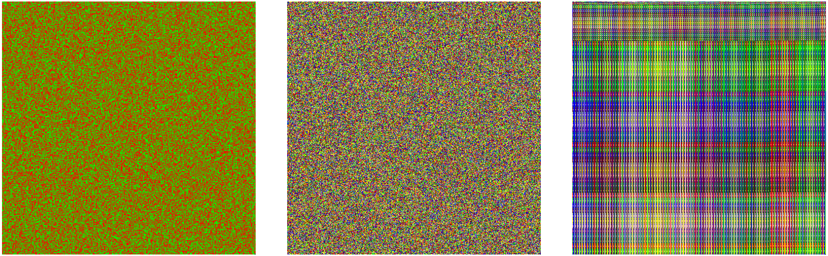
$0.12345678910111213141516171819202122232425\ldots$ *is normal to base* 10.

Normal to some base

Theorem (Champernowne, 1933)

$0.12345678910111213141516171819202122232425\ldots$ *is normal to base* 10*.*

It is unknown if it is normal to bases that are not powers of 10.



base 2 base 6 base 10

Plots of the first 250000 digits of Champernowne's number.

Normal to all bases

Theorem (Borel 1909)

Almost all real numbers are absolutely normal.

Problem (Borel 1909)

Give one example.

Normal to all bases

Theorem (Borel 1909)

Almost all real numbers are absolutely normal.

Problem (Borel 1909)

Give one example. Is any of the usual mathematical constants, such as π, e, or $\sqrt{2}$, simply normal to some base?

Normal to all bases

First ones by Lebesgue and independently Sierpiński in 1917. Not computable.
There are many other constructions 1917–2013.

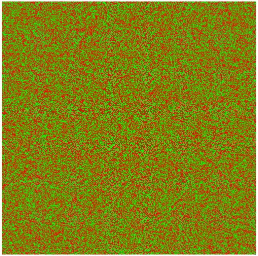
Theorem (Becher, Heiber, Slaman, 2013)

There is an algorithm that computes an absolutely normal number with just above quadratic complexity.

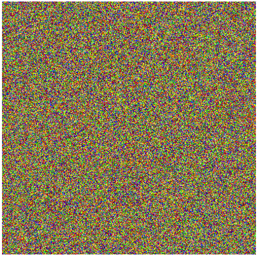
Normal to all bases

Output of our algorithm programmed by Martin Epszteyn, 2013.

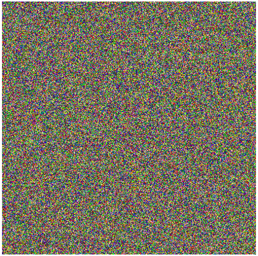
$0.40312905420038091323714283808270591027651167776241897751108966\ldots$



base 2



base 6



base10

Plots of the first 250000 digits of the output of our algorithm.

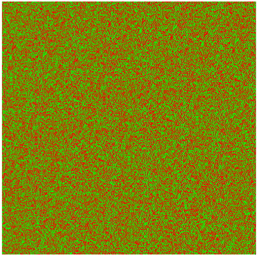
Also in 2013 polynomial-time algorithms by Lutz & Mayordomo and Figueira & Nies.

Normal to one base but not to another

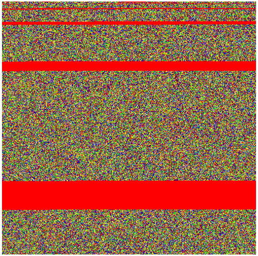
Bailey and Borwein (2012) proved that the Stoneham number $\alpha_{2,3}$,

$$\alpha_{2,3} = \sum_{k \geq 1} \frac{1}{3^k \; 2^{3^k}}$$

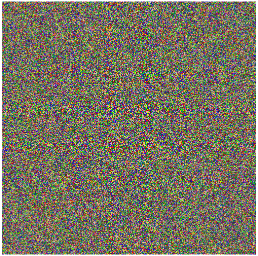
is normal to base 2 but not simply normal to base 6.



base 2



base 6



base 10

Plots of the first 250000 digits of Stoneham number $\alpha_{2,3}$.

Constructions can not be done by just concatenation

Suppose x, y are b-adic rationals with precision n and z is a b-adic rational less than b^{-n} and with precision m. Then,

$$(x + y)_b[n + 1 \ldots (n + m)] = (y + z)_b[n + 1 \ldots (n + m)].$$

However,

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$$(x+y)_b[n+1\ldots(n+m)] = (y+z)_b[n+1\ldots(n+m)].$$

However, for a base a that is not a power of b, in general,

$$(x+z)_a[n\ldots n+m] \neq (y+z)_a[n\ldots n+m].$$

For instance, for $b = 10$ and $a = 3$, $x = 25/100$, $y = 50/100$, $z = 17/10000$

$x = (0.25)_3 =$	$0.020202020202\ldots$	$y = (0.50)_3 =$	$0.110111200011\ldots$
$x + z = (0.2517)_3 =$	$0.020210111012\ldots$	$y + z = (0.5017)_3 =$	$0.111112201221\ldots$

Normality to different bases

Definition

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The positive integers that are not perfect powers, $2, 3, 5, 6, 7, 10, 11, \ldots$, are pairwise multiplicatively independent.

Theorem (Maxfield 1953)

Let b_1 and b_2 multiplicatively dependent. For any real number x, x is normal to base b_1 if and only if x is normal to base b_2.

Normality to different bases

Theorem (Cassels, 1959)

Almost every real number in the middle third Cantor set is normal to every base which is not a power of 3.

Theorem (Schmidt 1961/1962)

For any given set B *of bases closed under multiplicative dependence, there are real numbers normal to every base in* B *and not normal to any base in its complement. Furthermore, there is a real* x *is computable from* B.

Are there arithmetical properties tied to normality to different bases?

Borel hierarchy for subsets of the real numbers

Recall Borel hierarchy for subsets of the real numbers is the stratification of the σ-algebra generated by the open sets with the usual interval topology.

- A set is $\mathbf{\Sigma}^0_1$ iff it is open.
- A set is $\mathbf{\Pi}^0_1$ iff it is closed.
- A set is $\mathbf{\Sigma}^0_{n+1}$ iff it is countable union of $\mathbf{\Pi}^0_n$ sets.
- A set is $\mathbf{\Pi}^0_{n+1}$ iff it is a countable intersection of $\mathbf{\Sigma}^0_n$ sets.

A set is is hard for a Borel class if every set in the class is reducible to it by a continuous map.

A set is complete for a class if it is hard for this class and belongs to the class.

Effective Borel hierarchy

Arithmetic hierarchy of formulas in the language of second-order arithmetic:

- formulas involve only quantification over integers.
- atomic formulas assert algebraic identities between integers or membership of real numbers in intervals with rational endpoints.
- a formula is Π^0_0 and Σ^0_0 if all its quantifiers are bounded.
- a formula is Σ^0_{n+1} if it has the form $\exists x\, \theta$ where θ is Π^0_n.
- a formula is Π^0_{n+1} if it has the form $\forall x\, \theta$ where θ is Σ^0_n.

A set of real numbers is Σ^0_n (respectively Π^0_n) in the effective Borel hierarchy if membership in that set is definable by a formula Σ^0_n (respectively Π^0_n).

Effective reductions are computable maps.

Normality in the effective Borel hierarchy

A real number x is normal to base b if, and only if,

- for every $k \geq 1$, x is simply normal to base b^k,
- for every $k \geq 1$, and for every block d of k digits,

$$\lim_{n\to\infty} \frac{\#occ(a_1 a_2 \ldots a_n, d)}{n} = \frac{1}{b^k}$$

where $a_1 a_2 a_3 \ldots$ is the expansion of x in base b. Thus,

$$\forall k\ \forall d\ \forall \epsilon\ \exists n\ \forall m \geq n\ \ \theta(a_1 a_2 .. a_m, b)$$

where $\theta(a_1 a_2 .. a_m, b)$ is computable in x.

This is $\forall\exists\forall$ formula with one free real x and one free integer b, quantification only on the integers.

Normality in the effective Borel Hierarchy

In the effective Borel hierarchy for susbets of real numbers,

- Normal to a fixed base b is Π^0_3.
- Normal to all bases is also Π^0_3.
- Normal to some base is Σ^0_4.

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Are they complete?

Normal to all bases is complete at the third level

Asked first by Kechris 1994.

Theorem (Ki and Linton 1994)

The set of real numbers that are normal to any fixed base is Π^0_3*-complete.*

Theorem (Becher, Heiber, Slaman 2014)

The set of real numbers that are absolutely normal is Π^0_3*-complete.*

Normal to all bases is complete at the third level

To prove it we give a computable function f taking Π^0_3 sentences in first order arithmetic (no free variables) to indices for computable real numbers.

- On input φ, the function f constructs a real number x so that x is normal to all bases if and only if φ is true.

Effective case implies the general case

- Every Σ^0_n set is $\mathbf{\Sigma}^0_n$ and every Π^0_n set is $\mathbf{\Pi}^0_n$.
- For every $\mathbf{\Sigma}^0_n$ set A there is a Σ^0_n formula and a real parameter such that membership in A is defined by that Σ^0_n formula relative to that real parameter.

Since computable maps are continuous, proofs of hardness in the effective hierarchy yield proofs of hardness in general by relativization.

Normality to different bases

Theorem (Becher, Slaman 2014)

Let B be a set of bases closed under multiplicative dependence defined by a Π^0_3 formula. There is a real number x that is normal to every base in B and not simply normal to any of the bases outside B. Furthermore, x is uniformly computable in the Π^0_3 formula that defines B.

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The assertion on simple normality answers to Brown, Moran & Pearce.

We further show that the discrepancy functions for multiplicatively independent bases are pairwise independent.

Normality to different bases

In the proof we give a computable function f taking a Π^0_3 formula in first order arithmetic with one free variable $\varphi(b) = \forall n \exists m \forall p \ \theta(n, m, p, b)$, to indices for computable numbers.

- On input φ, the function f constructs a real number x so that, for every base b, x is normal to base b if and only if $\varphi(b)$ is true.

By relativization, the theorem holds for the non-effective case.

Normal to some base is Σ_4^0-complete

We confirmed a conjecture by Achim Ditzen, 1994.

Theorem (Becher, Slaman 2014)

The set of real numbers that are normal to some base is Σ_4^0*-complete.*

Normal to some base is Σ_4^0-complete

To prove it we give a computable function f taking Σ_4^0 sentences (no free variables) to indices for computable real numbers.

- On input φ, f constructs a real number x such that x is normal to at least one base if and only if φ is true.
- Let φ be a Σ_4^0 sentence and let θ be the Π_3^0 formula with one free integer variable such that $\varphi = \exists b\ \theta(b)$. Apply the previous theorem with input B as defined by θ.

By relativization, the proof implies the theorem holds for the non-effective case.

Arithmetical independence in normality to different bases

The set of bases for which x is normal can coincide with any other property of elements of the set of numbers that are not perfect powers definable by a Π^0_3 formula relative to x.

Thus, other than being closed by multiplicative dependence, the set of bases for normality can be arbitrary.

A fixed point

Theorem (Becher, Slaman 2014)

For any Π^0_3 formula φ in second order arithmetic there is a computable real number x such that, for any non-perfect power b, x is normal to base b if and only if $\varphi(x,b)$ is true.

(recall that a formula in arithmetic involves only quantification over integers.)

Simple normality to different bases

From the definition of simple normality, for any base b,

- Simply normality to base b^k implies simple normality to base b^ℓ, for each ℓ that divides k.
- Simple normality to infinitely many powers of base b implies normality to base b (Long, 1957)

Bugeaud asked:

What are the necessary and sufficient conditions on a set of bases so that there is a real number which is simply normal exactly to the bases in such a set?

Simple normality to different bases

Theorem (Becher, Bugeaud, Slaman 2013)

Let M be any function from the set of integers that are not perfect powers to their subsets such that

- *for each b, if $b^{km} \in M(b)$ then $b^k \in M(b)$*
- *if $M(b)$ is infinite then $M(b) = \{b^k : k \geq 1\}$.*

Then, there is a real x which is simply normal to exactly the bases specified by M. Furthermore, the real x is computable from the function M.

The theorem gives a complete characterization (necessary and sufficient conditions).

Normality to different bases

The theorems establish the logical independence of normality to multiplicatively independent bases.

The set of bases to which a real number can be normal is not tied to any arithmetical properties other than multiplicative dependence.

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The End

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