On the Repairman Problem with Controlled Servers and Switch-over Costs

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Abstract

This paper considers a maintenance system in which the queue size at the repair facility can be controlled by changing the repair time distribution. The production line consists of a finite number of identical and independent machines working in parallel; they are subject to failure with an exponential time-to-failure distribution. Each breakdown is repaired at a single server repair facility using one of two possible repair time distributions whose choice is based on the number of machines waiting to be repaired. The cost structure includes a holding cost, a repair cost, and a fixed switch-over cost when the repair time distribution is changed from one distribution to the other. The control problem is represented by a semi-Markov decision model in which the decision epochs are the repair completion epochs. The optimality criterion to be considered is the long-run average cost per unit time. A policy-iteration algorithm is used to compute the optimal stationary policy within a class of two-parameter policies. Numerical results are reported using an exponential distribution for the repair time.

Resumo

Este trabalho considera um sistema de manutenção em que o tamanho da fila na estação de reparo pode ser controlado pela escolha da distribuição do tempo de reparo entre duas distribuições disponíveis. A linha de produção consiste de um número finito de máquinas idênticas e independentes trabalhando em paralelo; estas máquinas estão sujeitas a falhas de acordo com uma distribuição exponencial para o tempo até a falha. Cada máquina quebrada é reparada na estação de reparo, que contém um único servidor, usando uma das duas distribuições disponíveis para o tempo de reparo. A escolha da distribuição utilizada é baseada no número de máquinas quebradas. A estrutura de custos inclui um custo de espera, um custo de reparo e um custo fixo de troca incorrido quando a distribuição do tempo de reparo é trocada. O problema de controle é representado por um modelo semi-markoviano de decisão em que os instantes de decisão são os instantes de término de reparo. O critério de otimalidade considerado é o custo médio por unidade de tempo a longo prazo. Um algoritmo de *iteração de políticas é usado para obter a política ótima estacionária dentro de uma classe de políticas de dois parâmetros. Resultados numéricos são obtidos admitindo uma distribuição exponencial para o tempo de reparo.*

Keywords: Maintenance; optimization; semi-Markov decision model.

1 Introduction

This paper considers a maintenance system in which the queue size at the repair facility can be controlled by changing the repair time distribution. The maintenance system consists of a production line, a repair facility, and a repair queue. The production line comprises a finite number M of identical and independent machines working in parallel. The machines are subject to failure during operation according to an exponential time-to-failure distribution with a mean failure time of $1/\lambda$. The production line can continue to operate with less than M machines.

When a machine breaks down, it is immediately sent to the repair facility which functions on a first-come, first-served basis and a queue forms for the single repairman. If the repairman is busy, the faulty machine waits in the repair queue until the repairman is available. After being repaired, the machine is as good as new and can be sent back to the production line.

Each machine is repaired using one of two possible repair time distributions whose choice is based on the number of machines waiting to be repaired. It is assumed that the repair time distribution can only be changed when a repair is completed. The repair time for a machine has a probability distribution of F_k (mean $1/\mu_k$) for repair type $k, k = 1, 2$. It is assumed that $F_k(0) < 1$, for $k = 1, 2$, and that $F_2(t) \geq F_1(t)$ for all $t > 0$ so that repair type 2 is (probabilistically) quicker than repair type 1. Also, it is assumed that $\lambda/\mu_2 < 1$.

The following costs are incurred. There is a holding cost at rate $h.i$ when i machines are waiting or being repaired at the repair facility, and a repair cost at rate r_k when the repairman is busy and is using repair type $k, k = 1, 2$. Further, a fixed switch-over cost R_k is incurred when the system manager decides to switch from repair type k to the other one, $k = 1, 2$. The cost parameters are assumed to be nonnegative. For an example of a switch-over cost, consider a computer centre with several terminals. The terminals can be repaired either by an internal technician or by a skilled external technician. If the system manager decides to call the external technician to speed up a repair, an extra cost will be involved in the switch-over.

The problem is to choose an optimal policy which minimises the long-run average cost per unit time. The control problem is represented by a semi-Markov decision model in which the decision epochs are the repair completion epochs. A policy-iteration algorithm is used to compute the optimal stationary policy within a class of two-parameter policies.

Previously, several researchers have studied similar control problems. Crabill [5] and Winston [10] considered the control of maintenance systems with variable repair rates. However, in both of those papers the maintenance systems were analysed without considering a switch-over cost when the repair rate is changed at the repair facility. On the other hand, Cohen [4] and Tijms [8] studied the control of queuing systems with variable service rates and with switch-over costs. However, in both of those papers the service rate was changed on the basis of the total workload accumulated in the system, not on queue size. Tijms [9] analysed an M/G/1 queuing system with two types of service time distributions and with switch-over costs. The service type was changed on the basis of the queue size at the service facility. He developed a policy-iteration algorithm to compute the optimal service type for each queue size so as to minimise the long-run average cost per unit time of the system. Other related papers are Çinlar [3], Goheen [6], Albright [1], Van Der Duyn Schouten and Wartenhorst [10], and Wang [11]. A good survey on maintenance models can be found in Cho and Parlar [2].

The contribution of the present paper is the introduction of a fixed switch-over cost in a maintenance system when the repair type is changed at the repair facility on the basis of queue size.

The rest of this paper is organised as follows: in the next section a semi-Markov decision process is formulated for the maintenance model and the policy-iteration algorithm used to compute the optimal policy is described. Following this, in section 3, the evaluation of expected transition times, transition probabilities, and expected costs are described. Finally, in section 4, some numerical results are presented for an exponential repair time distribution, and some suggestions for future studies are given.

2 The Semi-Markov Decision Process (SMDP)

The controlled maintenance problem described in the previous section can be represented by a SMDP in which the decision epochs are the repair completion epochs. The state space of the SMDP is taken to be

$$I = \{i|i = 0, 1, ..., M-1\} \cup \{i'|i' = 0, 1, ..., M-1\},$$

where state i (resp. i') corresponds to the situation in which the number of machines at the repair facility is i and repair type 1 (resp. 2) was used for the repair just completed. Note that the maximum number of failed machines at a repair completion epoch is $M-1$.

For any state $i \in I$ the set of available actions is given by $A(i) = A = \{1, 2\}$, where action $k (k = 1, 2)$ means that repair type k should be used for the next repair. Next the set of admissible policies is defined. A stationary policy is a function $f : I \to A$ such that if the observed state at a decision epoch is $i \in I$, then the single action $f(i) \in A$ is taken. Denote by F_0 the class of stationary policies having the following form. Any policy $f \in F_0$ is characterised by two switch-over levels I_1 and I_2 with $0 \leq I_2 \leq I_1$ and $I_1 \geq 1$. Under this policy, denoted by $f = (I_1, I_2)$, the server switches from repair type 1 to repair type 2 only at those repair completion epochs where the queue size is larger than I_1 and the server switches from repair type 2 to repair type 1 only at those repair completion epochs where the queue size is less than or equal to I_2. We will be concerned with the problem of finding the optimal policy within the class F_0.

The following quantities are needed to initiate the algorithm which will be used to compute the optimal policy. Given that at a decision epoch the state is $i \in I$ and action $k \in A$ is chosen, define

$p(i, j, k)$ = probability that at the next decision epoch the state will be $j \in I$;

$\tau(i, k)$ = expected transition time until the next decision epoch;

$c(i, k)$ = expected costs incurred until the next decision epoch.

In the next section the evaluation of the expected transition times, transition probabilities, and expected costs are described. In order to obtain the minimum cost policy, the following version of Howard's policy-iteration algorithm, proposed by Tijms [9], is used. This special purpose algorithm is used to compute the optimal stationary policy within the class F_0 of two-parameter policies. Given an initial stationary policy f, in step 1 the algorithm evaluates the average cost of this policy and a set of relative values which will be used in the next step. In step 2 these relative values are compared with a policy-improvement quantity in order to determine a better action for each state. Finally, in step 3 the new policy is compared with the current policy: if they are the same policy, this is the optimal policy; otherwise return to step 1 with the new policy replacing the current policy.

Policy-Iteration Algorithm:

***Step 1*:** (Value Determination Step) Let the current policy be $f = (I_1, I_2)$ with $0 \leq I_2 \leq I_1$, $I_1 \geq 1$. Solve the system of equations

$$w(i, f) = c(i, f(i)) - g(f)\tau(i, f(i)) + \textstyle\sum_{j \in I} p(i, j, f(i))w(j, f),$$

for all $i \in I$, in $g(f)$ and $\{w(i, f) : i \in I\}$, by making $w(i_f, f) = 0$ for any state i_f recurrent under policy f. The quantity $g(f)$ is the average cost rate for policy f and $\{w(i, f) : i \in I\}$ are relative values (see Mine and Osaki [7], p. 94).

***Step 2*:** (Policy-improvement Step) Using the values obtained in step 1, evaluate the policy-improvement quantity $T(i, k, f)$ defined as

$$T(i,k,f) = c(i,k) - g(f)\tau(i,k) + \sum_{j \in I} p(i,j,k)w(j,f) \tag{1}$$

for $0 \leq i \leq I_2$ and $k = 2$, and for $I_2 \leq i \leq I_1$ and $k = 1$. Next determine an integer $\bar{I}_2$ with $0 \leq \bar{I}_2 \leq I_1$. Define $\bar{I}_2$ as the largest integer n such that $I_2 < n \leq I_1$ and $T(i',1,f) < w(i',f)$ for all $I_2 < i \leq n$ if such an integer exists, otherwise let $\bar{I}_2$ be equal to $m-1$, with m the smallest integer such that $1 \leq m \leq I_2$ and $T(i',2,f) < w(i',f)$ for all $m \leq i \leq I_2$ if such an integer exists, otherwise let $\bar{I}_2 = I_2$. Now evaluate the test quantity (1) for $i > I_1$ and $k = 1$, and for $\bar{I}_2 \leq i \leq I_1$ and $k = 2$. Now determine an integer $\bar{I}_1$ with $\bar{I}_2 \leq \bar{I}_1$ and $\bar{I}_1 \geq 1$. Define $\bar{I}_1$ as the largest integer n such that $I_1 \leq n \leq M-1$ and $T(i,1,f) < w(i,f)$ for all $I_1 + 1 \leq i \leq n$ if such an integer exists, otherwise let $\bar{I}_1$ be equal to $m-1$, with m the smallest integer such that $\bar{I}_2 < m \leq I_1$, $m \geq 2$, and $T(i,2,f) < w(i,f)$ for all $m \leq i \leq I_1$ if such an integer exists, otherwise let $\bar{I}_1 = I_1$.

Step 3: Let $\bar{f} = (\bar{I}_1, \bar{I}_2)$. If $\bar{f} = f$, then stop; f is the optimal stationary policy. Otherwise go to step 1 with the previous policy $f = (I_1, I_2)$ replaced by the new policy $\bar{f} = (\bar{I}_1, \bar{I}_2)$.

3 Determination of Transition Probabilities and Expected Transition Times and Costs

In this section the expressions for $p(i,j,k)$, $\tau(i,k)$, and $c(i,k)$ are obtained. These quantities, together with an initial policy, are needed to initiate the policy-iteration algorithm.

If action $k \in A$ is chosen in state $i \in I$, then the distribution of the time until the next decision epoch is the distribution of a repair time under repair type k, provided that $i \neq 0, 0'$. If $i = 0, 0'$, then the distribution of the time until the next decision epoch is the distribution of the sum of the repair time under repair type k and the shortest remaining running time of all M machines. Thus, the expected time until the next decision epoch, $\tau(i,k)$, is given by, for $k = 1, 2$:

$$\tau(i,k) = \tau(i',k) = 1/\mu_k, \qquad i \geq 1, \text{ and}$$

$$\tau(0,k) = \tau(0',k) = 1/(M.\lambda) + 1/\mu_k.$$

Now to obtain the transition probabilities $p(i,j,k)$, let N be the number of working machines at the production line. For $k = 1, 2$, define

$$p_k(n) = \int_0^\infty \begin{pmatrix} N \\ n \end{pmatrix} (1 - e^{-\lambda t})^n (e^{-\lambda t})^{N-n} dF_k(t),$$

i.e., $p_k(n)$ is the probability that n machines fail during a repair under repair type k when $N > 0$ machines are working at the beginning of the repair. The transition probabilities $p(i,j,k)$ can be directly expressed in terms of the probabilities $p_k(n)$. Let f be a stationary policy. Then the transition probabilities under this policy are found as follows.

a) If the system is in state $i \in \{0,1,...,M-1\}$ and policy f chooses action $f(i) \in \{1,2\}$, then for $j \in \{0,1,...,M-1\}$:

$$
\begin{aligned}
p(i,j,f(i)) &= p_1(j-i+1) && \text{if} \quad f(i)=1,\\
p(i,j',f(i)) &= p_2(j-i+1) && \text{if} \quad f(i)=2,\\
p(i,j',f(i)) &= 0 && \text{if} \quad f(i)=1,\\
p(i,j,f(i)) &= 0 && \text{if} \quad f(i)=2.
\end{aligned}
$$

b) If the system is in state $i' \in \{0,1,...,M-1\}$ and policy f chooses action $f(i') \in \{1,2\}$, then for $j \in \{0,1,...,M-1\}$:

$$
\begin{aligned}
p(i',j,f(i')) &= p_1(j-i+1) && \text{if} \quad f(i')=1,\\
p(i',j',f(i')) &= p_2(j-i+1) && \text{if} \quad f(i')=2,\\
p(i',j',f(i')) &= 0 && \text{if} \quad f(i')=1,\\
p(i',j,f(i')) &= 0 && \text{if} \quad f(i')=2.
\end{aligned}
$$

Suppose, now, that exactly one failure has occurred in the interval $[0,t]$. In order to determine the distribution of the time at which this failure occurred, let S be the time of failure and $N(t)$ the number of failures during time t. Then, using the fact that

$$Pr(N(t)=n) = \begin{pmatrix} N \\ \\ n \end{pmatrix} (1-e^{-\lambda t})^n (e^{-\lambda t})^{N-n},$$

where N is the number of working machines at the production line, it follows that:

$$Pr(S \leq s | N(t)=1) = \frac{1-e^{-\lambda t}}{1-e^{-\lambda t}}\ , \quad s \leq t \tag{2}$$

Thus,

$$E(S|N(t)=1) = \frac{1}{\lambda} - \frac{te^{-\lambda t}}{I-e^{-\lambda t}}\ . \tag{3}$$

The next theorem generalises this last result.

***Theorem 1*:** Given that $N(t)=n$, the arrival times $S_1,...,S_n$ have the same distribution as the order statistics corresponding to n independent random variables distributed as in (2) on the interval $[0,t]$.

***Proof*:** Suppose $0 < t_1 < t_2 < ... < t_n < t_{n+1} = t$, and let h_i be small enough so that $t_i + h_i < t_{i+1}$, $i = 1, 2, ..., n$. Now,

Pr(exactly one event in $[t_i, t_i + h_i], i = 1, 2, ..., n$, no events elsewhere in $[0, t]) / Pr(N(t) = n) =$

$$= \frac{(e^{-\lambda t_1})^N \; N \; d_1.(N-1) \; d_2 ... \; (N-n+1) \; dn}{\begin{pmatrix} N \\ n \end{pmatrix} (1 - e^{-\lambda t})^n \, (e^{-\lambda t})^{N-n}}$$

$$= \frac{n! e^{-\lambda(t_1 + t_2 + ... + t_n)} \left(1 - e^{-\lambda h_1}\right) \left(1 - e^{-\lambda h_2}\right) ... \left(1 - e^{-\lambda h_n}\right)}{(1 - e^{-\lambda t})^n},$$

where

$$d_k = (1 - e^{-\lambda h_k})(e^{-\lambda h_k})^{N-k} \left[e^{-\lambda(t_{k+1} - t_k - h_k)}\right]^{N-k}, \quad k = 1, 2, ..., n.$$

$$\frac{Pr(t_i < S_i < t_i + h_i, i = 1, 2, ..., n | N(t) = n)}{h_1 h_2 ... h_n} =$$

$$= \left[\frac{n! e^{-\lambda(t_1 + t_2 + ... + t_n)}}{(1 - e^{-\lambda t})^n}\right] \cdot \left[\frac{(1 - e^{-\lambda h_1})}{h_1}\right] \cdot \left[\frac{(1 - e^{-\lambda h_2})}{h_2}\right] ... \left[\frac{(1 - e^{-\lambda h_n})}{h_n}\right].$$

Then,

Letting $h_1, h_2, ..., h_n \to 0$, it follows that

$$f_{S_1 S_2 ... S_n}(t_1, t_2, ..., t_n | N(t) = n) = \frac{n! \lambda^n e^{-\lambda(t_1 + t_2 + ... + t_n)}}{(1 - e^{-\lambda t}) n},$$

and the result is obtained. ∎

Given that at epoch 0 a repair starts when $i \geq 1$ machines are waiting or being repaired at the repair facility, define

ξ_i = total amount of time spent by machines at the repair facility during the first repair.

***Theorem 2*:** For any $i \geq 1$,

$$E(\xi_i) = \frac{1}{\mu_k} + (M - 1) \left[\frac{1}{\mu_k} - \frac{1}{\lambda} + \frac{1}{\lambda} \int_0^\infty e^{-\lambda t} dF_k(t)\right]. \tag{4}$$

***Proof*:** Let T_1 be the length of the first repair and let N_1 be the number of arrivals during the first repair. Then, using (3), it follows that for any $i \geq 1$:

$$E(\xi_i|N_1 = n, T_1 = t) = it + n(t - \frac{1}{\lambda} + \frac{te^{-\lambda t}}{1 - e^{-\lambda t}}),$$

which gives $E(\xi_i)$ after using properties of conditional expectations. ∎

Using (4) the expected cost incurred until the next decision epoch, $c(i,k)$, can be computed. Let $(iv1) = \max(i,1)$. Then, for $i \geq 0$:

$$c(i,1) = \frac{h.(iv1)}{\mu_1} + h.(M - (iv1)) \left[\frac{1}{\mu_1} - \frac{1}{\lambda} + \frac{1}{\lambda}\int_0^\infty e^{-\lambda t} dF_1(t)\right] + \frac{r_1}{\mu_1},$$

$$c(i',1) = R_2 + c(i,1),$$

$$c(i',2) = \frac{h.(iv1)}{\mu_2} + h.(M - (iv1)) \left[\frac{1}{\mu_2} - \frac{1}{\lambda} + \frac{1}{\lambda}\int_0^\infty e^{-\lambda t} dF_2(t)\right] + \frac{r_2}{\mu_2},$$

$$c(i,2) = R_1 + c(i',2).$$

Now the computation of the transition probabilities and expected times and costs is completed. In the next section these quantities are going to be used in the policy-iteration algorithm to provide some numerical results.

4 Numerical Results and Discussion

From now on the repair times are assumed to be exponentially distributed with mean $1/\mu_k$, $k = 1,2$. As an illustration consider the following numerical example. Let $M = 3$, $\mu_1 = 1.25$, $\mu_2 = 1.875$, $r_1 = 5$, $\lambda = 1$. Table 1 gives the optimal switch-over levels, $(I_1{}^*, I_2{}^*)$, and the optimal average cost per unit time, $g(I_1{}^*, I_2{}^*)$, for different values of the repair type 2 cost, holding cost, and switch-over costs.

Table 1: Numerical results

r_2	h	R_1	R_2	$g(I_1{}^*, I_2{}^*)$	$(I_1{}^*, I_2{}^*)$
10	15	2	3	32.31	$(1,0)$
25	15	2	3	32.58	$(2,2)$
40	15	2	3	32.58	$(2,2)$
10	15	50	3	32.58	$(2,0)$
10	15	2	60	32.58	$(2,0)$
10	10	2	3	23.23	$(2,0)$
10	20	2	3	40.73	$(1,0)$
10	30	2	3	57.58	$(1,0)$

It can be seen from Table 1 (see the first three lines) that when the repair cost for repair type 2, r_2, is increased, the algorithm computes a policy such that the system will never use repair type 2. Note that when $I_1 = 2$, repair type 2 is never used in the long-run, and thus policies $(2,0)$ and $(2,2)$ have the same cost. Moreover, if the switch-over costs (R_1 or R_2) are increased (see lines 4 and 5 in Table 1), the algorithm selects an optimal policy which avoids this high switch-over costs, i.e. policy $(2,0)$, which never uses repair type 2. It can also be seen (see the last three lines in Table 1) that, when the holding cost is increased, the algorithm finds an optimal policy such that the machines are repaired quicker, i.e. policy $(1, 0)$, which allows the use of repair type 2.

This model can be extended and modified in different ways. One possible modification would be to consider a maintenance system with more than one server at the repair facility. Note that in this case one must assume that the repair times are exponentially distributed. Then a comparison could be made between adding more servers and increasing the repair rate. This case will be studied in a forthcoming paper. See Tijms [9] for a similar analysis in a queueing system.

Another modification would be to analyse a maintenance system with spare machines. When the system has spare machines, the machines in excess that are in good condition wait in a production queue and each of them may enter the production line as a substitute for a broken down machine during a repair. This fact leads to a greater complexity in the determination of the transition probabilities and expected costs. The present model does not account for that, and this case will also be studied in a forthcoming paper.

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