A Qualitative Method for Multicriteria Decision Aid

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Abstract
The notion of relative importance of criteria is central in multicriteria decision aid. In this work we define the concept of comparative coalition structure, as an approach for formally discussing the notion of relative importance of criteria. We also present a multicriteria decision aid method that does not require the assignment of weights to the criteria.

1 Introduction

In a multicriteria decision problem, the criteria considered are generally in conflict and have different importances for the decision maker. Multicriteria decision aid consists in providing a recommendation to the decision maker related to the decision being taken, using for that a clearly specified mathematical model.

The notion of relative importance is central in multicriteria decision aid (see for example [4] and [7]). In most methods this relative importance is represented by numbers, commonly called weights. The interpretation of the weights depends fundamentally on the decision model. In a multicriteria linear value model, the weights can be interpreted as constant relative trade-offs [3]. In the ELECTRE methods [6] and in the Prométhée methods [2], the weights can be interpreted as importance coefficients.

In this work we present a multicriteria decision aid method that does not require the assignment of weights to the criteria. The paper is organized as follows. In section 2 we define the notions of partial preference structure and multicriteria preference structure. In section 3 we define the concept of comparative coalition structure as an approach for formally discussing the notion of relative importance of criteria. In section 4 we describe the qualitative method proposed.

2 Preferences Structures

Let A a finite set of two or more potential actions (or alternatives). Let $C = \{1, 2, ..., n\}$ $(n \geq 2)$ be the set of criteria considered. A *partial preference structure* E_j corresponding to the criterion j is an ordered pair $E_j = (X_j, \succ_j)$, where X_j is a set of impact levels which serve to describe plausible impacts of potential actions with respect to criterion j, and $\succ_j$ is an asymmetric binary relation modeling the partial preferences of the decision maker with respect to criterion j.

A partial preference structure $E_j = (X_j, \succ_j)$ is *essential* if there exist x_j, $y_j \in X_j$ such that $x_j \succ_j y_j$. Since nonessential criteria contribute nothing to the analysis of a decision problem, we will assume henceforth that every E_j is essential.

Let $E_j = (X_j, \succ_j)$ be a partial preference structure and let $\sim_j$be the binary relation defined in X_j as

$$x_j \sim_j y_j \Leftrightarrow \neg(x_j \succ_j y_j) \text{ and } \neg(y_j \succ_j x_j)$$

(the symbol $\neg$ represents the logical negation). It is easy to see that $\sim_j$ is reflexive and symmetric. The relation $\sim_j$ is called the relation of partial indifference corresponding to criterion j.

An *evaluator* g_j corresponding to criterion j is a function $g_j : A \to X_j$, where $g_j(a)$ represents the impact of alternative a with respect to criterion j. The function g_j induces a partial preference relation P_j and a partial indifference relation I_j on A as follows:

$$aP_jb \Longleftrightarrow g_j(a) \succ_j g_j(b),$$
$$aI_jb \Longleftrightarrow g_j(a) \sim_j g_j(b).$$

Suppose that for every $j \in C$, a partial preference structure E_j is defined. Let $X = X_1 \times X_2 \times ... \times X_n$; the set X is called the *consequence space*. Let U be a proper subset of $C = \{1, 2, ..., n\}$. For notational convenience we will write $\mathbf{x} \in X$ as $\mathbf{x} = ((x_j)_{j \in U}, (x_j)_{j \notin U})$.

Let $\succ$ be a binary relation modeling the global preference in X. We will say that $(X, \succ, E_1, ..., E_n)$ is a *multicriteria preference structure* iff it satisfies the following axioms:

M1) $\succ$ is asymmetric.

M2) $x_j \succ_j y_j \Longleftrightarrow (x_j, (a_i)_{i \neq j}) \succ (y_j, (a_i)_{i \neq j})$ for every $(a_i)_{i \neq j} \in \prod_{i \neq j} X_j$.

M3) For every $\mathbf{x}, \mathbf{y} \in X$, such that $\mathbf{x} \succ \mathbf{y}$, and for every $k \in C$:

M3.1) if $z_k \succ_k x_k$ then $(z_k, (x_j)_{j \neq k}) \succ \mathbf{y}$;

M3.2) if $y_k \succ_k z_k$ then $\mathbf{x} \succ (z_k, (y_j)_{j \neq k})$.

Proposition 1: Let $(X, \succ, E_1, ..., E_n)$ be a multicriteria preference structure, and let W be a nonempty subset of C. If $\mathbf{x}, \mathbf{y} \in X$ are such that $x_j \succ_j y_j$ for every $j \in W$ and $x_j = y_j$ for every $j \in C \backslash W$, then $\mathbf{x} \succ \mathbf{y}$.

Proof. The proof will be by induction on $m = |W|$. If $m = 1$, that is, if $W = \{k\}$, then $x_k \succ_k y_k$ and $x_j = y_j$ for all $j \neq k$. Hence

$$\mathbf{x} = (x_k, (x_j)_{j \neq k}) = (x_k, (y_j)_{j \neq k}) \succ (y_k, (y_j)_{j \neq k}) = \mathbf{y}.$$

Now, assume that the result holds for every subset of criteria with m elements. Let $W \subseteq C$ such that $|W| = m + 1$ and let $\mathbf{x}, \mathbf{y} \in X$ such that

$$x_j \succ_j y_j \text{ for every } j \in W \text{ and } x_j = y_j \text{ for every } j \in C \backslash W.$$

Let $k \in W$ and put $W' = W \backslash \{k\}$. Hence $|W'| = m$. Let $\mathbf{x}' \in X$ be defined as

$$x'_j = x_j \text{ for } j \neq k \text{ and } x'_k = y_k.$$

It follows that $x'_j \succ_j y_j$ for every $j \in W'$ and $x'_j = y_j$ for every $j \in C \backslash W'$. By our assumption, $\mathbf{x}' \succ \mathbf{y}$. Since $x_k \succ_k y_k = x'_k$, we have by axiom M3 that $(x_k, (x'_j)_{j \neq k}) \succ \mathbf{y}$. Since $x'_j = x_j$ for every $j \neq k$, we see that $\mathbf{x} \succ \mathbf{y}$. ■

3 Comparative Coalition Structure

Let $(X, \succ, E_1, ..., E_n)$ be a multicriteria preference structure. For every $j \in C$ let $x_j^+ \in X_j$ be a good level of impact; that is, an impact level considered attractive for the decision maker, and $x_j^0 \in X_j$ a neutral level of impact, that is, an impact level considered neither attractive nor unattractive for the decision maker.

Let $\wp(C)$ be the set of all the subsets of criteria. The elements of $\wp(C)$ will be called *coalitions*. We will say that "the coalition U is more important than coalition V" (notation: $U \rhd V$) if it satisfies the following condition:

$$((x_j^+)_{j \in U}, (x_j^0)_{j \notin U}) \succ ((x_j^+)_{j \in V}, (x_j^0)_{j \notin V}).$$

The ordered pair $(\wp(C), \rhd)$ will be called the *comparative coalition structure* determined by $(X, \succ, E_1, ..., E_n)$.

Proposition 2: Let $(\wp(C), \rhd)$ be the *comparative coalition structure* determined by $(X, \succ, E_1, ..., E_n)$. Then for every $U, V \in \wp(C)$ such that $V \subset U$, we have $U \rhd V$.

Proof. Let $U, V \in \wp(C)$ such that $V \subset U$ and put $W = U \backslash V$. Then W is nonempty. Since

$$((x_j^+)_{j\in U}, (x_j^0)_{j\notin U}) = ((x_j^+)_{j\in W}, (a)_{j\notin W})$$

and

$$((x_j^+)_{j\in V}, (x_j^0)_{j\notin V}) \succ ((x_j^0)_{j\in W}, (a_j)_{j\notin W}),$$

where $a_j = x_j^+$ for every $j \in V$ and $a_j = x_j^0$ for every $j \in C\backslash U$, we have by proposition 1 that

$$((x_j^+)_{j\in U}, (x_j^0)_{j\notin U}) \succ ((x_j^+)_{j\in V}, (x_j^0)_{j\notin V}),$$

so that $U \rhd V$. ■

Corollary 3: Let $(\wp(C), \rhd)$ be the *comparative coalition structure* determined by $(X, \succ, E_1, ..., E_n)$, then $C \rhd V \rhd \varnothing$ for every $U \in \wp(C), \varnothing \neq U \neq C$.

A multicriteria preference structure is *independent*, iff for every $U \subset C$ if

$$((x_j)_{j\in U}, (a_j)_{j\notin U}) \succ ((y_j)_{j\in U}, (a_j)_{j\notin U}) \text{ for some } (a_j)_{j\notin U} \in \prod_{j\notin U} X_j$$

then

$$((x_j)_{j\in U}, (b_j)_{j\notin U}) \succ ((y_j)_{j\in U}, (b_j)_{j\notin U}) \text{ for some } (b_j)_{j\notin U} \in \prod_{j\notin U} X_j.$$

Proposition 4: Let $(X, \succ, E_1, ..., E_n)$ be an independent multicriteria preference structure, and let $(\wp(C), \rhd)$ be the corresponding *comparative coalition structure*. If U and V are two coalitions, such that, $U \rhd V$, then $(U \cup Z) \rhd (V \cup Z)$ for every $Z \in \wp(C)$, such that $Z \cap U = \varnothing = Z \cap V$.

Proof. Let $U, V \in \wp(C)$. Since the multicriteria preference structure is independent, we can suppose, without loss of generality, that U and V are disjoint.

Assume that $U \rhd V$. Hence

$$((x_j^+)_{j\in U}, (x_j^0)_{j\in V}, (a_j)_{j\notin U\cup V}) \succ ((x_j^0)_{j\in U}, (x_j^+)_{j\in V}, (a_j)_{j\notin U\cup V})$$

where $b_j = x_j^+$ for every $j \in Z$ and $b_j = x_j^0$ for every $j \in [C\backslash(U \cup V)]\backslash Z$. Hence

$$((x_j^+)_{j\in U\cup Z}, (x_j^0)_{j\notin U\cup Z}) \succ ((x_j^+)_{j\in V\cup Z}, (x_j^0)_{j\notin V\cup Z}).$$

Thus $(U \cup Z) \rhd (V \cup Z)$. ■

Proposition 5: Let $(X, \succ, E_1, ..., E_n)$ be a transitive multicriteria preference structure, then $(\wp(C), \rhd)$ is transitive.

Proof. Let U, V, W be three coalitions such that, $U \rhd V$ and $V \rhd W$. That is,

$$((x_j^+)_{j\in U}, (x_j^0)_{j\notin U}) \succ ((x_j^+)_{j\in V}, (x_j^0)_{j\notin V})$$

and

$$((x_j^+)_{j\in V}, (x_j^0)_{j\notin V}) \succ ((x_j^+)_{j\in W}, (x_j^0)_{j\notin W}).$$

It follows, from the transitivity of $\succ$, that

$$((x_j^+)_{j\in U}, (x_j^0)_{j\notin U}) \succ ((x_j^+)_{j\in W}, (x_j^0)_{j\notin W}).$$

■

4 Description of the Method

In this section we present a method for multicriteria decision aid that does not require the assignment of weights to the criteria. We will suppose that the partial preferences of the decision maker satisfy the following model:

$$aP_jb \Longleftrightarrow g_j(a) > g_j(b) \quad \text{and} \quad aI_jb \Longleftrightarrow g_j(a) = g_j(b)$$

where $g_j(a) \in \Re$ represents the impact of alternative a with respect to criterion j.

We will also assume that for every criterion j there are six preference thresholds:

$$0 = u_j^0 < u_j^1 < u_j^2 < u_j^3 < u_j^4 < u_j^5$$

such that

1) if $g_j(b) + u_j^0 < g_j(a) \leq g_j(b) + u_j^1$ the preference aPb is *negligible*.

2) if $g_j(b) + u_j^1 < g_j(a) \leq g_j(b) + u_j^2$ the preference aPb is *weak*.

3) if $g_j(b) + u_j^2 < g_j(a) \leq g_j(b) + u_j^3$ the preference aPb is *moderate*.

4) if $g_j(b) + u_j^3 < g_j(a) \leq g_j(b) + u_j^4$ the preference aPb is *strong*.

5) if $g_j(b) + u_j^4 < g_j(a) \leq g_j(b) + u_j^5$ the preference aPb is *very strong*.

6) if $g_j(b) + u_j^5 < g_j(a)$ the preference aPb is *extreme*.

The functions g_j and the thresholds u_j^k can be built with the MACBETH method [1]. We define

$$C_k(a,b) = \{j \in C : g_j(b) + u_j^{k-1} < g_j(a) \leq g_j(b) + u_j^k\} \qquad \text{for } k = 1, 2, ..., 5,$$

and

$$C_6(a,b) = \{j \in C : g_j(b) + u_j^5 < g_j(a)\}.$$

Let S_k be the binary relation defined in A as aS_kb iff the following conditions hold:

(C1) $C_r(b,a) = \varnothing$ for every $r \geq k$.

(C2) If $\bigcup_{j=1}^{k-1} C_j(b,a) \neq \varnothing$, then $\bigcup_{j=1}^{6} C_j(a,b) \rhd \bigcup_{j=1}^{k-1} C_j(b,a)$.

The next result shows that the relations S_k are a nested sequence.

Theorem 6: $S_k \subseteq S_{k+1}$ for every $k = 1, ..., 5$.

Proof. If aS_kb then $C_r(b,a) = \varnothing$ for every $r \geq k$. Hence $C_r(b,a) = \varnothing$ for every $r \geq k+1$, and the condition (C1) is satisfied. Furthermore,

$$\bigcup_{j=1}^{6} C_j(a,b) \rhd \bigcup_{j=1}^{k-1} C_j(b,a) = \bigcup_{j=1}^{k} C_j(b,a)$$

(because $C_k(b,a) = \varnothing$). Thus the condition (C2) is satisfied, and it follows that $aS_{k+1}b$.

■

Let D be the binary relation defined in A as:

$$aDb \Longleftrightarrow g_j(a) \geq g_j(b)$$

for every $j \in C$. The relation D is called *dominance relation.*

A binary relation S in A is called an *outranking relation* [5] if it satisfies the following conditions:

(O1) If aDb then aSb.

(O2) If $aSb$and bDc then aSc.

(O3) If aDb and aSb then aSc.

Intuitively, aSb (*a outranks b*) if we have good reasons to admit the hypothesis that action a is at least as good as action b, and there are no good reasons to refuse it.

Theorem 7: Every S_k is an outranking relation.

Proof. (O1) If aDb, then $g_j(a) \geq g_j(b)$ for every $j \in C$, hence $C_r(b,a) = \varnothing$ for every $r = 1, 2, ..., 6$. Thus conditions (C1) and (C2) are clearly satisfied, so that aS_kb.

(O2) Suppose aS_kb and bDc. We will prove that aS_kc. Let $r \geq k$. By hypothesis, $C_r(b,a) = \varnothing$; that is, $g_j(b) \leq g_j(a) + u_j^{r-1}$. Since bDc, we have $g_j(c) \leq g_j(a) + u_j^{r-1}$, hence $C_r(c,a) = \varnothing$. Thus condition (C1) is satisfied. To prove condition (C2) we note that if $j \in \bigcup_{j=1}^{k-1} C_j(c,a)$ then

$$g_j(a) + u_j^{r-1} < g_j(c) \leq g_j(a) + u_j^r$$

for some $r \leq k-1$. Since bDc, we have $g_j(a) + u_j^{r-1} < g_j(b)$. Furthermore it cannot be that $g_j(b) > g_j(a) + u_j^{k-1}$, because $C_r(b,a) = \varnothing$ for every $r \geq k$. Hence,

$$g_j(a) + u_j^{r-1} < g_j(c) \leq g_j(a) + u_j^{k-1}$$

that is, $j \in \bigcup_{j=1}^{k-1} C_j(b,a)$. Thus $\bigcup_{j=1}^{k-1} C_j(c,a) \subseteq \bigcup_{j=1}^{k-1} C_j(b,a)$. It follows that either

$$\bigcup_{j=1}^{k-1} C_j(b,a) \rhd \bigcup_{j=1}^{k-1} C_j(c,a) \quad \text{or} \quad \bigcup_{j=1}^{k-1} C_j(b,a) = \bigcup_{j=1}^{k-1} C_j(c,a).$$

Next, if $j \in \bigcup_{j=1}^{6} C_j(a,b)$, then $g_j(b) + u_j^r < g_j(a)$ for some r. Since bDc, we also have $g_j(c) + u_j^r < g_j(a)$. hence $j \in \bigcup_{j=1}^{6} C_j(a,c)$; that is,

$$\bigcup_{j=1}^{6} C_j(a,b) \subseteq \bigcup_{j=1}^{6} C_j(a,c).$$

Thus $\bigcup_{j=1}^{6} C_j(a,c) \rhd \bigcup_{j=1}^{6} C_j(a,b)$. Since aS_kb, we have

$$\bigcup_{j=1}^{6} C_j(a,b) \rhd \bigcup_{j=1}^{k-1} C_j(b,a).$$

Finally, since either $\bigcup_{j=1}^{k-1} C_j(b,a) \rhd \bigcup_{j=1}^{k-1} C_j(c,a)$ or $\bigcup_{j=1}^{k-1} C_j(b,a) = \bigcup_{j=1}^{k-1} C_j(c,a)$ it follows from the transitivity of $\rhd$ that

$$\bigcup_{j=1}^{6} C_j(a,c) \rhd \bigcup_{j=1}^{k-1} C_j(c,a).$$

Thus the condition (C2) is satisfied and we conclude that aS_kc. Thus (O2) is true. The proof that S_k satisfies condition (O3) is analogous.

■

Let $S_1, ..., S_6$ the outranking relations defined above. For every $a \in A$ and for every $k = 1, 2, ..., 6$ put

$$d_k^+(a) = |\{b \in A : b \neq a \text{ and } aS_kb\}|$$

and

$$d_k^-(a) = |\{b \in A : b \neq a \text{ and } bS_ka\}|.$$

Let $f : A \to \Re$ be the function defined as:

$$f(a) = \sum_{k=1}^{6} d_k^+(a) - \sum_{k=1}^{6} d_k^-(a).$$

We will say that action a is better than b iff $f(a) > f(b)$. The function f defines a weak order on A.

5 Summary and Conclusions

In this work we presented the notion of comparative coalition structure as an approach for formally discussing the notion of relative importance of criteria. We also have presented an outranking method for multicriteria decision aid. Our method differs from the ELECTRE methods in the following:

(i) In ELECTRE I, II and III the importance of criterion j is represented by means of a weight $k_j > 0$, which can be regarded as an importance coefficient. our method does not require the assignment of weights to the criteria, instead a relation of importance between criteria is considered.

(ii) In ELECTRE methods the partial preferences are modeling using an indifference threshold and a weak preference threshold. We use five preference thresholds with the purpose of modeling strength of preference.

(iii) In ELECTRE methods the outranking relations are built by relaxing a concordance condition, while in our method the outranking relations are obtained by relaxing a discordance condition.

(iv) The ELECTRE IV method supposes, implicitly, that a coalition of criteria is more important than another if and only if its cardinality is greater or equal, while in our method this is not necessarily true.

An advantage of the method proposed is that the calculations involved are few an easy, moreover it is possible to built each outranking relation independently of the others.

A disadvantage of the method is that, in order to finding the comparative coalition structure it is necessary that the decision maker expresses her or his preferences between every pair of coalitions of criteria. However, if we assume that the multicriteria preference structure is independent and transitive, we can built the comparative coalition structure with a relative small number of comparisons.

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References

[1] Bana e Costa, C. and J.C. Vansnick. "A theoretical framework for measuring attractiveness by a categorical based evaluation technique (MACBETH)", in *Advances in Multicriteria Analysis*, P.M. Pardalos, Y. Siskos and C. Zopounidis (eds.), Kluver Academic Publishers, 1995, 93-100.

[2] Brans, J. P., B. Maréschal and Ph. Vincke. "How to select and rank projects. The Prométhée method", *European Journal of Operational Research*, **59**, 1986, 248-261.

[3] French, S. *Decision Theory: An Introduction to the Mathematics of Rationality*, Halsted Press-Wiley, New York, 1988.

[4] Mousseau, V. "Analyse et classification de la littérature traitant de l'importance relative des critères en aide multicritère à la décision", *Recherche Opérationnelle*, **26**, 4, 1992, 367-389.

[5] Roy, B. "How outranking relation helps multiple criteria decision making", in Cochrane, J. and Zeleni M. (eds.), *Multiple Criteria Decision Making*, University of South Carolina Press, Columbia, South Carolina, 1973, 179-201.

[6] Roy, B. "The outranking approach and the foundations of ELECTRE methods", in C. Bana e Costa (ed.), *Readings in Multiple Criteria Decision Aid*, Springer Verlag, Berlin, 1990, 155-183.

[7] Roy, B. and V. Mousseau. "A theoretical framework for analysing the notion of relative importance of criteria", *Journal of Multicriteria Decision Analysis*, **5**, 1996, 145-159.