Algebraically expandable classes of implication algebras

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In this work we solve the following problem:

Characterize the subclasses of implication algebras that can be axiomatized by sentences of the form $\forall \exists! \land p = q$.

In the process we obtain a representation result for finite implication algebras, and as a by-product of our solution a number of interesting classes of implication algebras arise. We also obtain a characterization of the congruence permutable implication algebras.

Implication algebras, also known as Tarski algebras, have been introduced and studied by J. C. Abbott in [1], [2]. They are the $\{\rightarrow\}$-subreducts of Boolean algebras. It is also known that implication algebras are the algebraic counterpart of the implicational fragment of classical propositional logic [4].

An implication algebra is an algebra $(L, \rightarrow, 1)$ satisfying:

1. $1 \rightarrow x \approx x$,
2. $x \rightarrow 1 \approx 1$,
3. $x \rightarrow (y \rightarrow z) \approx y \rightarrow (x \rightarrow z)$,
4. $(x \rightarrow y) \rightarrow y \approx (y \rightarrow x) \rightarrow x$.

We write $I$ to denote the variety of implication algebras. The algebra $2 = (\{0, 1\}, \rightarrow, 1)$, where $x \rightarrow y = 0$ iff $x = 1$ and $y = 0$, is the only (up to isomorphisms) subdirectly irreducible in $I$.

An equational function definition sentence (EFD-sentence) is a sentence of the form

$$\forall x_1, \ldots, x_n \exists! z_1, \ldots, z_m \varepsilon(\bar{x}, \bar{z}),$$

where $n \geq 0$, $m \geq 1$, and $\varepsilon(\bar{x}, \bar{z})$ is a conjunction of equations. Observe that for every algebra $A$ that satisfies $\varphi$ we can define a function $[\varphi]^A : A^n \rightarrow A^m$ by

$$[\varphi]^A(\bar{a}) = \text{the only } \bar{b} \in A^m \text{ such that } A \vDash \varepsilon(\bar{a}, \bar{b}).$$

If $\pi_j : A^m \rightarrow A$ is the $j$th canonical projection we write $[\varphi]^A$ to denote $\pi_j \circ [\varphi]^A$, for $j = 1, \ldots, m$.

We call the structure $(A, [\varphi]^A_1, \ldots, [\varphi]^A_n)$ an algebraic expansion of $A$, since the new operations are defined as unique solutions to systems of equations. If a class $C$ of algebras of the same language satisfies an EFD-sentence, then every algebra in $C$ can be expanded in this way, thus, we call a class axiomatizable by
EFD-sentences an *algebraically expandable* class. A study of this kind of classes for several other varieties can be found in [3].

Let us begin by defining a class of finite implication algebras that play a key rôle in our work. For $n \geq 2$ let

$$F_n = \{0, 1\}^n - \{(0, \ldots, 0)\},$$

and let $F_n$ be the subalgebra of $2^n$ whose universe is $F_n$. For notational purposes it will be convenient to define $F_1 = 2$. Let

$$\mathcal{F} = \{F_n : n \geq 1\}.$$  

In other words, $\mathcal{F}$ is the class containing the implicational reducts of one of each finite Boolean algebra without its bottom element, plus the implication algebra $2$. The following proposition is one of the main tools in the solution of our problem.

**Proposition 1** Every finite member of $\mathcal{I}$ is isomorphic to a global subdirect product with factors in $\mathcal{F} = \{F_n : n \geq 1\}$.

The rôle of the members of $\mathcal{F}$ in the study of axiomatizability by EFD-sentences in $\mathcal{I}$ closely resembles the rôle subdirectly irreducibles would play in studying axiomatizability by identities in a given variety. This is due to the following:

**Lemma 2 ([5])** Suppose $A \subseteq \Pi\{A_i : i \in I\}$ is a global subdirect product, and let $\varphi$ be an EFD-sentence. If $A_i \models \varphi$, for all $i \in I$ then $A \models \varphi$.

Let $n \geq 2$, and let $x_1, \ldots, x_n$ be variables. For $i = 1, \ldots, n$ define the terms

$$s^n_i(x_1, \ldots, x_n) = \bigvee_{j=1, j\neq i}^n x_j,$$

and let

$$\varphi_n = \forall x_1, \ldots, x_n \exists z \left( z \leq s^n_i(x) \land \ldots \land z \leq s^n_i(x) \land \left( \bigvee_{i=1}^n (s^n_i(x) \rightarrow z) = 1 \right) \right).$$

Though $\varphi_n$ and $s^n_i$ are defined using the symbols the $\leq$ and $\lor$, these symbols are just shorthand, and it should be clear that $\varphi_n$ is a sentence in the language ($\rightarrow, 1$).

Let $\mathcal{I}_n$ be the class of algebras in $\mathcal{I}$ that satisfy $\varphi_n$, i.e.,

$$\mathcal{I}_n = \text{Mod}(\varphi) \cap \mathcal{I}.$$  

The theorem below presents the solution to the problem we stated in the beginning of this abstract.

**Theorem 3** If $C \subseteq \mathcal{I}$ is an algebraically expandable class then either $C = \{\text{trivial algebras in } \mathcal{I}\}$, $C = \mathcal{I}$ or $C = \mathcal{I}_n$ for some $n \geq 2$. Furthermore we have

$$\{\text{trivial algebras in } \mathcal{I}\} \subsetneq \mathcal{I}_2 \subsetneq \mathcal{I}_3 \subsetneq \cdots \subsetneq \mathcal{I}.$$
We show next that each class \( I_n \) is actually the reduct of a variety to the language of \( I \). For every \( L \in I_n \) a new \( n \)-ary operation \( \mu_n : L^n \to L \) can be defined by
\[
\mu_n(a_1, \ldots, a_n) = \bigwedge_{j=1}^{n} s_j^n(a_1, \ldots, a_n).
\]
Now, extend the language of \( I \) with the \( n \)-ary function symbol \( \mu_n \), and define the following class of algebras in this new language:
\[
\mathcal{M}_n = \{ (L, \mu_n) : L \in I_n \}.
\]

**Proposition 4** The class \( \mathcal{M}_n \) is a variety axiomatizable by

\[
(I_1), (I_2), (I_3), (I_4),
\]
\[
\forall \bar{x} \, \mu_n(\bar{x}) \leq s_j^n(\bar{x}), \text{ for } j = 1, \ldots, n,
\]
\[
\forall \bar{x} \, \bigwedge_{i=1}^{n} (s_i^n(\bar{x}) \to \mu_n(\bar{x})) = 1.
\]

Furthermore, \( \text{Con}(L, \mu_n) = \text{Con}(L) \), for \( L \in I_n \).

We conclude this work with a characterization of congruence permutable implication algebras.

**Theorem 5** Let \( L \in I \). T.f.a.e.:

1. \( L \) is congruence permutable.
2. The meet of any two elements of \( L \) exists.
3. \( L \models \forall x_1, x_2 \exists z \, z \leq x_1 \land z \leq x_2 \land ((x_1 \rightarrow z) \lor (x_2 \rightarrow z) = 1) \).
4. \( L \) is isomorphic to a global subdirect product whose factors are all isomorphic to \( 2 \).

Furthermore, if \( L \) is finite the above are equivalent to:

5. \( L \) has no homomorphic images in \( F - \{ 2 \} \).

**References**


