New normal numbers

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Seminario de Probabilidades del Departamento de Matemática, FCEyN, UBA, julio 2014

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Equivalently, x is normal to base b if every block of digits occurs in the expansion of x in base b with limiting frequency equal to $1/b^k$, where k is the block length.

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- Stoneham number $\alpha_{2,3} = \sum_{k\geq 1} \frac{1}{3^k\ 2^{3^k}}$ is normal to base 2 but not simply normal to base 6 (Bailey, Borwein, 2012).

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Conjecture (Borel 1950)

Irrational algebraic numbers are absolutely normal.

Normality to different bases

Two integers are multiplicatively dependent if one is a rational power of the other. Example: 2 and 8 are multiplicatively dependent.

The set of positive integers that are not perfect powers $\{2, 3, 5, 6, 7, 10, 11, \ldots\}$ are pairwise mutually independent.

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Theorem (Cassels 1959; Schmidt 1961/1962; Becher, Slaman 2013)

For any subset S of the multiplicative dependence classes, there is a real x which is normal to the bases in S and not simply normal to the bases in the complement of S. Furthermore, the real x is computable from S.

Simple normality to different bases

Theorem (Becher, Bugeaud, Slaman 2013)

Let M be any function from the multiplicative dependence classes to their subsets such that

- *for each b, if $b^{km} \in M(b)$ then $b^k \in M(b)$*
- *if $M(b)$ is infinite then $M(b) = \{b^k : k \geq 1\}$.*

Then, there is a real x which is simply normal to exactly the bases specified by M. Furthermore, the real x is computable from the function M.

The theorem gives a complete characterization (necessary and sufficient conditions).

Normal numbers and uniform distribution modulo one

Normality as uniform distribution modulo one

Theorem (Wall 1949)

A real x is normal to base b if and only if $(b^k x)_{k \geq 0}$ equidistributes modulo one for Lebesgue measure.

Belief

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Taking this to the extreme and applying it to singletons one arrives at the folklore conjecture that constants such as π, e, $\sqrt{2}$ are absolutely normal.

Normality using Weyl's criterion

A real x is normal to base b if and ony if $(b^k x)_{k \geq 0}$ is uniformly distributed modulo one. That is, if and only if, for every non-zero integer t,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} e^{2\pi i t b^k x} = 0.$$

Measures whose Fourier transform decays quickly

Lemma (application of Davenport, Erdős, LeVeque's Theorem)

Let μ be a measure, I an interval and b a base. If for every non-zero integer t,

$$\sum_{n\geq 1} \frac{1}{n} \int_I \left| \frac{1}{n} \sum_{k=0}^{n-1} e^{2\pi i t b^k x} \right|^2 d\mu(x) \; < \infty$$

then for μ-almost all x in interval I are normal to base b.

Normal numbers and Diophantine approximations

Irrationality exponent

Definition (Liouville 1855)

The irrationality exponent of a real number x, is the supremum of the set of real numbers z for which

$$0 < \left| x - \frac{p}{q} \right| < \frac{1}{q^z}$$

is satisfied by an infinite number of integer pairs (p, q) with $q > 0$.

Irrationality exponent

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- Rational numbers have irrationality exponent equal to 1.

Jarník's fractal

Fix a real a greater than 2. Jarník gave a Cantor-like construction of a set in $[0, 1]$.
Let $(m_k)_{k\geq 1}$ be an appropriate increasing sequence of positive integers.
For each $k \geq 1$,

$$E(k) = \dot{\bigcup_{\substack{q \text{ prime} \\ m_k < q < 2m_k}}} \left\{ x \in \left(\frac{1}{q^a}, 1 - \frac{1}{q^a} \right) : \exists p \in \mathbb{N}, \left| \frac{p}{q} - x \right| < \frac{1}{q^a} \right\}$$

$E(k)$ has about $\dfrac{m_k^2}{\log m_k}$ disjoint intervals, each of length at least $\dfrac{2}{(2m_k)^a}$.

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Jarník's's fractal for the real a is

$$J = \bigcap_{k \geq 1} E(k).$$

Absolutely normal Liouville numbers

Kaufman (1981) defined for each a greater than 2, a measure on Jarník's fractal for a whose Fourier transform decays quickly.

Bluhm (2000) defined a measure such that it is supported by the Liouville numbers and its Fourier transform decays quickly.

Theorem (Bugeaud 2002)

There is an absolutely normal Liouville number.

Theorem (Becher, Heiber, Slaman 2014)

There is a computable absolutely normal Liouville number.

Uniform measure on Jarník's fractal

Consider the uniform measure on Jarník's fractal for a. The set of reals with irrationality exponent equal to a has uniform-measure equal to 1.

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Theorem (Jarník's 1929, Besicovich 1934)

For every real a greater than or equal to 2, the set of reals with irrationality exponent equal to a has Hausdorff dimension $2/a$.

Theorem (Mass Distribution Principle)

Let ν be a finite measure, d a positive real number. Suppose that there are positive reals c and ϵ such that for all intervals I with $|I| < \epsilon$,

$$\nu(I) \leq c\, |I|^d.$$

Then, for any set X with Hausdorff dimension less than d, we have $\nu(X) = 0$.

Take $d = 2/a$.

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- Its support set is strictly included Kaufman's support set.
- At each level of the Cantor-like construction, each new interval concentrates most of Kaufman's measure of the original interval.
- Thus, the support set for this uniform mesure is the set of reals which are absolutely normal and have irrationality exponent equal to a.

Normality and prescribed irrationality exponent

Theorem (Becher, Bugeaud, Slaman 2014)

For every real a greater than or equal to 2 and for every set S of bases satisfying the conditions for simple normality there is a real x such that

- *x has irrationality exponent equal to a,*
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Furthermore, the real x is computable from a and S.

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The End

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Uniform distribution modulo one

Let $\{x\}$ denote the fractional part of a real x.

Definition

A sequence of reals $(x_j)_{j\geq 1}$ is uniformly distributed modulo one if, for every subinterval I of the unit interval,

$$\lim_{n\to\infty} \frac{\#\{j : 1 \leq j \leq n \text{ and } \{x_j\} \in I\}}{n} = |I|.$$

Weyl's criterion

A sequence is u.d. in the unit interval if for any Riemann integrable function f, $\int_0^1 f(x)dx$ is the limit of the average values of f on the sequence.

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Theorem (Weyl's Criterion)

A sequence of reals $(x_j)_{j\geq 1}$ is uniformly distributed modulo one in the unit interval if and only if for every complex-valued 1-periodic continuous function f,

$$\int_0^1 f(x)dx = \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} f(x_j).$$

That is, if and only if, for every non-zero integer t,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} e^{2\pi i t x_j} = 0.$$