Randomizing Reals

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Introduction

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What happens if we replace Lebesgue measure on 2^ω with an arbitrary measure?

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Theorem (Reimann–Slaman, 2008)

For any real X, the following are equivalent:

1. $X >_T 0$.
2. There is a measure μ such that $\mu(X) = 0$ and X is μ-Martin-Löf-random.

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Question

Can we replace “Martin-Löf-random” with stronger notions of randomness?

Randomness Notions Stronger than MLR

- Martin-Löf Random (MLR):
- Difference Random (DiffR):
- Weak-2-Random (W2R):
- n-random (nR):

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  $V_k = W_{g(k)}$ where $g(k)$ is recursive, and $\lambda(V_k) \leq 2^{-k}$
- Difference Random (DiffR):
  $V_k = W_{g_1(k)} \setminus W_{g_2(k)}$ where $g_1(k), g_2(k)$ is a recursive functions, and $\lambda(V_k) \leq 2^{-k}$
- Weak-2-Random (W2R):
  $V_k = W_{g(k)}$ where $g(k)$ is recursive, and $\lim \lambda(V_k) = 0$
- n-random (nR):
  $V_k = W_{g(k)}^{0^{(n-1)}}$ where $g(k)$ is $0^{(n-1)}$-recursive, and $\lambda(V_k) \leq 2^{-k}$

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Heirarchy of Randomness

$$\text{Kurtz} \supset \text{MLR} \supset \text{DiffR} \;\begin{matrix} \rotatebox{45}{$\supset$}\ \text{DR}\ \rotatebox{-45}{$\supset$} \\ \rotatebox{-45}{$\supset$}\ \text{W2R}\ \rotatebox{45}{$\supset$} \end{matrix}\; 2R \supset 3R \supset \cdots$$

The Problem

Definition

A real X is Martin-Löf (DiffR, W2R, ...) **randomizable** if there is a measure μ such that $\mu(X) = 0$ and X is μ-Martin-Löf (DiffR, W2R, ...) random.

Question

What reals are DiffR (W2R, nR, ...) randomizable?

What about **continuous** measures?

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Theorem (Kjos-Hanssen–Montalban, 2005)

For every $\beta < \omega_1^{\mathrm{CK}}$ there is a real $X \equiv_T 0^{(\beta)}$ such that $X \in \mathrm{NCR}$.

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Theorem (Reimann–Slaman, 2008)

$\mathrm{NCR} \subset \mathrm{HYP}$

Some Useful Facts

Theorem (Franklin–Ng, 2010)

Suppose X is Martin-Löf random. Then the following are equivalent:

1. X is difference random.
2. $X \not\geq_T 0'$.

Theorem (Downey–Nies–Weber–Yu, 2006)

Suppose X is Martin-Löf random. Then the following are equivalent:

1. X is weakly-2-random.
2. X forms a minimal pair with $0'$, i.e. $X \geq_T Z$ and $0' \geq_T Z$ implies $0 \geq_T Z$.

Some Relativized Useful Facts

Theorem (Franklin–Ng, 2010)

Suppose X is μ-Martin-Löf random. Then the following are equivalent:

1. X is μ-difference random.
2. $X \oplus \mu \not\geq_T \mu'$.

Theorem (Downey–Nies–Weber–Yu, 2006)

Suppose X is μ-Martin-Löf random. Then the following are equivalent:

1. X is μ-weakly-2-random.
2. $X \oplus \mu$ forms a minimal pair with μ' over μ, i.e. $X \oplus \mu \geq_T Z$ and $\mu' \geq_T Z$ implies $\mu \geq_T Z$.

Initial Observations

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Proposition

If X is n-r.e. then X is μ-DiffR iff $\mu(X) > 0$.
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Proposition

There are no neutral measures for DiffR. That is, given any measure μ, there is a real X such that $\mu(X) = 0$ and X is captured in a DiffR test relative to every representation of μ.

Some Negative Results

Theorem (H.)

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Proposition (Reimann–Slaman, 2008)

If $n \geq 2$, then for all $k \geq 0$, $0^{(k)}$ is not n-random with respect to a continuous measure.

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- [*]There is a Π^0_2 predicate $H(a, Z)$ such that $H(a, Z)$ holds iff $Z = H_a$.

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- Fixing an ordinal representation a such that $\alpha = |a|_{\mathcal{O}}$ and an index e such that $\Phi_e^X = H_a$. Define

$$\mathcal{C} = \{Z : \Phi_e^Z \text{ is total} \wedge H(a, \Phi_e^Z)\}$$

Then $\mathcal{C}$ is a Π^0_2 class containing X.

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- X is μ-W2R implies $\mu(\mathcal{C}) > 0$, so $\mu \geq_T H_a$.

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  Then $\mathcal{C}$ is a Π^0_2 class containing X.
- X is μ-W2R implies $\mu(\mathcal{C}) > 0$, so $\mu \geq_T H_a$.
- $\mu(X) = 0 \Rightarrow \mu \not\geq_T X$, but $\mu' \geq_T 0^{(\alpha+1)} \geq_T X$.

□

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Building Measures

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$$\mu(\sigma) \geq c\lambda(\Psi^{-1}(\sigma))$$

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  then X is μ-MLR.

Proposition (Reimann–Slaman, 2008)

X is μ-MLR for a continuous measure μ iff there is a MLR real Z such that $X \equiv_{tt} Z$.

Analyzing the Reimann–Slaman proof for MLR

Theorem

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Proof.

1. Find a G such that $X \oplus G \equiv_T G'$.
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Proof.

1. Find a G such that $X \oplus G \equiv_T G'$.
2. Find a Z which is random relative to G and such that $X \equiv_{T(G)} Z$.
3. Let Φ, Ψ be Turing functionals (relative to G) such that $\Phi(X) = Z$ and $\Psi(Z) = X$. Define $\text{Pre}(\sigma) = \{\tau : \Psi^{-1}(\sigma) \subseteq \tau \wedge \Phi(\tau) \supseteq \sigma\}$. Find a measure μ such that
$$\lambda(\text{Pre}(\sigma)) \leq \mu(\sigma) \leq \lambda(\Phi(\sigma))$$

(more or less)

□

Is the Intermediate Step Necessary?

In step (2), we found a random real Z such that $X \equiv_{T(G)} Z$ using:

Theorem (Kučera)

If $X \oplus G \geq_T G'$ then there is a MLR relative to G real Z such that $X \equiv_{T(G)} Z$.

Note that this theorem does *not* extend to DiffR (or higher).

Question

Is this intermediate step necessary? That is, to randomize X, is it necessary to find G, Z such that Z is random in G and $X \equiv_{T(G)} Z$?

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Theorem (H.)

Suppose X is μ-random (DiffR, W2R, DR, ...) and that $\mu(X) = 0$. Then there are reals M, Z such that Z is λ-random (DiffR, W2R, DR, ...) relative to M and such that $X \equiv_{T(M)} Z$. Furthermore, if μ is continuous then $X \equiv_{tt(M)} Z$.

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Theorem (H.)

Suppose ν is a continuous measure, X is μ-random (DiffR, W2R, DR, ...) relative to ν and that $\mu(X) = 0$. Then there are reals M, N, Z such that Z is ν-random (DiffR, W2R, DR, ...) relative to $M \oplus N$ and such that $X \equiv_{T(M \oplus N)} Z$. Furthermore, if μ is continuous then $X \equiv_{tt(M \oplus N)} Z$.

Lemma

There is a one-to-one Turing functional Φ (relative to μ), computably invertible on its range, such that $\Phi(Y)\downarrow$ iff $\mu(Y) = 0$ and such that $\exists c \forall \sigma \left(\lambda(\sigma) \geq c \cdot \mu(\Phi^{-1}(\sigma))\right)$.

Proof.

ON BOARD

$\square$

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3. Are there reals $X \geq_T 0'$ (or even reals $X \equiv_T 0'$) for which there is a measure μ such that X is μ-DiffR?

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2. In the counter-examples given for W2R (that is, where there is no measure μ such that $\mu(X) = 0$ and X is μ-W2R), the real X is a Π^0_2 singleton. Are there other counter-examples?
3. Are there reals $X \geq_T 0'$ (or even reals $X \equiv_T 0'$) for which there is a measure μ such that X is μ-DiffR?
4. Can anything be said about reals which are not DiffR (DR, W2R, ...) random for any **continuous** measures?