A Generalization of Geodetic Graphs: K–Geodetic Graphs

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Abstract
An undirected graph $G = (V, E)$ is said to be geodetic, if between any pair of vertices $x, y \in V$ there is a unique shortest path. Generalizations of geodetic graphs are introduced in this paper. K-geodetic graphs are defined as graphs in which every pair of vertices has at most k paths of minimum length between them. Some properties and characterizations of k–geodetic graphs are studied.

Keywords: Graph Theory, Connected Graphs, Geodetic Graphs.

1 Introduction

The study of connectivity properties in graphs and digraphs is of special interest in the design of reliable interconnection networks. In particular for the network designer is useful to have some knowledge of those graphs that have high vertex connectivity. Thus, different types of graphs have attracted much interest in recent years, which are characterized by conditions determined in their configurations. A special class of such graphs is that formed by geodetic graphs, in which every pair of non-adjacent vertices has an unique shortest path between them (Ore (1962)).

The concept of geodetic graph is a natural generalization of a tree. A *tree* is a connected graph whose number of edges is $n - 1$. In a tree there is a unique path between any two vertices; in a geodetic graph, there is a unique shortest path between any two vertices.

This class of graphs has been studied by several authors. They have obtained some interesting properties of them, together with a number of results that connect with other types of graphs (see e.g.,[2], [3], [5] [6] [7], [9] and [11]).

Srinivasan, Opatrny and Alagar [10] in 1988 introduced a new type of graphs, called bigeodetic graphs, which are a generalization of geodetic graphs. *Bigeodetic graphs* are defined as graphs in which each pair of vertices has at most two paths of minimum length between them.

The class of bigeodetic graphs contains both geodetic graphs and interval-regular graphs of diameter two in which every pair of non-adjacent vertices has exactly two paths of length two between them ([1]). Also there are other bigeodetic graphs, i.e., in the form of a wheel with n spokes, $n \geq 6$, or even cycles of length $l \geq 6$, which are neither geodetic nor interval-regular graphs.

In this paper we present generalizations of geodetic graphs, which allow at most three, four, five or k paths of minimum length between any two non-adjacent vertices. These graphs are named *trigeodetic*, *quartergeodetic*,.. or, in general, *k–geodetic*.

The remainder of this section is devoted to introducing some basic concepts and simple results used throughout this paper. Let $G = (V, E)$ be an undirected simple graph, that is without loops or multiple edges, where V is the set of vertices and E the set of edges. The cardinalities $n = |V|$ and $m = |E|$ are, respectively, the order and size of G. Only simple connected graphs with at least two vertices are considered.

Two vertices x, y are *adjacent* if the edge (x, y) exists. A *path* is a sequence of adjacent vertices. A vertex x is a *predecessor* of y if there is a path from x to y. The *distance* between any pair of vertices x, y of the graph is the minimum length between both vertices and it is denoted $d(x, y)$. A path of minimum length between vertices x, y will be called a $(x - y)$ *distance path*. The *diameter* d of G is the maximum of distances $d(x, y)$ between any vertices x, y of the graph. The *eccentricity* of the vertex v, denoted by ecc(v), is the distance from v to the farthest vertex. So, ecc(v) $= max\{d(v, x)/x \in V\}$, and the diameter d is the maximum eccentricity of all the vertices.

Let $N_i(v) = \{x \in V/d(v, x) = i\}$ be the set of vertices at distance i from vertex v. $P(v) = \cup_{i=1}^{ecc(v)} N_i(v)$ denotes the set of all the predecessors of v.

A vertex v of a graph G is a *cutvertex* if, for any pair of edges in G incident to v, there is no circuit in G containing both edges. A *non-separable graph* is a connected graph which has not cutvertices. A *block* of a graph is a maximal non-separable subgraph. If G is non-separable, then G itself is often called a block.

In the following section we present some properties about geodetic and bigeodetic graphs obtained for several authors. Section 3 is devoted to study the new class of graphs: *k–geodetic graphs*. In this section we generalize some properties cited in section 2 for geodetic and bigeodetic graphs to k–geodetic graphs. In section 4 a characterization of k-geodetic graphs is given.

2 Some Results for Geodetic and Bigeodetic Graphs

Geodetic graphs have been studied by several authors [2], [3], [4], [9], [11] who analyzed various of their properties.

In particular, they obtained results on the general construction of geodetic graphs and on some properties relating to the diameter of the graph. Thus, in [7], an upper bound for the number of edges in a geodetic graph was obtained, using certain general properties of geodetic graphs. The result obtained by them was as follows:

1. If G is a connected geodetic graph on n points with m edges and diameter d, then

$$n-1 \leq m \leq (d-1) + \binom{n+1-d}{2}$$

A natural extension of geodetic graphs would be to define a new graph, where each pair of vertices has two paths of minimum length betweeen them. A simple graph with that condition is not possible. This is only possible when the graph is a multigraph of order two and these multigraphs have to be complete.

The situation is similar if the condition is extended when there are only k–paths of minimum length between two vertices. The only configuration possible is that of complete multigraphs. Therefore, the extension of geodetic graphs should be the relaxation of the condition "unique path". So, in Srinivasan, Opatrny and Alagar (1988) the concept of the bigeodetic graph is introduced, this being a graph in which each pair of vertices has at most two paths of minimum length between them. Below we present some results of bigeodetic graphs obtained by these authors, which will be referenced when are studied k–geodetic graphs in the next section.

1. If G is a bigeodetic graph then all its blocks are bigeodetic.

2. A separable graph G with diameter $d \geq 3$ is bigeodetic if, and only if, all its blocks are bigeodetic and satisfy the following property: All but at most one block B_1 of G are such that all vertices of each block B have unique distance paths from the cutvertex in that block B, which connects the vertices of that block with the vertices of B_1.

3. If G is a connected bigeodetic graph of diameter d, with n points and m edges, then

$$n-1 \leq m \leq d + \binom{n+1-d}{2}$$

Note: The second result is similar to the theorem obtained by Srinivasan, Opatrny and Alagar (1988). We have added the words "at most" in the statement of the theorem, since the graph on the fig. 1 is bigeodetic and does not satisfy the hypothesis of Srinivasan et al's theorem: in that graph there is not a block B_1 satisfying the statement of the theorem.

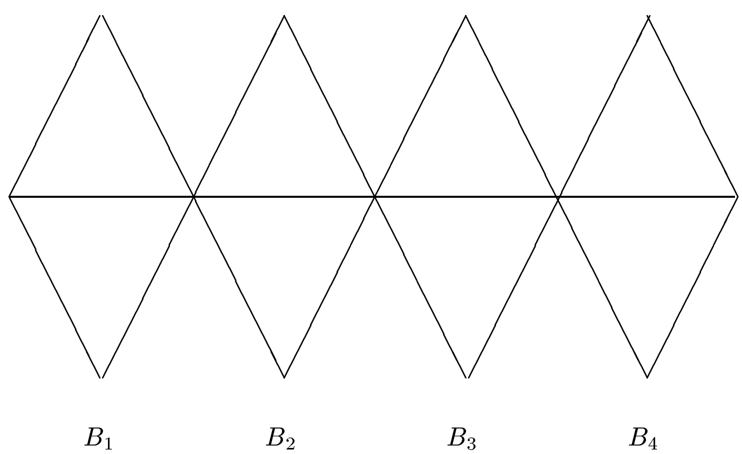

Fig. 1

The characterization problem for the classes of geodetic and bigeodetic graphs is solved (see [2], [5], [10] and [12]). In particular, Parthasarathy and Srinivasan [5] in 1982 proposed the following characterization of a geodetic graph:

- A graph G is geodetic iff for every $v \in V(G)$ each point of $N_r(v)$ is adjacent to a unique point of $N_{r-1}(v)$ for each r with $2 \leq r \leq d$, where d is the diameter of the graph.

Parthasarathy and Srinivasan said in their paper that they made use of this characterization to check various graphs for geodeticity using an IBM 370 computer, but they give neither the algorithm nor computational results.

Morelater, Srinivasan, Opatrny and Alagar [10] in 1988 studied two characterizations of bigeodetic graphs. One of those is cited below:

• A graph G is bigeodetic iff there do not exist a $v \in V(G)$ and $v_i \in N_i(v), 2 \leq i \leq ecc(v)$, with either of the following properties:

1. v_i has more than two predecessors in some $N_j(v), 1 \leq j \leq i-1$.
2. v_i has two predecessors v^1_{i-1}, v^2_{i-1} in $N_{i-1}(v)$ and one of v^1_{i-1}, v^2_{i-1} has more than one predecessor in some $N_j(v), 1 \leq j \leq i-2$.

In [10] the authors didn't give an algorithm to check graphs for bigeodeticity. However, following this characterization it is possible to build and algorithmic procedure for it.

In the following section we introduce the k–geodetic graphs. A characterization for the new class of k–geodetic graphs is proposed in section 4 of this paper.

3 Generalized Geodetic Graphs

We are now going to present a natural generalization of geodetic and bigeodetic graphs. The new concept of k–geodetic graph is introduced as follows:

Definition.

Let G be a simple graph, that is, without loops or multiple edges. We will say G is a *k–geodetic graph* if each pair of vertices has at most k paths of minimum length between them.

It is obvious that if a graph is k–geodetic then it will be p–geodetic with $p \geq k$. The inverse is not true.

Next, we present some properties on k–geodetic graphs.

Proposition 1: If G is a k–geodetic graph then all its blocks are k–geodetic.

Proof: If G is a k–geodetic graph, then each pair of vertices has at most k shortest paths.

Let us suppose that there exists a block, B, which is not k–geodetic. Then, there will be two vertices x and y in B, such that there will be $k + l$ $(l \geq 1)$ paths of minimum length between them. Now, in G there are $k + l$ $(l \geq 1)$ paths of minimum length between x and y. This contradicts the idea of G being a k–geodetic graph. Hence the blocks of G must be k–geodetic.

The converse of the above is not true. Graph G in fig. 2 is not k–geodetic, though the blocks of G are k–geodetic.

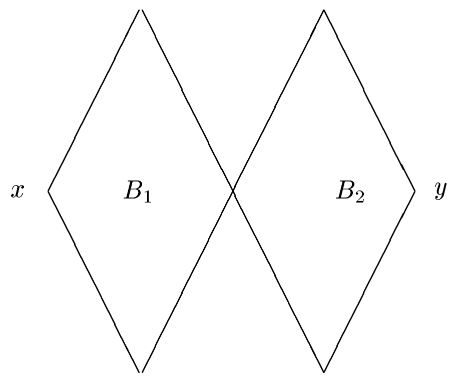

Fig. 2

Theorem 1: A separable graph of diameter two is k–geodetic if, and only if, G has exactly one cutvertex, all its blocks are k–geodetic of diameter two at most and all the vertices of G are adjacent to the cutvertex of G.

Proof: Let G be a separable k–geodetic graph of diameter two. Obviously it should have exactly one cutvertex z, because otherwise the diameter of G would be at least three (see fig 3). Since G is k–geodetic, all its blocks are k–geodetic. As the diameter of G is two, then each block of G has diameter two at most. Let $x, y \in V(G)$ and let $x \in V(B_1)$ and $y \in V(B_2)$, where B_1 and B_2 are two blocks of G. Since the diameter of G is two, both vertices x and y must be adjacent to the cutvertex z. Thus all the vertices of G are adjacent to the cutvertex z.

Conversely, let G be a separable graph with exactly one cutvertex z, all its k–geodetic blocks being of diameter two at most and all its vertices adjacent to the cutvertex z. It is obvious that the diameter of G is two. If we choose two non-adjacent vertices x, y of G such that $x,\ y \in\ V(B_i), 1 \leq\ i \leq\ k$, then as B_i is k–geodetic of diameter two at most, there will be at most k paths of length two between x and y in B_i and hence in G. If $x \in V(B_i), y \in V(B_j), i \neq j$, then $d(x,y) = 2$ since both are adjacent to the cutvertex z, and this path is unique. Hence G is a separable k–geodetic graph of diameter two.

In fig. 3, we can see a 4–geodetic graph of diameter two with three blocks. This graph has a unique cutvertex and any vertex is adjacent to the cutvertex.

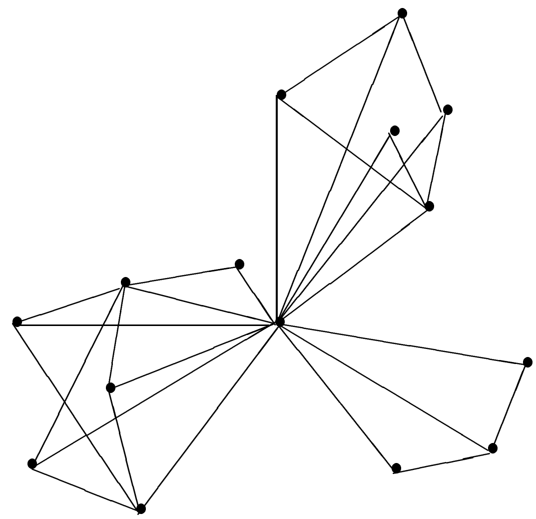

Fig. 3

Now, we will see additional properties on k–geodetic graphs. The following theorems give sufficient conditions to obtain a k–geodetic graph.

Theorem 2: Let G be a separable graph where all its blocks are k–geodetic and satisfy the following property: all the vertices of each block B_i are adjacent to any cutvertex of B_i. Then G is k–geodetic.

Proof: If all its blocks are k–geodetic then the graph is at least k–geodetic. We will see that the geodeticity of the graph cannot be greater than k. Let x, y be two vertices of the graph G.

(i) If $x \in V(B_i)$ and $y \in V(B_j)$ $i \neq j$ then there is only one path between x and y. That path is obtained as follows: the edge (x, z_i), the path linking z_i with z_j and the edge (z_j, y); where z_i and z_j are cutvertices in B_i and B_j, respectively.

(ii) If $x, y \in V(B_i)$ (the same block B_i) then there are at most k shortest paths between them because each block is k–geodetic.

Now, we are going to look for upper and lower bounds for the number of edges of a k–geodetic graph with n points. A trivial result is obtained for the lower bound, which is the number of edges of a tree. So, the minimum number of edges without violating the connectivity will be $n-1$ and, as any tree is geodetic, then the graph will be k–geodetic. Also an upper bound is $\binom{n}{2}$ which corresponds to the complete graph.

That does not mean that a k–geodetic graph can have any number of edges m. So, a k–geodetic graph with a determined number of edges is not always possible. For example, a graph which has four vertices and five edges cannot be geodetic.

The following result determines that it is possible to build a k–geodetic graph with a large number of edges if we require that G has diameter d.

Theorem 3: Given $k \geq 2$ and $n \geq k + d + 1$, it is possible to design a connected k–geodetic graph with n vertices and diameter d, in such a way that the number of edges is

$$m = d - 2 + k + \binom{k}{2} + k(n - k - d + 1) + \binom{n - k - d + 1}{2}$$

Remark. Obviously, the diameter must be greater or equal than two ($d \geq 2$). Otherwise if $d = 1$ then $d(x, y) \leq 1 \ \forall x, y \in V(G)$, but $d(x, y) \geq 1$ because the graph G is connected. So the graph G would be a complete graph with n vertices. Therefore the graph would be geodetic ($k = 1$) and it is not necessary to find out the number of edges, since m is known ($m = \binom{n}{2}$).

Proof: We will demonstrate the theorem by presenting a procedure to build the k–geodetic graphs having that number of edges.

As the diameter is d there will be a path P linking two vertices x and y with d edges. We will see how to add edges to the path P to build the graph so that there are k shortest paths at most between any vertices. The idea consists of setting the vertices of the path P on different planes. We will next set the remaining vertices on the planes and add the maximum number of edges so that the k–geodetic is not violated.

So, for $d = 2$, the maximum number of edges on the graph is:

$$k + \binom{k}{2} + k(n - k - 1) + \binom{n - k - 1}{2}$$

This number is obtained by setting the vertices on three parallel planes (see fig. 4). On the first plane we place only one vertex x. On the second plane we place a complete graph of k vertices and on the last plane a complete graph of (n - k - 1) vertices.

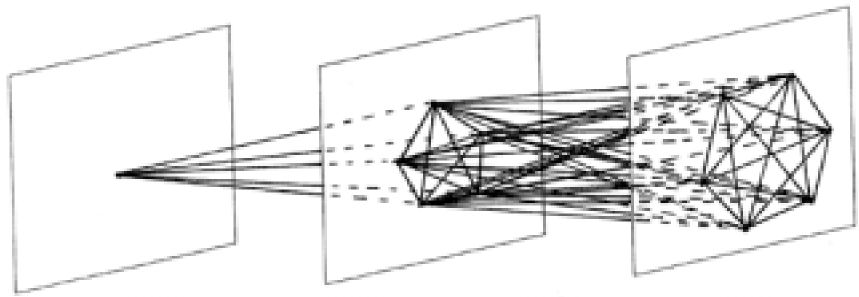

Fig. 4

k links $\binom{k}{2}$ links $k(n-k-1)$ links $\binom{n-k-1}{2}$ links

1 vertex k vertices n-k-1 vertices

Next, we place k edges between the first and second plane, and $k(n-k-1)$ edges between the second and the last plane. Thus, we will have at most k paths of length two between the x vertex and one vertex on the last plane.

The situation is similar to that of a ray of light projected from x on the first plane to the last plane and passing through the holes of the vertices found on its way.

Consequently, there will be at most k paths of length two between the vertex x and other vertex on the last plane. Hence it is not possible to add more edges because the graph would not be k–geodetic. Therefore for $d = 2$ the theorem is true:

$$m = k + \binom{k}{2} + k(n-k-1) + \binom{n-k-1}{2}$$

For $d = 3$ we will set the vertices of the graph between four parallel planes. The last planes maintain the above configuration (for $d = 2$) and the first of the planes must contain only one vertex. A possible graph is shown in figure 5.

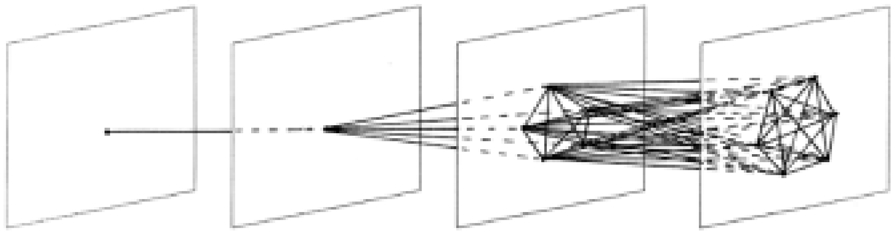

Fig. 5

1 link $\quad$ k links $\quad$ $\binom{k}{2}$ links $\quad$ $k(n-k-2)$ links $\quad$ $\binom{n-k-2}{2}$

1 vertex $\quad$ 1 vertex $\quad$ k vertices $\quad$ n-k-2 vertices

For $d = 3$ we obtain:

$$m = 1 + k + \binom{k}{2} + k(n-k-2) + \binom{n-k-2}{2}$$

If we repeat this procedure for $d = p$ we will obtain sucessive planes with a vertex on an extreme plane which is joined to any vertex on the another extreme plane (see fig. 6).

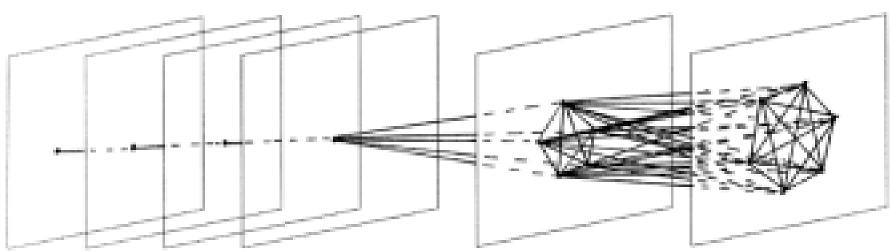

Fig. 6

Sequence of planes

1 link $\quad$ k links $\quad$ $\binom{k}{2}$ links $\quad$ $k(n-p-k+1)$ links $\quad$ $\binom{n-p-k+1}{2}$

1 vertex $\quad$ 1 vertex $\quad$ k vertices $\quad$ n-p-k+1 vertices

For $d = p$ the number of edges attained is:

$$m = p - 2 + k + \binom{k}{2} + k(n - k - p + 1) + \binom{n - k - p + 1}{2}$$

We cannot add any more edges to the graph, since the condition of k–geodeticity will be violated.

Note: It is interesting to remark that the upper bound $d + \binom{n+1-d}{2}$ obtained by Srinivasan, Opatrny & Alagar (1988) for the bigeodetic graphs of diameter d is also obtained in the theorem 6.

Proposition 2: Given k, all the connected graphs of diameter two with $n \leq k + 2$ vertices are k–geodetic graphs. Besides, this number of vertices is maximal, i.e. there exists at least a connected graph of diameter two with $n = k + 3$ vertices which is not k–geodetic.

Proof: We can apply the induction principle on k.

For $k = 1$, all the connected graphs with $n \leq 3$ vertices are geodetic and the first graph which is not geodetic requires $n = 4$ vertices. This graph is the circuit graph C_4.

For $k = 2$, all the connected graphs with $n \leq 4$ vertices are bigeodetic and it is possible to obtain a graph with $n = 5$ vertices, which is not bigeodetic (see fig. 7).

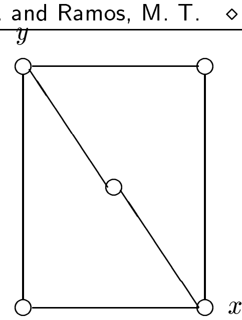

Fig. 7 Three paths of minimum length exist between x and y.

For $k = 3$, all the connected graphs with $n \leq 5$ vertices are trigeodetic and the graph on fig. 8 with $n = 6$ vertices is not trigeodetic.

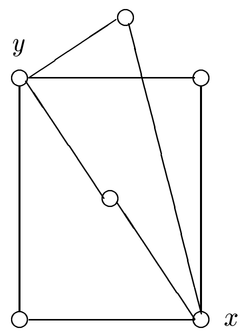

Fig. 8 Four paths of minimum length exist between x and y.

In general, we suppose all the connected graphs with $n \leq (k-1) + 2$ vertices are $(k-1)$-geodetic. If we add a new vertex and all necesary edges to join that vertex with the remaining vertices, we will have a new graph with $n = k + 2$ vertices. This graph has at most k paths of minimum length between each pair of vertices. Therefore, this graph will be k–geodetic. Besides for $n = k + 3$ it is always possible to build a new graph which is not k–geodetic.

4 Characterizing K–geodetic Graphs

Some results and properties on k–geodetic graphs are discussed in this section.

We propose a characterization which follows closely to the one proposed in [10]. This characterization generalizes for k–geodetic graphs the results obtained by them.

Lemma 1: Let G be an undirected graph and $v \in V(G)$. If there exists a $v_i \in N_i(v)$, $2 \leq i \leq ecc(v)$ with more than k predecessors in $N_j(v)$, $1 \leq j \leq (i-1)$, then G is not k–geodetic.

Proof: For any $v \in V(G)$ and $v_i \in N_i(v)$ let $w_1, w_2, w_3, \ldots, w_k, w_{k+1}$ be the predecessors of v_i in $N_j(v)$, $1 \leq j \leq i-1$. There will be $k+1$ shortest paths from v to v_i, each one, respectively, through the vertices $w_1, w_2, \ldots, w_{k+1}$, hence G is not k–geodetic.

Lemma 2: Let G an undirected graph and $v \in V(G)$. If $v_i \in N_i(v)$, $3 \leq i \leq ecc(v)$, has k predecessors $v^1_{i-1}, v^2_{i-1}, ..., v^k_{i-1} \in N_{i-1}(v)$ and if at least one of $v^1_{i-1}, v^2_{i-1}, ..., v^k_{i-1}$ has more than one predecessor in some $N_j(v)$, $1 \leq j \leq (i-2)$, then G is not k–geodetic.

Proof: Let $v \in V(G)$ and let $v_i \in N_i(v)$, $3 \leq i \leq ecc(v)$. Each v_i has $v^1_{i-1}, v^2_{i-1}, ...,$ $v^k_{i-1} \in N_{i-1}(v)$ as predecessors. Besides, v^1_{i-1} has two predecessors $v^1_j, v^2_j \in N_j(v)$, $1 \leq j \leq (i-2)$. Now there are $k+1$ shortest paths between v and v_i, one throughs v^1_j, v^1_{i-1}, another throughs v^2_j, v^1_{i-1} and the remaining one through $v^2_{i-1}, v^3_{i-1}, \ldots, v^k_{i-1}$. Thus G is not k–geodetic.

Lemma 3: Let G be an undirected graph and $v \in V(G)$. If $v_i \in N_i(v)$, $3 \leq i \leq ecc(v)$ has l predecessors $v^1_{i-1}, v^2_{i-1}, ..., v^l_{i-1} \in N_{i-1}(v)$, $1 \leq l \leq k-1$, and if r $(1 \leq r \leq l)$ of them have $s_1, s_2, ..., s_r$ predecessors, respectively, in some $N_j(v)$, $1 \leq j \leq i-2$, so that

$$s_1 + s_2 + ... + s_r > k - l + r$$

then G is not k–geodetic.

Proof: If a vertex v_i exists so that $v_i \in N_i(v)$, $3 \leq i \leq ecc(v)$, where v_i has l predecessors $v^1_{i-1}, v^2_{i-1}, \ldots, v^l_{i-1} \in N_{i-1}(v)$, $1 \leq l \leq k-1$, and if r of them have $s_1, s_2, \ldots, s_r$ predecessors, respectively, in some $N_j(v)$, $1 \leq j \leq i-2$, so that

$$s_1 + s_2 + \ldots + s_r > k - l + r$$

then we will show that there exist at least $k+1$ shortest paths between v and v_i.

Without losing generality, let $v^1_{i-1}, v^2_{i-1}, .., v^r_{i-1}$ be the vertices of the set $v^1_{i-1}, v^2_{i-1},$ $..., v^l_{i-1}$ so that they have $s_1, s_2, \ldots, s_r$ predecessors, respectively. Then the $k+1$ paths are obtained as follows:

(i) $s_1 + s_2 + \ldots + s_r$ paths are obtained across $v^j_{i-1}, 1 \leq j \leq r$, and its predecessors.

(ii) The remaining paths, $(l-r$ paths$)$ are obtained across v^j_{i-1}, $r+1 \leq j \leq l$.

So, we have $s_1 + s_2 + \ldots, s_r + l - r$ paths between v and v_i. But, in accordance with the hypothesis this number is greater than k. Therefore there exist at least $k+1$ paths, and so the graph is not k–geodetic.

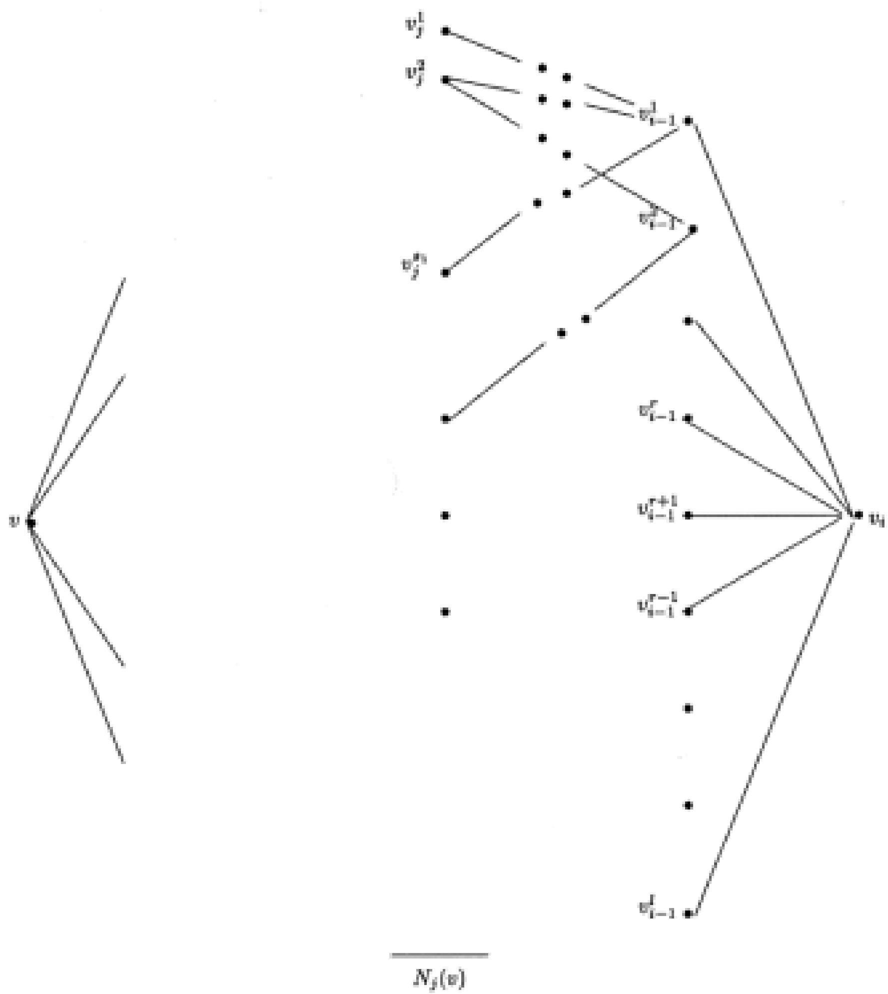

Fig. 9

Theorem 4: An undirected graph G is k–geodetic if, and only if, given a vertex $v \in V(G)$ do not exist a vertex $v_i \in N_i(v)$, $2 \leq i \leq ecc(v)$, satisfying some of following properties:

1. v_i has more than k predecessors in some $N_j(v)$, $1 \leq j \leq (i-1)$.

2. v_i has k predecessors $v_{i-1}^1, v_{i-1}^2, ..., v_{i-1}^k$ in $N_{i-1}(v)$ and at least one of v_{i-1}^1, v_{i-1}^2, ..., v_{i-1}^k has more than one predecessor in some $N_j(v)$, $1 \leq j \leq i-2$.

3. v_i has l predecessors $v_{i-1}^1, v_{i-1}^2, ..., v_{i-1}^l$ in some $N_{i-1}(v), 1 \leq l \leq k-1$, and r of these vertices $v_{i-1}^1, v_{i-1}^2, ..., v_{i-1}^l$ has $s_1, s_2, ..., s_r$ predecessors, respectively, in some $N_j(v),\ \ 1 \leq j \leq i-2$, so that

$$s_1 + s_2 + ... + s_r > k - l + r$$

Proof: If there exists a $v \in V(G)$ and $v_i \in N_i(v),\ \ 2 \leq i \leq$ ecc(v) satisfying any of the above conditions, then G is not a k–geodetic graph by lemmas 1, 2 and 3.

Conversely, let G be a non k–geodetic graph. Then there exists a pair (u, v) of vertices having at least $k+1$ paths of minimum length between them. Suppose $d(u, v) = i$, then $v_i = u \in N_i(v)$. Now, some of the following cases are possible:

1. v_i has more than k predecessors in some $N_j(v)$, $1 \leq j \leq i-1$.

2. v_i has k predecessors $v_{i-1}^1, v_{i-1}^2, ..., v_{i-1}^k$ in $N_{i-1}(v)$.

3. v_i has l predecessors $v_{i-1}^1, v_{i-1}^2, ..., v_{i-1}^l$ in $N_{i-1}(v)$ with $1 \leq l \leq (k-1)$.

Case 1 is the Lemma 1 above. If case 2 occurs and there are $k+1$ paths of minimum length between v and u, then it must be that at least one of $v_{i-1}^1, v_{i-1}^2, ..., v_{i-1}^k$ has more than one predecessor in some $N_j(v)$, $1 \leq j \leq i-2$, because if that were not the case the graph would be k–geodetic. Also, the graph requires r $(1 \leq r \leq l)$ vertices of $v_{i-1}^1, v_{i-1}^2, ..., v_{i-1}^l$ with $s_1, s_2, \ldots, s_r$ predecessors, respectively, in some $N_j(v)$, $1 \leq j \leq i-2$, so that $s_1 + s_2 + \ldots + s_r > k - l + r$. If that condition is not true then any r vertices of $v_{i-1}^1, v_{i-1}^2, ..., v_{i-1}^l$ with $s_1, s_2, \ldots, s_r$ predecessors respectively, in some $N_j(v), 1 \leq j \leq i-2$, satisfy

$$s_1 + s_2 + \ldots + s_r \leq k - l + r$$

So, the number of paths of minimum length between v and v_i could be

$$l - r + s_1 + s_2 + \ldots + s_r$$

As that number is bound by k, the graph G would be k- geodetic. Therefore, graph G must satisfy some of the above properties.

The characterization of k–geodetic graphs proposed leads to a polynomial algorithm for k–geodetic graphs, since it is based on the problem to determine the number of predecessors of a vertex. It is possible to compute this number using the adjacency matrix and the distance matrix of the graph which can be calculated by any algorithm for obtaining the shortest paths between any pair of vertices, i.e. Dijkstra or Floyd's algorithm.

5 Conclusion

In this paper k–geodetic graphs are defined as a natural extension of geodetic graphs. Certain properties of these graphs considering blocks and cutvertices have been studied. Also, a characterization of k–geodetic graphs has been proposed. Future directions of this paper might be the generalization of other properties of geodetic graphs to k–geodetic graphs, i.e. the construction of k–geodetic blocks with given girth and diameter.

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