On normal numbers

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Normal numbers

Normality is a basic form of randomness for real numbers. It asks that the expansions of real numbers obey the Law of Large Numbers.

Definition

- A base is an integer b greater than or equal to 2.
- For a real number x, the expansion of x in base b is a sequence $(a_k)_{k\geq 1}$ of integers a_k from $\{0, 1, \ldots, b-1\}$ such that

$$x = \lfloor x \rfloor + \sum_{k\geq 1} \frac{a_k}{b^k} = \lfloor x \rfloor + 0.a_1 a_2 a_3 \ldots$$

where infinitely many of the a_k are not equal to $b - 1$.

Normal numbers

Definition (Borel, 1909)

- A real number x is simply normal to base b if in the expansion of x in base b, each digit occurs with limiting frequency equal to $1/b$.
- A real number x is normal to base b if x is simply normal to every base b^k, for every positive integer k.
- A real number x is absolutely normal if x is normal to every base.

Normal numbers

Theorem (Borel 1922, Pillai 1940)

A real number x is normal to base b if, for every $k \geq 1$, every block of k digits occurs in the expansion of x in base b with the limiting frequency $1/b^k$.

Examples

$0.01\ 002\ 0003\ 00004\ 000005\ 0000006\ 00000007\ 000000008\ldots$
is not simply normal to base 10.

$0.0123456789\ 0123456789\ 0123456789\ 0123456789\ 0123456789\ldots$
is simply normal to base 10, but not simply normal to base 100.

Problem (Borel, 1909)

Is any of the usual mathematical constants, such as π, e, or $\sqrt{2}$, normal to some base?

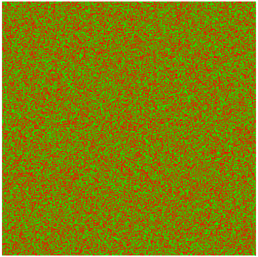
Examples

Theorem (Champernowne, 1933)

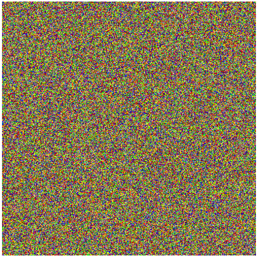
$0.12345678910111213141516171819202122232425\ldots$ *is normal to base* 10.

It is unknown if it is normal to bases that are not powers of 10.

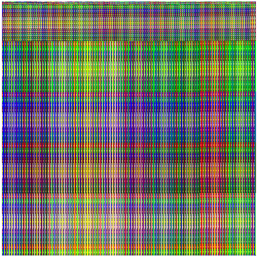
Plot of the first 250000 digits of Champernowne's number.



base 2



base 6



base 10

Existence of absolutely normal numbers

Theorem (Borel 1909)

Almost all real numbers are absolutely normal.

Problem (Borel 1909)

Give one example.

Conjecture (Borel 1950)

Irrational algebraic numbers are absolutely normal.

Old problems, some new results

- Exhibit an absolutely normal number.
  We gave an algorithm to compute one in just over quadratic time
- How much independence there is between normality to different bases?
  We gave a logical analysis of normality to different bases.
- Construct normal numbers also with properties other than normality (geometric, algebraic or number-theoretic).
  We gave algorithms for normal numbers with Diophantine properties.

Joint work of Becher and Slaman, partially with Bugeaud and partially with Heiber, in 2013 and 2014.

Today I will only talk abut the first two problems.

First announcement: exhibit an absolutely normal number

Examples of absolutely normal numbers

First constructions of absolutely normal numbers were done by Lebesgue and Sierpiński, independently, in 1917. They were not computable.

A real number x is computable when there is a computable function that outputs each of its digits in its expansion in some base.

Theorem (Turing 1937; see Becher, Figueira, Picchi 2007)

There is a computable absolutely normal number.

Other computable instances Schmidt 1961/1962; Becher, Figueira 2002, among others.

Absolutely normal numbers in polynomial time

Theorem (Lutz, Mayordomo 2013; Figueira, Nies 2013; Becher, Heiber, Slaman 2013)

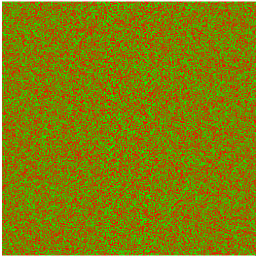
There is a polynomial-time algorithm to compute an absolutely normal number.

The algorithm by Becher, Heiber, Slaman (2013) has complexity just above quadratic: for any computable non-decreasing unbounded function f, there is an algorithm with complexity $O(f(n)\, n^2)$.

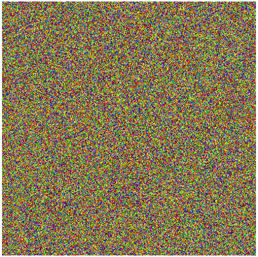
The output of our algorithm

Programmed by Martin Epszteyn, 2013.

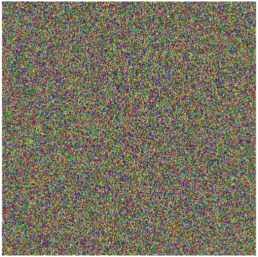
$0.40312905420038091323714283808270591027651167776241897751108963 66\ldots$



base 2



base 6

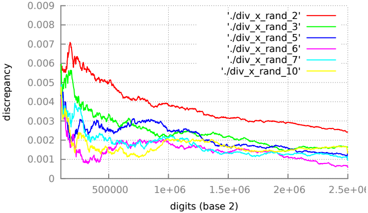


base10

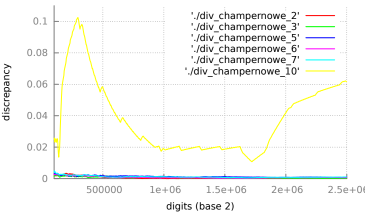
Parameters $t_i = (3 * \log(i)) + 3$; $\epsilon_i = 1/t_i$ Initial values $t_1 = 3$; $\epsilon_1 = 1$.

Discrepancy of simple normality

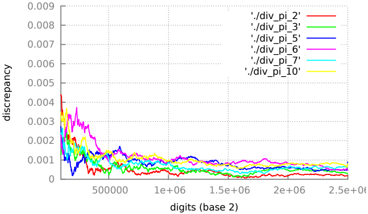
$$D_{b,n}(x) = \max_{d \in \{0,1,\ldots b-1\}} \left| \frac{\#\text{occurences of } d \text{ in } x_{b,1}..x_{b,n}}{n} - \frac{1}{b} \right|$$



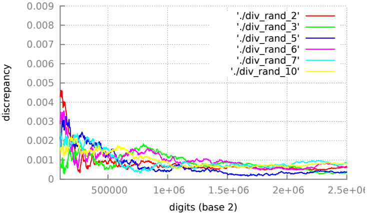
Output of our algorithm



Champernowne



π



pseudo-random

Fast computation versus fast convergence?

The three algorithms for computing absolutely normal numbers
(Lutz, Mayordomo 2013; Figueira, Nies 2013; Becher, Heiber, Slaman 2013)
achieve speed of computation by sacrificing speed of convergence to normality
(delaying new bases and allowing slow convergence of discrepancy)

Our algorithm is a descendant of Turing's algorithm (1937).
The other two are based on martigales, a usual tool in algorithmic randomness.

Problem (Becher, Heiber Slaman 2013)

Are there polynomial-time algorithms with fast convergence to normality?

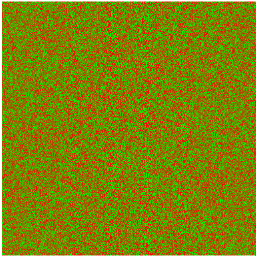
Second announcement: logic and normality to different bases

Normal to one base but not to another

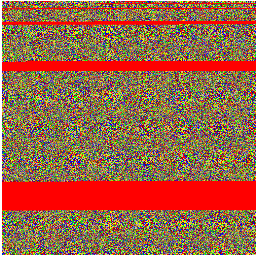
Bailey and Borwein (2012) proved that the Stoneham number $\alpha_{2,3}$,

$$\alpha_{2,3} = \sum_{k \geq 1} \frac{1}{3^k \ 2^{3^k}}$$

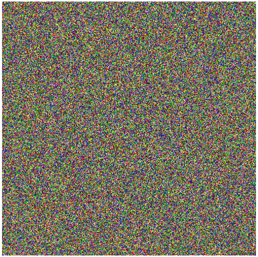
is normal to base 2 but not simply normal to base 6.



base 2

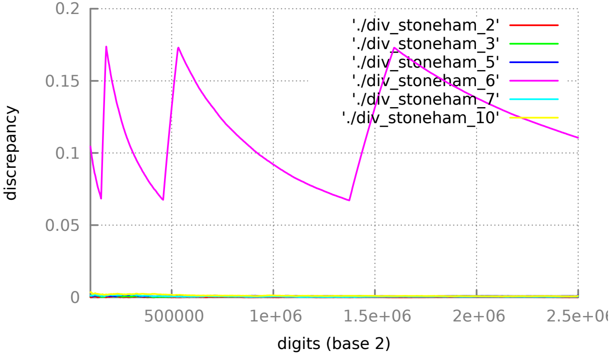


base 6



base 10

Simple Discrepancy of Stoneham $\alpha_{2,3}$



Normality to different bases

Definition

Two positive integers are multiplicatively dependent if one is a rational power of the other. Thus, 2 and 8 are multiplicatively dependent. But 2 and 6 are independent.

The positive integers that are not perfect powers, $2, 3, 5, 6, 7, 10, 11, \ldots$, are pairwise multiplicatively independent.

Theorem (Maxfield 1953)

Let b_1 and b_2 multiplicatively dependent. For any real number x, x is normal to base b_1 if and only if x is normal to base b_2.

Normality to different bases

Theorem (Cassels, 1959)

Almost every real number in the middle third Cantor set is normal to every base which is not a power of 3.

Theorem (Schmidt 1961/1962)

For any given set B *of bases closed under multiplicative dependence, there are real numbers normal to every base in* B *and not normal to any base in its complement. Furthermore, there is a real* x *is computable from* B.

We further show that the discrepancy functions for multiplicatively independent bases are pairwise independent.

Normality to different bases

Recall that the Borel hierarchy for subsets of the real numbers is the stratification of the σ-algebra generated by the open sets with the usual interval topology.

When we restrict to intervals with rational endpoints and computable countable unions and intersections, we obtain the effective Borel hierarchy.

Normality to different bases

Theorem (Becher, Slaman 2014)

Let B be a set of bases closed under multiplicative dependence defined by a Π^0_3 formula. There is a real number x that is normal to every base in B and not simply normal to any of the bases outside B. Furthermore, x is uniformly computable in the Π^0_3 formula that defines B.

We confirmed a conjecture by Achim Ditzen (1994):

Theorem (Becher, Slaman 2014)

The set of real numbers that are normal to some base is Σ^0_4-complete.

Normality to different bases

A fixed point!

Theorem (Becher, Slaman 2014)

For any Π^0_3 formula φ in second order arithmetic there is a computable real number x such that, for any non-perfect power b, x is normal to base b if and only if $\varphi(x, b)$ is true.

(Recall that a formula in arithmetic involves only quantification over integers.)

Simple normality to different bases

From the definition of simple normality, for any base b,

- Simply normality to base b^k implies simple normality to base b^ℓ, for each ℓ that divides k.
- Simple normality to infinitely many powers of base b implies normality to base b. (Long, 1957)

Bugeaud asked: For which sets of bases there are real numbers that are simply normal exactly to the bases those sets?

Simple normality to different bases

Theorem (Becher, Bugeaud, Slaman 2013)

Let M be any function from the multiplicative dependence classes to their subsets such that

- *for each b, if $b^{km} \in M(b)$ then $b^k \in M(b)$*
- *if $M(b)$ is infinite then $M(b) = \{b^k : k \geq 1\}$.*

Then, there is a real x which is simply normal to exactly the bases specified by M. Furthermore, the real x is computable from the function M.

The theorem gives a complete characterization (necessary and sufficient conditions).

Normality to different bases

The theorems establish the logical independence of normality to multiplicatively independent bases.

The set of bases to which a real number can be normal is not tied to any arithmetical properties other than multiplicative dependence.

The End

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