Random measures

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Motivating Question

Is there a natural measure *on the space of measures* (on 2^ω) as, say, the Lebesgue measure is natural?

Preliminaries

- A (Borel probability) **measure** on 2^ω is a function $\mu\colon 2^{<\omega} \to [0,1]$ satisfying
  - $\mu(\varnothing) = 1$ and
  - $\mu(\sigma) = \mu(\sigma 0) + \mu(\sigma 1)$.
- $\mathcal{P}(2^\omega)$ = (Borel probability) measures on 2^ω
  - This space is nice enough to do computability on.
- $\mathrm{MLR}_\lambda \subset 2^\omega$ is the set of (Lebesgue) Martin-Löf randoms.

Introduction

- A measure is determined by the sequence $\langle \mu(0|\sigma) \rangle_{\sigma \in 2^{<\omega}}$ of conditional probabilities, where $\mu(0|\sigma)\mu(\sigma) = \mu(\sigma 0)$.
- This is easy to work with.

A Natural Map

Definition (Porter)

The map $\Phi\colon 2^\omega \to \mathcal{P}(2^\omega)$, $x \mapsto \mu_x$ is defined by

- $\mu_x(\varnothing) = 1$,
- $\mu_x(0|\sigma_i) = x_i$,

where $x_i(j) = x(\langle i, j\rangle)$ is the i^{th} **column** of x (thought of as a real number).

This map is

- computable: $\mu_x(\sigma)$ is computable, to desired precision, uniformly in (the oracle) x and σ;
- surjective
- (strongly) almost injective: y has no dyadic columns and $x \neq y \implies \mu_x \neq \mu_y$

Gloria in excelsis Φ

"A function is glorified into a random variable as soon as its domain is assigned a probability distribution with respect to which the function is measurable" – Joseph Doob

- Φ pushes forward λ to $P = \lambda \circ \Phi^{-1} \in \mathcal{P}(\mathcal{P}(2^\omega))$
- P corresponds to the stochastic process that uses an IID uniform sequence $\langle X_i \rangle_{i \in \omega}$ to assign conditional probabilities (as before)
- Any $Q \in \mathcal{P}(\mathcal{P}(2^\omega))$ is the push forward of *some* measure on $[0, 1]$ via Φ.
  - Does that make P natural?

Random measures

- Now we have $\mathrm{MLR}_P = \Phi(\mathrm{MLR}_\lambda)$
- What do elements of MLR_P look like?
- What do elements of MLR_μ look like for $\mu \in \mathrm{MLR}_P$?
  - $y \in \mathrm{MLR}_\mu$ if $y \notin \bigcap U_n$ whenever $\langle U_n \rangle_{n \in \omega}$ is uniformly $\Sigma_1^{0,x}$ with $\mu U_n \leq 2^{-n}$, where $\mu = \mu_x$.

Barycenter of P

Fact

$$\lambda(\sigma) = \int_{\mathcal{P}(2^\omega)} \mu(\sigma)\, dP$$

Proof.

$$\begin{aligned}
\int_{\mathcal{P}(2^\omega)} \mu(\sigma)\, dP &= \int_{2^\omega} \mu_x(\sigma)\, d\lambda && \text{(Change of variables.)} \\
&= \int_{2^\omega} \prod_{i=0}^{|\sigma|-1} \mu_x(\sigma(i) | \sigma \restriction i)\, d\lambda \\
&= \prod_{i=0}^{|\sigma|-1} \int_{2^\omega} \mu_x(\sigma(i) | \sigma \restriction i)\, d\lambda && \text{(Independence.)} \\
&= 2^{-|\sigma|}
\end{aligned}$$

Randoms' randoms are random...

Fact (Hoyrup)

$$\bigcup_{\mu \in \mathsf{MLR}_P} \mathsf{MLR}_\mu = \mathsf{MLR}_\lambda$$

Proof.

($\subseteq$) A λ Solovay test $\langle A_n \rangle_{n\in\omega}$ is a μ Solovay test for each $\mu \in \mathsf{MLR}_P$: Build a P-ML test $V_k := \{\mu\colon \sum_n \mu(A_n) > 2^k\} \in \Sigma^0_1$.

$$1 \geq \sum \lambda A_n = \int \sum \mu(A_n)\, dP \geq \int_{V_k} \sum \mu(A_n)\, dP$$
$$\geq 2^k P(V_k)$$

...and randoms are randoms' randoms

$$\bigcup_{\mu \in \mathrm{MLR}_P} \mathrm{MLR}_\mu = \mathrm{MLR}_\lambda$$

Proof (cont.)

($\supseteq$)

- $\exists$ lower semicomputable (l.s.c.) $t_P \colon \mathcal{P}(2^\omega) \to [0,\infty]$ and $t_\mu \colon 2^\omega \to [0,\infty]$ that are finite iff input is random.
- $t_P(\mu) \cdot t_\mu(x)$ is a l.s.c. function of (μ, x)
- $f(x) := \inf_\mu t_P(\mu) \cdot t_\mu(x)$ is l.s.c.
- $\int_{2^\omega} f(x)\, d\lambda \leq 1$
- So $x \in \mathrm{MLR}_\lambda \Rightarrow f(x) < \infty \Rightarrow t_P(\mu) < \infty$ and $t_\mu(x) < \infty$ □

Random measures are atomless

Fact (Quinn)

$\mu \in \mathrm{MLR}_P \Rightarrow \forall x\, \mu\{x\} = 0.$

Proof.

- $\{x\colon \mu_x$ has an atom with mass $\geq 1/n\} = \{x\colon \forall k \exists \sigma \in 2^k [\mu_x(\sigma) \geq 1/n]\} \in \Pi^0_1$.
- $m(x) :=$ max mass of an atom of μ_x,
- $m_i(x) :=$ max mass of an atom strictly above $i = 0, 1$

Random measures are atomless

Atomlessness proof continued.

- $m(x) :=$ max mass of an atom of μ_x,
- $m_i(x) :=$ max mass of an atom strictly above $i = 0, 1$

Facts:

- $m(x) = \max\{x_0 m_0(x), (1 - x_0) m_1(x)\}$
- m and m_i have the same distribution.
- Kolmogorov's 0-1 law $\Rightarrow m, m_i = 0$ a.s. or $m, m_i > 0$ a.s.

Random measures are atomless

Atomlessness proof continued.

$M := \{x_0 m_0 \geq (1 - x_0) m_1\}$.

$$\begin{aligned}
\int_{2^\omega} m(x) &= \int_M x_0 m_0(x) + \int_{M^c} (1 - x_0) m_1(x) \\
&\leq \int_{2^\omega} x_0 m_0(x) + \int_{M^c} (1 - x_0) m_1(x) \\
&= \int_{2^\omega} \texttt{fc}(x) m(x) + \int_{M^c} (1 - x_0) m_1(x) \qquad (\texttt{fc} \equiv_d x_0) \\
&= \int_{2^\omega} \texttt{fc}(x) m(x) + \int_{2^\omega} (1 - x_0) m_1(x) - \int_M (1 - x_0) m_1(x) \\
&= \int_{2^\omega} \texttt{fc}(x) m(x) + \int_{2^\omega} (1 - \texttt{fc}(x)) m(x) - \int_M (1 - x_0) m_1(x) \\
&= \int_{2^\omega} m(x) - \int_M (1 - x_0) m_1(x).
\end{aligned}$$

Random measures are atomless

Atomlessness proof continued.

$\int_{2^\omega} m(x) \leq \int_{2^\omega} m(x) - \int_M (1 - x_0) m_1(x)$

- $\Rightarrow \lambda M = 0$ or $m_1(x) = 0$ a.s. on M.
- Similarly $\lambda(M^c) = 0$ or $m_0(x) = 0$ a.s. on M^c.
- $0 < \lambda M \Rightarrow m_1 = 0$ is zero on a positive measure set, hence a.e.
- Similarly if $0 < \lambda M^c$.

Thus having an atom is a null Π^0_1 class, so even Kurtz random measures are atomless. □

Random measures are mutually singular (wrt λ)

Definition

- **Absolute continuity**: $\mu \ll \lambda$ iff $\lambda A = 0 \Rightarrow \mu A = 0$
  - Note: $\mu \ll \lambda \Rightarrow \mu$ atomless
- **Mutual singularity**: $\mu \perp \lambda$ iff $\exists A \; \lambda A = 1, \; \mu A = 0$.

Random measures are mutually singular (wrt λ)

Fix $\mu \in \text{MLR}_P$.

Fact (Laurent)

$$\underbrace{\text{MLR}_\mu}_{\mu-full} \cap \underbrace{\text{MLR}^\mu}_{\lambda-full} = \emptyset$$

Proof idea.

- Given x, look at places where $\mu(0|x \upharpoonright n) > \frac{3}{4}$.
- $x \in \text{MLR}_\mu$ obeys the μ measure, so more than $\frac{3}{4}$ of the time, $x(n)$ will be 0.
- $x \in \text{MLR}^\mu$ obeys λ, so $\frac{1}{2}$ of the time, $x(n)$ will be 0. □

B/t absolute continuity & atomlessness

- Motivation: If $\mu \ll \lambda$, then by Radon-Nikdodym and Lebesgue differentiation

$$-\frac{\log \mu(x \upharpoonright n)}{n} \to 1 \quad \mu\text{-a.e.} \,\&\, \lambda\text{-a.e.}$$

- Notice: $\limsup -\frac{\log \mu(x \upharpoonright n)}{n} > 0\, \mu$-a.e. $\Rightarrow \mu$ is atomless.

B/t absolute continuity & atomlessness

- $c_i(\mu, x) := \mu(x(i)|x \upharpoonright i)$
- $\mu(x \upharpoonright n) = \prod_{i<n} c_i(\mu, x)$
- The c_i's are uniformly distributed IID

B/t absolute continuity & atomlessness

For all x and P-almost every μ,

$$\begin{aligned} -\frac{1}{n}\log\mu(x \restriction n) &= -\frac{1}{n}\log\prod_{i<n} c_i(\mu, x) \\ &= -\frac{1}{n}\sum_{i<n}\log c_i(\mu, x) \\ &\to -\mathbb{E}(\log c_i) = 1 \qquad \text{(By LLN.)} \end{aligned}$$

B/t absolute continuity & atomlessness

Fact (Effective LLN)

Since $\langle \log c_i \rangle_{i \in \omega}$ *is an IID sequence of computable random variables on* $\mathcal{P}(2^\omega) \times 2^\omega$*, for each* $(\mu, x) \in \mathrm{MLR}_{P \otimes \lambda}$

$$-\frac{1}{n} \sum_{i<n} \log c_i(\mu, x) \to 1.$$

B/t absolute continuity & atomlessness

Fact (Van Lambalgen)

$(\mu, x) \in \mathrm{MLR}_{P \otimes \lambda}$ *iff* $\mu \in \mathrm{MLR}_P$ *and* $x \in \mathrm{MLR}^{\mu}$

B/t absolute continuity & atomlessness

- $(\mu, x) \in \mathrm{MLR}_{P \otimes \lambda}$ iff $\mu \in \mathrm{MLR}_P$ and $x \in \mathrm{MLR}^\mu$

Fact

If $\mu \in \mathrm{MLR}_P$, then $-\frac{1}{n} \log \mu(x \upharpoonright n) \to 1$ for $x \in \mathrm{MLR}^\mu$ (in particular λ-a.e.).

- So λ-a.e. x has lots of μ-information (on average).

B/t absolute continuity & atomlessness

- $-\frac{1}{n}\log \mu(x \upharpoonright n) \to 1 \quad \lambda$-a.e.
- But this doesn't generalize atomlessness, we want convergence μ-a.e.

Conjecture

If $\mu \in \mathrm{MLR}_P$ *and* $x \in \mathrm{MLR}_\mu$, *then*

$$-\frac{1}{n}\log \mu(x \upharpoonright n) \to \int H(p)\,d\lambda,$$

where $H(p) = -p\log p - (1-p)\log(1-p)$.

Questions & To-do list

- Are there $\mu, \nu \in \text{MLR}_P$ with $\text{MLR}_\mu \cap \text{MLR}_\nu = \emptyset$?
- If not, are there $\mu_i \in \text{MLR}_P$ with $\bigcap \text{MLR}_{\mu_i} = \emptyset$?
- What is $P\{\mu : x \in \text{MLR}_\mu\}$ for a fixed $x \in \text{MLR}_\lambda$?
- Investigate the more general theory of pushing forward not-necessarily λ.
- The maps $T_\sigma : \mathcal{P}(2^\omega) \to \mathcal{P}(2^\omega),\ T(\mu)(\tau) = \mu(\sigma\tau)/\mu(\sigma)$ are P-invariant for each σ.
- Computability-theoretic questions?