Heuristic Approach for Assigning Multi-job Adopted Machines

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Abstract
An heuristic method is presented to approximate the number of machines as well as the time-interval required to produce given volumes of works by employing multi-job adopted machines. A transportation model is described as an approximation of the stated problem.

Keywords: Linear programming; multi-job adopted machines; production; transportation problem.

1 Statement and NP-hardness of the problem

We have to realize, in the shortest possible time-interval, known volumes $V_1, V_2, ..., V_n$ in n different categories of works. For this, there are available m different types of identical multi-job adopted machines. Each type of a_i machines is multi-job adopted in the sense that it could perform works of some of the n categories. Let p_{ij} be the non-negative productivity per unit time of a machine of type i when assigned to perform category j of works, and x_{ij} the number of machines so assigned.

With the advancement of technology and the consequent specialization, each type of machines is generally best adopted to only one or few of the n works. However, several factors often necessitate the employment of at least some of the machines of a given type on categories of works for which they are not best adopted. The problem posed is that of determining the optimal assignment of the machines so as to realize all the given volumes of works in the shortest possible time-interval.

Problems of this nature are known from many applications [1,3-6,8]. They are often encountered in many public works such as construction of buildings, highways..., as well as in many production systems. Our objective in the present paper is to formulate and approximate such problems in the framework of Linear

Programming [7].

Let $I = \{1, 2, ..., m\}$ and $J = \{1, 2, ..., n\}$. The problem posed, called P, is

$$\begin{aligned} \min t & & \\ \sum_{j \in J} x_{ij} &\leq a_i, \quad i \in I, \\ t \sum_{i \in I} p_{ij} x_{ij} &\geq V_j, \quad j \in J, \\ x_{ij} &\in Z^+, \quad i \in I,\ j \in J, \\ t &\geq 0. \end{aligned}$$

Proposition 1: Problem P is NP-hard.

Proof. Consider the Partition Problem, known to be NP-complete [2]. Given m positive integer numbers $e_1, e_2, ..., e_m$, whose sum is $2S$. Can we partition them into two disjoint subsets both having the same sum of their elements. The problem is to find $x_{i1},\ x_{i2},\ i = 1, 2, ..., m$, such that

$$\begin{aligned} x_{i1} + x_{i2} &\leq 1, \quad i = 1, 2, ..., m, \\ \sum_{i=1}^{m} e_i x_{ij} &\geq S, \quad j = 1, 2, \\ x_{ij} &\in \{0, 1\}, \quad i = 1, ..., m, \quad j = 1, 2. \end{aligned}$$

Clearly, Partition is a special case ($t = 1$, $n = 2$, $a_i = 1$, $p_{ij} = e_i$, $V_j = S$) of problem P and may be interpreted as follows: Is it possible to realize, in one unit time, the volumes S, using one machine of each type i, which has productivity e_i when assigned to perform task j? Therefore problem P is at least as hard as Partition.

Problem P may have no solution (take for example: $m = 1$, $a_1 = 1$, $n = 2$.) So, we assume that $\sum_{i \in I} a_i \geq n$.

The set of feasible solutions of problem P, if not empty, is bounded. Therefore, problem P has an optimal solution if and only if it is feasible. The next proposition gives a necessary and sufficient condition for problem P to be feasible.

Consider the following transportation problem defined from problem P instance:

$$\min \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij}$$

$$\begin{aligned}\sum_{j\in J} x_{ij} + x_{i,n+1} &= a_i,\ i\in I,\\ \sum_{i\in I} x_{ij} &= 1,\ j\in J,\\ \sum_{i\in I} x_{i,n+1} &= \sum_{i\in I} a_i - n,\\ x_{ij} &\geq 0,\end{aligned}$$

where

$$c_{ij} = \begin{cases} 0 & \text{if} \quad p_{ij} \succ 0,\\ 1 & \text{if} \quad p_{ij} = 0,\end{cases}$$

for $i \in I,\ j \in J$, and $c_{i,n+1} = 0$ for $i \in I$, and let c^* be the value of an optimal solution.

Proposition 2: Problem P is feasible if and only if $c^* = 0$.

Proof. (i) Assume that $c^* = 0$. An optimal solution of this transportation problem (omitting the destination labeled $n+1$) is a feasible solution of problem P. Indeed, the first set of constraints of P is satisfied. In order to satisfy the second, set

$$t = \max_{j\in J} \frac{V_j}{p_{i_j j}},$$

where i_j is that unique i for which $x_{ij} = 1$.

(ii) Assume $c^* > 0$. This means that there is at least one category of works for which no machine could be assigned with a positive productivity. Therefore the corresponding volume cannot be realized.

The following proposition shows that all the constraints of the LP-relaxation of problem P are binding at an optimal solution $\{x^*_{ij}\}$.

Proposition 3: For each $i \in I$, either there exists $j_0 \in J$ such that $p_{i_0 j} = 0$ or $\sum_{j\in J} x^*_{ij} = a_i$.

Proof. Let i_0 be an arbitrary type of machines. Assume that $p_{i_0 j} \succ 0,\ \forall j \in J$ and

$$f_{i_0} = a_{i_0} - \sum_{j\in J} x^*_{i_0 j} \succ 0.$$

Consider the system S of linear equations

$$\sum_{j\in J} y_{i_0 j} = f_{i_0} \qquad (1)$$

$$\frac{p_{i_0 1} y_{i_0 1}}{V_1} = \cdots = \frac{p_{i_0 n} y_{i_0 n}}{V_n}. \tag{2}$$

Set $J_1 = J \backslash \{1\}$ and

$$h_j = V_1 / V_j \ , \quad j \in J_1. \tag{3}$$

We can express system S as follows

$$\begin{aligned} \sum_{j \in J} y_{i_0 j} &= f_{i_0} \\ p_{i_0 1} y_{i_0 1} - h_j p_{i_0 j} y_{i_0 j} &= 0, \quad j \in J_1. \end{aligned}$$

Let α_j, $j \in J$, be scalar numbers. Set

$$\begin{aligned} \alpha_1 + \sum_{j \in J_1} \alpha_j &= 0 \\ \alpha_1 p_{i_0 1} - \alpha_j h_j p_{i_0 j} &= 0, \quad j \in J_1. \end{aligned}$$

The $n-1$ last equations imply that

$$\alpha_j = \alpha_1 \frac{p_{i_0 1}}{h_j p_{i_0 j}}, \quad j \in J_1.$$

Hence, the first equation becomes

$$\alpha_1 \left(1 + p_{i_0 1} \sum_{j \in J_1} \frac{1}{h_j p_{i_0 j}} \right) = 0$$

implying that α_1 =0. Consequently, $\alpha_j = 0$, $j \in J$, which proves that system S is not singular. From equation (1), there exists at least one category j such that $y_{i_0 j} > 0$. From that, and from equations (2), it follows that $y_{i_0 j} > 0$, $j \in J$. By building, $\{x^*_{ij} + y_{ij}\}$ is a feasible solution of the LP-relaxation of problem P. It is easy to see that this solution gives a shorter time-interval. But this contradicts the optimality of $\{x^*_{ij}\}$. Thence $f_{i_0} = 0$.

Corollary 4: We can replace the first inequality constraints of the LP-relaxation of problem P by equations.

Invoking similar arguments, we prove that the second inequality constraints of the LP-relaxation of problem P could be replaced by equations. Therefore, the LP-relaxation of problem P is equivalent to the following problem Q

$$\min t$$

$$\sum_{j \in J} x_{ij} = a_i, \quad i \in I, \tag{4}$$

$$t \sum_{i \in I} p_{ij} x_{ij} = V_j, \quad j \in J, \tag{5}$$

$$x_{ij} \geq 0, \quad i \in I, \quad j \in J, \tag{6}$$

$$t \geq 0$$

2 Linearization of problem Q

Let $p = \sum_{i \in I} \sum_{j \in J} p_{ij} x_{ij}$ be the total production per unit time. By adding equations (5), it follows that

$$tp = \sum_{j \in J} V_j. \tag{7}$$

Hence, for minimizing t, any set of $\{x_{ij}\}$ satisfying (4), (5), and (6) must maximize p. Problem Q can easily be transformed into a linear programming problem. We eliminate t by setting

$$\frac{\sum_{i \in I} p_{i1} x_{i1}}{\sum_{i \in I} p_{ij} x_{ij}} = h_j, \quad j \in J_1.$$

Now, the minimization of t could be achieved by solving the linear program PP

$$\min p = \sum_{i \in I} \sum_{j \in J} p_{ij} x_{ij} \text{ subject to (4), (6), and}$$

$$\sum_{i \in I} p_{i1} x_{i1} - h_j \sum_{i \in I} p_{ij} x_{ij} = 0, \quad j \in J_1. \tag{8}$$

The optimal value of t is deducible from (7).

Proposition 5: The rank of the constraints matrix of problem PP is $m + n - 1$.

Proof. This matrix has $m + n - 1$ rows and $m \times n$ columns. The coefficients h_j defined in (3) are positive. So, any set of $n - 1$ column-vectors indexed by (i,j) with $j > 1$ and $p_{ij} > 0$ is linearly independent. Add to this set the m column-vectors indexed by $(i, 1)$, $i \in I$. The resulting set of $m+n-1$ vectors is linearly independent.

Define

$$\begin{aligned} p'_i &= \max_{j\in J} p_{ij},\ i\in I, \quad \text{and} \\ r_{ij} &= \frac{p_{ij}}{p'_i},\ i\in I,\ j\in J. \end{aligned}$$

Therefore $0 \le r_{ij} \le 1,\ i\in I,\ j\in J$. Define a linear program designated as PR as follows

$$\begin{aligned} \max\ r &= \sum_{i\in I}\sum_{j\in J} r_{ij}x_{ij} \\ \sum_{j\in J} x_{ij} &= a_i, \quad i\in I, \\ \sum_{i\in I} r_{i1}x_{i1} - h_j \sum_{i\in I} r_{ij}x_{ij} &= 0, \quad j\in J_1, \\ x_{ij} &\ge 0, \quad i\in I, \quad j\in J. \end{aligned}$$

From proposition 4 we know that the rank of the constraints matrix of problem PR is $m+n-1$. Problem PR has two nice properties: (i) The standard transportation problem is a special case of problem PR as we shall see, and (ii).

Proposition 6: Every dual-feasible basis of problem PR is a dual-feasible basis of problem PP.

Proof. Let $u_i,\ i\in I$, and $s_j,\ j\in J_1$, be a dual basic feasible solution of problem PR. It must satisfy

$$u_i + r_{i1}\sum_{j\in J_1} s_j \ge r_{i1},\ i\in I, \tag{9}$$

$$u_i - h_j r_{ij} s_j \ge r_{ij},\ i\in I,\ j\in J_1. \tag{10}$$

Multiplying the two sides of (9) and (10) by p'_i, we obtain

$$\begin{aligned} p'_i u_i + p_{i1}\sum_{j\in J_1} s_j &\ge p_{i1},\ i\in I, \\ p'_i u_i - h_j p_{ij} s_j &\ge p_{ij},\ i\in I,\ j\in J_1. \end{aligned}$$

The values $u'_i = p'_i u_i,\ i\in I,\ s'_j = s_j,\ j\in J_1$, constitute a dual basic feasible solution of problem PP, since the dual constraints are satisfied.

3 Heuristic approach

Set

$$b'_j = V_j \frac{\sum_{i\in I} a_i}{\sum_{j\in J} V_j},\ j \in J.$$

Then $\sum_{i\in I} a_i = \sum_{j\in J} b'_j$ and

$$\frac{b'_1}{b'_j} = h_j,\ j \in J_1. \tag{11}$$

Since the equations $\sum_{i\in I} x_{ij} = b'_j,\ j \in J$, are equivalent to

$$\sum_{i\in I} x_{i1} - h_j \sum_{i\in I} x_{ij} = 0,\ j \in J_1,$$

the following problem TP

$$\begin{aligned} \max\ r &= \sum_{i\in I}\sum_{j\in J} r_{ij}x_{ij} \\ \sum_{j\in J} x_{ij} &= a_i, \quad i \in I, \\ \sum_{i\in I} x_{i1} - h_j \sum_{i\in I} x_{ij} &= 0, \quad j \in J_1, \\ x_{ij} &\geq 0, \quad i \in I, \quad j \in J, \end{aligned}$$

is a transportation problem. Let B be an optimal basis. By setting

$$\begin{aligned} s_1 &= \sum_{j\in J_1} s_j \quad \text{and} \\ \delta_j &= \begin{cases} h_1 = 1 & \text{if } j = 1 \\ -h_j & \text{if } j \in J_1, \end{cases} \end{aligned}$$

a feasible and optimal dual solution $u_i,\ i \in I$, and $s_j,\ j \in J$, must satisfy

$$u_i + \delta_j s_j - f_{ij} = r_{ij},\ i \in I,\ j \in J, \tag{12}$$

where f_{ij} are slack variables satisfying

$$f_{ij} \begin{cases} = 0 & \text{if } (i,j) \in B \\ \geq 0 & \text{if } (i,j) \notin B. \end{cases}$$

From proposition 4, if $r_{ij} > 0$, $(i,j) \in B$, then B would be a basis of problem PR. We can write equations (12) as follows

$$u_i + r_{ij}\delta_j s_j - \Delta_{ij} = r_{ij},\ i \in I,\ j \in J \tag{13}$$

with $\Delta_{ij} = (r_{ij} - 1)\delta_j s_j + f_{ij}$.

Equations (13) represent the dual constraints of problem PR, Δ_{ij} being slack variables. We see that if $\Delta_{ij} \geq 0$, $\forall i, j$, then B would be a dual-feasible basis of problem PR. Note that most of the coefficients r_{ij} through the optimal basis B would be close to unity since the transportation problem TP is solved with maximization form of the objective function. However, if we define

$$\Delta = \min_{i,j} \ \{(r_{ij} - 1)\,\delta_j s_j + f_{ij}\},$$

we can accept B as a "Δ-dual-feasible" basis of problem PR and, from proposition 5, of problem PP. Set

$$b_j = \max\left\{1, \lfloor b'_j \rfloor\right\},\ j \in J. \tag{14}$$

This definition ensures an integer optimal solution for problem TP though the coefficients h_j in formula (11) are slightly modified.

4 Algorithm outline

We have shown in the last section that an optimal solution of problem TP, when accepted as an approximate solution of problem P, satisfies the integrity constraints as well as the first set of constraints of P. It is not "far from optimality" as B is a Δ-dual-feasible basis. In order to take into account the second set of constraints of P , several policies might be adopted. One of which is the following:

Step1. Solve the transportation problem TP and accept its optimal solution as an approximate solution of problem P.

Step 2. Work with this solution until one (or more) volume(s) is (are) fully realized, and define a new smaller problem whose volumes are the shortages and return to step 1.

Step 2 avoids shortages as well as excesses in the volumes realized. It provides a good time-interval (see table 6) but needs several (often $n - 1$) changes of planning.

5 Numerical example (from [4])

Three volumes of works have to be realized according to the values given in the first row of table 1. For this, there are available four types of machines (values a_i are given in the first column.) The remaining cells of the same table refer to the productivities p_{ij}.

	$V_1 = 5000$	$V_2 = 10000$	$V_3 = 10000$
$a_1 = 20$	4	10	11
$a_2 = 50$	0.4	0	10
$a_3 = 30$	0	4	6
$a_4 = 100$	0.4	2.5	2.5

Table 1

Table 2 presents the transportation problem with the coefficients b_j defined in (14).

	$b_1 = 40$	$b_2 = 80$	$b_3 = 80$
$a_1 = 20$	0.36	0.91	1
$a_2 = 50$	0.04	0	1
$a_3 = 30$	0	0.67	1
$a_4 = 100$	0.16	1	1

Table 2

Table 3 gives an optimal solution of problem TP.

	$b_1 = 40$	$b_2 = 80$	$b_3 = 80$
$a_1 = 20$	20		
$a_2 = 50$			50
$a_3 = 30$			30
$a_4 = 100$	20	80	

Table 3

We work with this solution then we stop the planning after 14.7 units of time, having fully realized volume V_3. The shortages are shown in table 4.

Jobs	Volumes	Volumes realized	Shortages
1	5000	1294	3706
2	10000	2940	7060
3	10000	10000	0

Table 4

Table 5 gives the new problem instance.

	$V_1 = 3706$	$V_2 = 7060$
$a_1 = 20$	4	10
$a_2 = 50$	0.4	0
$a_3 = 30$	0	4
$a4 = 100$	0.4	2.5

Table 5

Summarizing, after two changes and 46.5 units of time, each of the volumes is fully realized. Note that the optimal solution of problem PP, using the simplex method, gives a time-interval of about 44 units of time.

6 Computational experience

We coded our heuristic in Turbo-Pascal and run on a Systems International 386 SX (33 MHz). Twenty-four problems with up to 200 integer variables were randomly generated according to the uniform law. All the coefficients are non-negative integers. The a_i's belong to the range [1,20], the V_j's to [100,10000], and the p_{ij}'s to [0,100].

Problem PP is solved using the simplex method. The corresponding optimal time-interval is reported in the second column of table 6. The third column shows the time-interval provided by our heuristic. Comparing these two values, we observe that our heuristic provides a time-interval that is close to the optimal time-interval of problem PP. The drawback, as previously mentioned, is that several changes of the planning of the machines are necessary. Unless the production program is a long-term one, such a policy might be prohibitive.

We point out that our heuristic is polynomial-time since the major effort is spent in solving at most $(n-1)$ transportation problems.

7 References

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Size $m \times n$	Problem PP solution	Heuristic
5×5	4.25	4.28
	11.61	11.93
	6.26	6.26
	11.80	13.15
5×10	16.85	18.65
	17.27	17.42
	24.05	24.10
	12.13	12.17
5×20	23.74	23.85
	17.12	17.27
	25.89	26.07
	15.75	15.78
10×5	5.33	5.36
	1.85	1.90
	1.91	1.93
	2.59	2.67
10×10	7.31	7.32
	7.77	7.82
	7.74	7.95
	7.87	7.95
10×20	9.27	9.27
	10.49	10.54
	11.14	11.15
	10.36	10.45

Table 6. Computational results