A Generalization of the Graph Coloring Problem*

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Abstract

This paper presents a generalization of the graph coloring problem. We study its properties and generalize some well-known bounds and definitions.We propose a generalization of perfect graphs and analize the complexity of the problem. An heuristic is developed to seek "good" solutions and we report their results for random graphs. An IP formulation allows us to evaluate the heuristic in small dimension instances.

1 Introduction

Suppose we want to group persons in n commissions of k individuals to give their point of view on some topics. The topics were assigned to one or more commissions depending on its importance. To be sure of having several opinions on each topic, it would be good that if two commissions share at least one subject, they don't have more than i persons in common. In order to save money, we want to hire the minimum number of persons to study the topics.

To model this problem, let $G = (V, E)$ be a graph where each vertex corresponds to a commission and there is an edge between a pair of vertices if the corresponding commissions share at least one topic. Now, if we associate the persons with col ors, the problem is to find the minimum number of colors such that there are k colors assigned to each vertex and adjacents vertices don't share more than i colors.

The purpose of this paper is to study the properties of this problem. It is formally defined in the next section and the case for complete graphs is analized. Upper

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bounds are given at section 3 while in section 4 and 5 we generalize some well-known defin itions of graph theory in order to improve the upper bounds for special cases. In section 6 we deal with the complexity of the problem. A mathematical model is presented at section 7 and an heuristic algorithm is developed at section 8. The numerical results are presented in section 9. Finally, in section 10 some conclusions are stated and ideas for future research are devised.

In the rest of the paper we will follow the definitions and notation of [15].

2 The k,i-coloring problem

The problem described above can be seen as a generalization of the well-known coloring problem, which is the case for $k = 1$ and $i = 0$. This particular case has received much attention as it combines interesting theoretical features with a wide range of pr actical applications. For other cases, we only found the work of Hilton, Rado and Scott [16] who introduced the problem for $i = 0$ on planar graphs.

Formally, we make de following definition :

Definition: Given k and i, with $k, i \in \{1, 2, \dots\}$, the k,i-coloring problem of G is to find the smallest integer j such that there is a mapping of V onto k elements of $\{1...j\}$ such that adjacents vertices map don't share mo re than i elements. We call this number k,i-cromatic number of G and we denote it by $\chi_k^i(G)$.

We start by exploring the k,i-coloring of complete graphs.

Proposition 2.1. $\chi_k^i(K_n) = kn - \frac{n(n-1)i}{2}$ *for* $k \geq (n-1)i$

Proof: First, we will see that there is a k,i-coloring with $kn - \frac{n(n-1)i}{2}$ colors. This can be shown using the following coloring procedure :

assign k colors to v_1
for $j = 2$ **to** n
    for $l = 1$ **to** $j - 1$
        color v_j with i colors assigned to v_l and not shared with any other
    end for
    complete this coloring with $k - (j-1)i$ new colors
end for

From the new set of colors assigned to a vertex v_j, we allways can take $(n - j)$ sets of i colors to share with each one of the $n - j$ vertices that have not been colored yet. This is true because $k \geq (n-1)i$ and so the procedure is correct. It is easy to see that this procedure assigns K_n with $kn - \frac{n(n-1)i}{2}$ colors.

Now, we will prove by induction that any k, i-coloring of K_n uses at least $kn - \frac{n(n-1)i}{2}$ colors.

Case $n = 2$ is trivial.

We suppose by inductive hypothesis that every k, i-coloring of K_{n-1} uses at least $k(n-1) - \frac{(n-1)(n-2)i}{2}$ colors. Let see what happens with K_n. For any k, i-coloring of K_{n-1}, it will be necessary at least $k - i(n-1)$ new colors to color v_n.
As we have

$$\chi_k^i(K_{n-1}) \geq k(n-1) - \frac{(n-1)(n-2)i}{2}$$

then

$$\chi_k^i(K_n) \geq k(n-1) - \frac{(n-1)(n-2)i}{2} + k - i(n-1)$$

□

For general graphs, we were not able to give the optimal coloring solution, but we have derivated some upper bounds. An easy way to find upper bounds is to apply a greedy algorithm. Following this idea, we have:

Proposition 2.2. $\chi_k^i(G) \leq k(\Delta(G) + 1) - i.$

Proof: This bound follows by ordering the vertices of G arbitrarily and then color them greedily. □

Actually, we can do better.

Proposition 2.3. $\chi_k^i(G) \leq k(\max\{\delta(H)/H \textit{ induced subgraph of } G\} + 1) - i.$

Proof: We order the vertices by repeatedly removing a vertex of minimum degree in the subgraph induced by the vertices not yet chosen and placing it after all the remaining vertices but before all the vertices already removed. Now, the boun d follows by coloring greedily on this order . □

Proposition 2.4. *If* $\chi_k^i(G) = j$ *and* $\chi_k^i(\overline{G}) = \overline{j}$ *then* $n \leq \binom{j}{k}\binom{\overline{j}}{k}$

Proof: Let $\wp_k(j) = \{C : C \subseteq \{1 \ldots j\} \textit{ and } |C| = k\}$ and f, g the functions defining a k, i-coloring for G and $\overline{G}$ respectively

$$f : V \to \wp_k(\chi_k^i(G)) \qquad g : V \to \wp_k(\chi_k^i(\overline{G}))$$

Let $h : V \to \wp_k(\chi_k^i(G)) \times \wp_k(\chi_k^i(\overline{G}))$, $h(v) = (f(v), g(v))$

Clearly, h is injective, so $|V| \leq |\wp_k(\chi_k^i(G)) \times \wp_k(\chi_k^i(\overline{G}))|$ □

New concepts are developed in the next two sections that will help us to improve the upper bounds.

3 k,i-Critical Graphs

Critical graphs were first defined by Dirac in 1951 [7], as follows.

Definition: G is critical if $\chi(G - \{v\}) < \chi(G)$ for every vertex v of G.

This notion is very useful because some problems for chromatic graphs may often be reduced to problems for critical graphs, which are more restricted than chromatic graphs in general. Extending this concept, we propose this new definition:

Definition: G is k,i-critical if $\chi_k^i(G - \{v\}) < \chi_k^i(G)$ for every vertex v of G.

From this definition, we can generalize some properties developed for critical graphs.

Proposition 3.1. *If G is k,i-critical then G is a connected graph.*

Proof: Suppose that G is not connected. Let H be a connected component such that $\chi_k^i(H) = \chi_k^i(G)$. Then, if v $\notin H$, $\chi_k^i(G - \{v\}) = \chi_k^i(H)$, and this is a contradiction. □

Proposition 3.2. *If G is k,i-critical then G has no cutpoints.*

Proof: Suppose that v, vertex of G, is a cutpoint. Let G_1 be a connected component of $G - \{v\}$ and $G_2 = (G - G_1) - \{v\}$. G is k,i-critical, then there are a k,i-coloring of $G_1 \cup \{v\}$ and a k,i-coloring of $G_2 \cup \{v\}$ with $\chi_k^i(G) - 1$ colors. If the same set of k colors were assigned to v for both k,i-coloring, then a k,i-coloring of G using $\chi_k^i(G) - 1$ colors could be derived, but it is not possible. It is a contradiction.

In case that there is at least one different color assigned to v for the two k,i-coloring, the colors can be renamed to be in the previous case. □

Proposition 3.3. *If G is k,0-critical then every cutset of vertices of G contains two nonadjacents vertices*

Proof: Suppose that K_h is a complete subgraph of G and it is a cutset. Let G_1 be a connected component of $G - K_h$ and $G_2 = (G - G_1) - K_h$.

G is $k, 0$-critical, then there are a $k, 0$-coloring of $G_1 \cup K_h$ and a $k, 0$-coloring of $G_2 \cup K_h$ with $\chi_k^0(G) - 1$ colors. All optimal $k, 0$-coloring of K_h are equivalent, so we can rename the colors assigned to $G_2 \cup K_h$ such that K_h has the same coloring in $G_1 \cup K_h$ and $G_2 \cup K_h$. From these colorings we can derive a $k, 0$-coloring of G with $\chi_k^0(G) - 1$ colors.

This is a contradiction. □

Proposition 3.4. *If G is k,i-critical then* $\chi_k^i(G) \leq k\delta(G) + k - i$

Proof: Let $\chi_k^i(G) = j$.

Suppose there is a vertex v of G such that $d(v) \leq \lfloor j - k + i - 1/k \rfloor$. Since G is critical, there is a coloring of $G - v$ with $j - 1$ colors. In this coloring, at most $d(v)k$ colors are used by the neighbors of v, so at most $\lfloor j - k + i - 1/k \rfloor k$ colors which is less or equal than $j - k + i - 1$. Therefore, there are $k - i$ colors not being used by the neighbors of G in the coloring of $G - \{v\}$. To color v in G, we can assign i colors that are being used by the neighbors and $k - i$ colors that are not. Then we have a k, i-coloring of G wi th $j - 1$ colors.

This is a contradiction. □

From these properties we can derivated upper bounds.

Proposition 3.5. $\chi_k^i(G) + \chi_k^i(\overline{G}) \leq k(n+1) - 2i$ *for* $n \geq 3$

Proof: We will prove it by induction.

Case $n = 3$

It is enough to analize K_3 and K_3 without an edge because the others are their complements. It's easy to see that :

$$\chi_k^i(K_3) \leq 3 * k - 2 * i \quad \textit{and} \quad \chi_k^i(\overline{K_3}) = k$$

$$\chi_k^i(K_3 - \{e\}) = 2 * k - i \quad \textit{and} \quad \chi_k^i(\overline{K_3 - \{e\}}) = 2 * k - i$$

and the bound follows from this.

We suppose by inductive hypothesis, that the property is true for graphs of $n-1$ vertices and we have to prove it for graphs of n vertices.

Case i- G_n and $\overline{G_n}$ are k,i-critical.

By Proposition 3.4, $kd_{G_n}(v) \geq \chi_k^i(G_n) - k + i$ and $kd_{\overline{G_n}}(v) \geq \chi_k^i(\overline{G_n}) - k + i$ for every vertex v. From this

$$k(n-1) \geq \chi_k^i(G_n) + \chi_k^i(\overline{G_n}) - 2k + 2i$$

then

$$\chi_k^i(G_n) + \chi_k^i(\overline{G_n}) \leq k(n+1) - 2i$$

Case ii- G_n or $\overline{G_n}$ are not k,i-critical

W.l.g. we suppose that G_n is not k,i-critical. Then, there is a vertex v such that $\chi_k^i(G_n - \{v\}) = \chi_k^i(G_n)$
On the other hand,

$$\chi_k^i(\overline{G_n}) - k + i \leq \chi_k^i(\overline{G_n - \{v\}}) \leq \chi_k^i(\overline{G_n})$$

By the commutative property of complementing a graph and deleting a vertex, we obtain

$$\chi_k^i(G_n) + \chi_k^i(\overline{G_n}) \leq \chi_k^i(G_n - \{v\}) + \chi_k^i(\overline{G_n - \{v\}}) + k - i$$

and by inductive hypothesis

$$\chi_k^i(G_n - \{v\}) + \chi_k^i(\overline{G_n - \{v\}}) + k - i \leq kn - 2i + k - i \leq k(n+1) - 2i$$

then

$$\chi_k^i(G_n) + \chi_k^i(\overline{G_n}) \leq k(n+1) - 2i$$

□

Lemma 3.1. *If $\chi_k^i(G) + \chi_k^i(\overline{G}) = k(n+1) - 2i$ then $\chi(G) + \chi(\overline{G}) = n + 1$ for $n \geq 3$*

Proof: It is easy to see that

$$\chi_k^i(G) \leq k\chi(G) - i$$

then

$$\chi_k^i(G) + \chi_k^i(\overline{G}) \leq k(\chi(G) + \chi(\overline{G})) - 2i$$

From this, $\chi(G) + \chi(\overline{G}) = n + 1$. □

For the case $i = 0$ and $k = 1$, Finck [10] has shown that only two families of graphs attain the bound. It is true also for $i = 0$ and $k \geq 1$. However if $i, k \geq 1$ there are members of these families do not attain it.

Proposition 3.6. $\chi_k^i(G)\ \chi_k^i(\overline{G}) \leq [(k(n+1) - 2i)/2]^2$ *for* $n \geq 3$

Proof: It is known that $(ab)^{1/2} \leq (a+b)/2 \quad \forall\, a, b \geq 0$.

The bound follows taking $a = \chi_k^i(G)$ and $b = \chi_k^i(\overline{G})$ and using the result of proposition 3.5. □

Note that these results are generalization of the bounds of the classic coloring problem.

4 Coloring Number

The coloring number was defined by Erdös and Hajnal in 1966 [8], as follows:

Definition:

$$col(G) = \min_{p \in \Pi_n} \max_{1 \leq j \leq n} \{d(v_{p(j)}, G_{p(j)})\} + 1$$

where $\Pi_n = \{P : P \textit{ is a permutation of } \{1, 2, \ldots, n\}\}$, $G_{p(j)}$ is the subgraph of G induced by the vertices $v_{p(1)}, \ldots, v_{p(j)}$, i.e. $G_{[v_{p(1)}, \ldots, v_{p(j)}]}$, and d(v,H) denotes the degree of v in the graph H.

We generalize this concept.

Definition: the k, i-coloring number is

$$col_k^i(G) = k(\min_{p \in \Pi_n} \max_{1 \leq j \leq n} \{d(v_{p(j)}, G_{p(j)})\} + 1) - i$$

Consider the permutation $P^* = (v_{p(1)}, v_{p(2)}, \ldots, v_{p(n)})$ where $v_{p(j)}$ is a vertex of minimum degree in $G - \{v_{p(n)}, \ldots, v_{p(j+1)}\}$. The following theorem shows that P^* is the permutation where the minimum is achieved.

Theorem 4.1.

$$col_k^i(G) = k(\max_{1 \le j \le n}\{d(G - \{v_{p(1)}, .., v_{p(j)}\})\} + 1) - i$$

$$= k * (\max\{d(H)/H \ induced \ subgraphs \ of \ G\} + 1) - i$$

This result arises naturally from the definition and the case $k = 1$ and $i = 0$ which was proved by Halin [14] and independently by Matula[21], Finck and Sachs [9] and Lick and White [19].

The bound for $\chi_k^i(G)$ obtained by Proposition 2.2 can be improved by using $col_k^i(G)$.

Proposition 4.1. *If G has not a $\Delta(G)$-regular connected component, then*

$$\chi_k^i(G) \le k\Delta(G) - i$$

Proof: By hypothesis, $\delta(H) \le \Delta(G) - 1 \ \forall \ H$ induced subgraph of G. Using the theorem 4.1 , $col_k^i(G) \le k\Delta(G) - i$. On the other hand, $\chi_k^i(G) \le col_k^i$ by Proposition 2.3 and Theorem 4.1. So $\chi_k^i(G) \le k\Delta(G) - i$ □

5 k,i-Perfect Graphs

Perfect graphs were defined by Berge in 1960 [2] as follows:

Definition: G is perfect if each induced subgraphs of G, H, has chromatic number equal to the size of its largest complete subgraph , i.e. $\chi(H) = \omega(H)$.

With the new concept of k, i-chromatic number, we introduce the following definitions:

Definition: The k, i-clique number $\omega_k^i(G)$ is the k, i-chromatic number of the largest complete subgraph of G, i.e. $\omega_k^i(G) = \chi_k^i(C)$, where C is a maximum clique.

Definition: G is a k, i-perfect graph if G and each of the induced subgraphs of G has k, i-chromatic number equal to the k, i-clique number.

Note that for $k = 1$ and $i = 0$, the definition coincides with Berge's definition.

The following proposition shows a relation between perfect graphs and k,i-perfect graphs.

Proposition 5.1. *If G is a perfect graph then G is a k,i-perfect graph.*

Proof: Let H be an induced subgraph of G. A k,i-coloring of a maximum clique C of H needs $\omega_k^i(H)$ colors. For $v = 1, 2, \ldots, \omega_H$, let S_v be the set of colors assigned to each vertex v of C. Consider a $1,0$-coloring of H with $\chi(H)$ colors. $\chi(H) = \omega(H)$ because H is perfect. We can define a k,i-coloring of H assigning the set S_j to the vertices which were colored with the color j in the $1,0$-coloring considered ab ove. Then $\chi_k^i(H) = \omega_k^i(H)$. □

The converse is false. For example an induced cycle (hole) of 5 vertices is $2,1$-perfect but it is not a $1,0$-perfect graph.

In 1963, Berge made a conjecture for perfect graphs that remains a challenging unsolved problem. The evidence in support of the conjecture has been steadily increasing over the years.

Strong perfect graph conjecture (Berge[3]): G is a perfect graph iif G has not odd holes of length greather than or equal to 5 nor their complements as induced subgraphs.

From the new notion of k,i-perfect graphs, we can generalize the conjecture of Berge.

Conjecture 5.1. *G is k,0-perfect for $k \geq 1$ iif G has not odd holes of length greather than or equal to 5 nor their complements as induced subgraphs.*

Like in the case of Berge's conjecture, we were only able to proof the *if* implication.

Lemma 5.1. *The odd holes of length greather than or equal to 5 and their complements are not $k,0$-perfects*

Proof: Let $C_{2j+1},\ \ j \geq 2$ be an odd hole of length $2j+1$ with vertices $v_1, v_2, \ldots, v_{2j+1}$. Then $\omega_k^0(C_{2j+1}) = 2k$ and it is easy to see that $\chi_k^0(C_{2j+1}) > 2k$.

Now, let $\overline{C_{2j+1}}$ be the complement of an odd hole of length greather than or equal to 5.

It is true that $\omega_k^0(\overline{C_{2j+1}}) = kj$ and $\chi_k^0(\overline{C_{2j+1}}) > kj$.

The former is trivial. To see the later, let $v_1, v_3, \ldots, v_{2j-1}$ be the maximum clique of $\overline{C_{2j+1}}$. If $\chi_k^0(\overline{C_{2j+1}}) = kj$, then a $k,0$-coloring of the maximum clique assigns $\{1, \ldots, k\}$ to v_1, $\{k+1, \ldots, 2k\}$ to $v_3, \ldots, \{(j-1)k, \ldots, jk\}$ to v_{2j-1}. All the vertices of the maximum clique but v_1 belong to the neighborhood of v_{2j+1}, then a k,0-coloring of $\overline{C_{2j+1}}$ must assign $\{1,\ldots,k\}$ to v_{2j+1}. Wi th the same criterion $\{k+1,\ldots,2k\}$ is assigned to v_2 and so on $\{(j-1)k,\ldots,jk\}$ to v_{2j-2}. But there are no colors left to assign to v_{2j}. □

6 Complexity

The general k,i-coloring problem is NP-Hard because the coloring problem (which is a special case for $k=1$ and $i=0$) is a known NP-Hard problem.

An important question that arises when studying the complexity of a problem is if there are special cases in which the complexity changes. For example is the problem still NP-Hard for a given k,i? The answer is no, since for $k=i$ the problem is trivial . But we conjecture that this the only case, i.e for any given k,i with $k>i$ the problem is NP-Hard. We could only prove it for $i=k-1$.

Proposition 6.1. *The k,k-1-coloring problem is NP-Hard*

Proof: Let $A(G)$ be the number of colors used by the algorithm A to color G. Consider two constants: r and d, with $r<2$.

If a polinomial algorithm exists with $A(G)\leq r\chi(G)+d$, then a polinomial algorithm $\tilde{A}$ will exist with $\tilde{A}(G)=\chi(G)$. But $\chi_k^{k-1}(G)\leq\chi(G)+k-1$, so if there is a polinomial algorithm to solve the $k,k-1$-coloring problem, there will be a polinomial algorithm for the coloring problem. □

7 Mathematical Model

The k,i-coloring problem can be formulated as a linear integer programming problem (IP).

Let c be an upper bound of $\chi_k^i(G)$ and let n,m be the number of vertices and edges of G respectivily. We consider the following binary variables :

$\forall j\in\{1,\ldots,c\}$

$$w_j=\begin{cases}1 & \text{if color j was assigned to any vertex}\\ 0 & \text{otherwise}\end{cases}$$

$\forall j\in\{1,\ldots,c\},\ p\in\{1,\ldots,n\}$

$$x_{pj} = \begin{cases} 1 & \text{if color j was assigned to vertex p} \\ 0 & \text{otherwise} \end{cases}$$

$\forall$ pair (p,q) of adjacent vertices, $\forall j \in \{1, \ldots, c\}$

$$y_{pqj} = \begin{cases} 1 & \text{if color j was assigned to vertices p and q} \\ 0 & \text{otherwise} \end{cases}$$

With these variables we make the following IP model:

Min $\sum_{j=1}^{c} w_j$

s.t.

k colors have to be assigned to every vertex

$$\sum_{j=1}^{c} x_{pj} = k \qquad \forall p \in \{1, \ldots, n\}$$

two adjacents vertex don't share more than i colors

$$\sum_{j=1}^{c} y_{pqj} \leq i \qquad \forall p,\ q \ \textit{adjacent vertices}$$

wj has to be 1 if color j was assigned to any vertex

$$x_{pj} \leq w_j \qquad \forall j \in \{1, \ldots, c\}\ p \in \{1, \ldots n\}$$

y_{pqj} has to be 0 if color j was no assigned to vertices p or q

$$y_{pqj} \leq \frac{x_{pj} + x_{qj}}{2}$$

y_{pqj} has to be 1 if color j was assigned to vertices p and q

$$y_{pqj} \geq \frac{x_{pj} + x_{qj} - 1}{2}$$

$$x_{pj}, w_j, y_{pqj} \in \{0, 1\} \quad \forall j \in \{1, \ldots, c\}, p \in \{1, \ldots, n\}, q \in \{1, \ldots, n\}$$

In the last years, the standard approach to solve a IP problem is the Branch and Cut algorithm. It is out of the scope of this work to describe this technique. For details the reader may refer to [23].

We have made much computational experience with two of the most important implementations of the mentioned algorithm: CPLEX [25] and MIPO [1]. Both IP solvers have the state-of-the-art algorithms for IP problems but the major part of the instances of the k, i-coloring problem could not be solved to optimality within a given reasonable time limit.

We believe that the study of the structure of the model will improve the computation time and will lead to optimal solutions for some of the unsolved instances. This is left to further research. Therefore, at the next section we present an heuristic algor ithm to tackle the problem.

8 An Heuristic Algorithm

Since we conjecture that the problem is NP-Hard, many of the instances of k,i-coloring would be potentially intractable from the standpoint of their computational complexity. In this section we describe a starting and improvement heuristic as a way of f inding "good" approximate solutions.

The starting heuristic takes an uncolored vertex, assigns it the first k feasible colors which are choosen from a list and puts them at the end of the list. It finishes when there are no vertices left to color.

STARTING HEURISTIC: generate an initial feasible coloring by a greedy algorithm
L $\leftarrow \emptyset$ (list of colors)
for $l = 1$ to n
    $j \leftarrow 1$
    while $j < k$
        $c \leftarrow$ choose_first_feasible_color_of_L
        if $c =$ **NULL**
            $c \leftarrow$ choose_new_color
        else
            $L \leftarrow L - c$
        end if
        $L \leftarrow$ put_at_the_end(L, c)
        assign color c to v_l
        $j \leftarrow j + 1$
    end while
end for
$colors \leftarrow \#(L)$

The improvement heuristic considers the list of colors used by the current solution except one. It takes an uncolored vertex and assigns it k feasible colors from the list by using the parameters α and β to choose the colors. It finishes after $iter$ trails.

IMPROVEMENT HEURISTIC: looking for better solution
Given $0 < \alpha < 1$,$0 < \beta < 1$ and $iter \in$ N
$loops \leftarrow 1$
while $loops < iter$
    $L \leftarrow 1, \ldots, colors - 1$ (list of colors)
    $flag \leftarrow$ TRUE

```
        l ← 1
        while l < n and flag
                j ← 1
                while j < k and flag
                        c ← choose_first_feasible_color_of_L
                        if c ≠ NULL
                                p1 ← random(0,1)
                                p2 ← random(0,1)
                                if p1 < α
                                        if p2 < β
                                                c1 ← choose_second_feasible_color_of_L
                                        else
                                                c1 ← choose_last_feasible_color_of_L
                                        end if
                                        if c1 ≠ NULL
                                                c ← c1
                                        end if
                                end if
                                L ← L − c
                                L ← put_at_the_end(L, c)
                                assign color c to v_l
                                j ← j + 1
                        else
                                flag ← FALSE
                                loops ← loops + 1
                        end if
                end while
                l ← l + 1
        end while
        if flag
                loops ← 1
                colors ← colors − 1
        end if
end while
return colors
```

The procedures of choosing colors from L consider that a color p is feasible if there are not of adjacent vertices of v_l sharing more than i colors when p is assigned to v_l.

We make three implementations where the uncolored vertices are choosen from a list sorted by each of the following criteria:

1. by the label of the vertices of the graph
2. by maximum to minimum degree
3. by repeatedly removing a vertex of minimum degree in the subgraph induced by the vertices not yet chosen and placing it after all the remaining vertices but before all the vertices already removed (*smallest-last*)

The complexity of the heuristic is $O(itern^3)$.

9 Computational Experience

This section is devoted to testing the heuristic computationally. For this purpose, we use a family of randomly generated graphs with different number of vertices and edge densities.

In Table 1 we report for each graph:

- the number of vertices
- the percentage of edge's density
- the theoretical upper bound
- the theoretical lower bound derivated by the k, i-coloring of K_2 and K_3.
- the optimal solution obtained by IP in the cases where it was possible to find it or a lower bound getting from the IP optimal solution of the k, i-coloring of subgraphs.
- the heuristic solution with $\alpha = 0.8$, $\beta = 0.4$ and the *smallest-last* criteria (the best performance of the heuristic was obtained with them).

The linear integer problems were solved using the CPLEX 6.0 or MIPO, running on a SUNW, SPARCstation-4. The heuristic was implemented in Borland C++ 3.1 and was run on a Pentium 166Mhz.

Table 1

n	density (%)	k	i	theoretical bound upper	theoretical bound lower	IP solution	heuristic solution
7	100	8	3	53	15	17	18
7	80	5	3	22	8	8*	8
7	60	10	3	37	19	22*	22
7	40	11	7	26	15	15*	15
7	20	15	5	25	25	25*	25
9	100	7	2	61	15	17	20
9	80	8	6	42	10	10*	11
9	60	10	2	48	18	27	28
9	40	6	1	23	11	18*	18
9	20	9	6	21	12	12*	12
12	100	10	8	112	12	12	13
12	80	9	6	66	12	13	14
12	60	11	7	59	15	15	17
12	40	9	3	33	15	18*	19
12	20	10	4	26	17	18*	18
15	100	7	4	101	10	11	14
15	80	12	9	111	15	15	17
15	60	10	8	62	12	12	13
15	40	6	3	27	9	10*	11
15	20	9	7	20	11	11*	11
30	100	10	3	297	21	22	40
30	80	8	4	156	12	14	19
30	60	15	10	215	20	20	25
30	40	6	5	55	7	7	8
30	20	8	6	34	10	10	11
50	100	8	3	397	15	17	31
50	80	10	1	339	19	34	90
50	60	5	3	117	7	7	11
50	40	9	2	133	16	16	38
50	20	13	5	99	21	21	32
80	100	4	2	318	6	8	16
80	80	10	8	562	12	12	15
80	60	12	10	458	14	14	16
80	40	9	2	223	16	16	46
80	20	11	9	112	13	13	15

n	density (%)	k	i	theoretical bound upper	theoretical bound lower	IP solution	heuristic solution
100	100	12	8	1192	16	16	24
100	80	7	5	499	9	10	14
100	60	6	1	311	11	15	49
100	40	10	7	323	13	13	18
100	20	14	3	207	25	25	60
150	100	7	4	1046	10	11	22
150	80	10	5	1095	15	17	32
150	60	9	4	707	14	15	32
150	40	12	5	595	19	21	41
150	20	3	1	71	5	6	14

* Optimal solution

The above experiments suggest the following conclusions:

1. The IP problem is very hard to solve. Even in small instances of the problem, the branch and cut algorithm spends too much time to obtain the optimal solution.
2. The theoretical bounds of section 2 are too bad in high density graphs like in the classic coloring problem.
3. The heuristic solution was very good in small instances. For high density graphs we know that the lower bounds are weak and therefore we do not have enough information to compare the solutions. The "hardness" of an instance is related to the densit y of the graph more than to the number of vertices of the graph.

10 Concluding Remarks

In this article we have introduced a generalization of the graph coloring problem. Some upper bounds and an extension of properties of the classic coloring problem are presented. We make an IP formulation and develope an heuristic algorithm.

Further research is needed mainly in two directions: the complete characterization of the complexity problem and the study of the feasible solution set of the IP formulation to improve the performance of a Branch&Cut algorithm.

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