Coarse Computability and Algorithmic Randomness

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Coarse reducibility

Upper density: $\overline{\rho}(S) = \limsup_n \dfrac{|S \cap [0,n)|}{n}$.

Lower density: $\underline{\rho}(S) = \liminf_n \dfrac{|S \cap [0,n)|}{n}$.

Density: If $\overline{\rho}(S) = \underline{\rho}(S)$ then $\rho(S) = \overline{\rho}(S)$.

A coarse description of X is a D s.t. $\rho(X \triangle D) = 0$.

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$Y \leqslant_{\mathbf{c}} X$ if there is a Γ s.t. for any coarse description D of X, Γ^D is a coarse description of Y.

A is coarsely computable if $A \leqslant_{\mathbf{c}} \emptyset$, i.e., if A has a computable coarse description.

Algorithmic randomness and Turing reducibility

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Thm (Kučera). If $X, Y \leqslant_{\mathbf{T}} \emptyset'$ are relatively 1-random then they do not form a minimal pair.

If $A \leqslant_{\mathbf{T}} X, Y$ then X is A-1-random but computes A.

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If $A \leqslant_{\mathbf{T}} X, Y$ then X is A-1-random but computes A.

If X is A-weakly 2-random then X does not compute A.

If X, Y are relatively weakly 2-random, they form a minimal pair.

K-triviality

A is low for 1-randomness if every 1-random set is A-1-random.

Such sets are usually called K-trivial. **(Nies)**

Let $\mathcal{K}$ be the class of K-trivials.

Thm (Nies).

1. Every K-trivial is (super)low.
2. $\mathcal{K}$ is a Turing ideal.
3. Every K-trivial is computable in a c.e. K-trivial.

A is a base for 1-randomness if there is an A-1-random $X \geqslant_{\mathrm{T}} A$.

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1-randomness, K-triviality, and Turing reducibility

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Thm (Hirschfeldt, Nies, and Stephan). Every base for 1-randomness is K-trivial.

Cor. If X, Y are relatively 1-random and $A \leqslant_{\mathrm{T}} X, Y$ then A is K-trivial.

Embedding the Turing degrees into the coarse degrees

Let $I_n = [2^n, 2^{n+1})$ and let $I(S) = \bigcup_{n \in S} I_n$.

Let $F(A) = \{\langle n, i \rangle : n \in A \wedge i \in \omega\}$.

Let $E(A) = I(F(A))$.

$A \leqslant_{\mathbf{T}} B$ iff $E(A) \leqslant_{\mathbf{c}} E(B)$.

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If X is random then we should have $E(A) \nleqslant_{\mathbf{c}} X$ for $A \nleqslant_{\mathbf{T}} \emptyset$.

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Let $X^{\mathfrak{c}} = \{A : \rho(X \triangle D) = 0 \rightarrow A \leqslant_{\mathbf{T}} D\}$.

If X is random then we should have $X^{\mathfrak{c}} = \mathbf{0}$.

Thm. If X is 1-random then $X^{\mathfrak{c}} \subseteq \mathcal{K}$.

Cor. If X is weakly 2-random then $X^{\mathfrak{c}} = \mathbf{0}$.

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Thm. Let $X \leqslant_{\mathrm{T}} \emptyset'$ be 1-random. There is a noncomputable c.e. A s.t. if $\overline{\rho}(X \triangle D) < \frac{1}{4}$ then $A \leqslant_{\mathrm{T}} D$. In particular, $X^{\mathfrak{c}} \neq \mathbf{0}$.

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Thm. If X, Y are relatively weakly 3-random then their coarse degrees form a minimal pair.

Proving that if X is 1-random then $X^{\mathfrak{c}} \subseteq \mathcal{K}$

Let $P_0, \ldots, P_k$ partition ω into infinite computable sets.

Write X_i for $X \upharpoonright P_i$ and $X_{\neq i}$ for $X \upharpoonright \bigcup_{j \neq i} P_i$.

Lem. If each X_i is $(X_{\neq i} \oplus A)$-1-random then X is A-1-random.

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So if $A \leqslant_{\mathrm{T}} X_{\neq i}$ for all i then X is A-1-random.

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Thus if $A \notin \mathcal{K}$ then there is an i s.t. $A \nleqslant_{\mathrm{T}} X_{\neq i}$.

We build a coarse description D of X s.t. $A \nleqslant_{\mathrm{T}} D$ in stages.

As we go along, we make $\overline{\rho}(X \triangle D)$ smaller and smaller.

At stage e, we ensure that $\exists n \neg(\Phi_e^D(n)\downarrow = A(n))$ by using a sufficiently thin partition.

Thm. Let $X \leqslant_{\mathrm{T}} \emptyset'$ be 1-random. There is a noncomputable c.e. A s.t. if $\overline{\rho}(X \triangle D) < \frac{1}{4}$ then $A \leqslant_{\mathrm{T}} D$. In particular, $X^{\mathfrak{c}} \neq \mathbf{0}$.

Thm. Let $X \leqslant_T \emptyset'$ be 1-random. There is a noncomputable c.e. A s.t. if $\overline{\rho}(X \triangle D) < \frac{1}{4}$ then $A \leqslant_T D$. In particular, $X^{\mathfrak{c}} \neq \mathbf{0}$.

The proof is related to that of the following result:

For $\mathcal{C} \subseteq 2^\omega$, let $\mathcal{C}^\diamond$ be the set of all A computable from every 1-random $X \in \mathcal{C}$.

Thm (Hirschfeldt and Miller). If $\mathcal{C}$ is Σ^0_3 and $\mu(\mathcal{C}) = 0$ then there is a noncomputable c.e. $A \in \mathcal{C}^\diamond$.

Minimal pairs in the coarse degrees revisited

Thm. If X, Y are relatively weakly 3-random then their coarse degrees form a minimal pair.

Follows from the fact that if X is weakly 3-random relative to $A \nleq_c \emptyset$ then X cannot compute a coarse description of A.

Proof is by a version of majority voting.

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Follows from the fact that if X is weakly 3-random relative to $A \nleqslant_{\mathbf{c}} \emptyset$ then X cannot compute a coarse description of A.

Proof is by a version of majority voting.

Thm. If $X, Y \leqslant_{\mathbf{T}} \emptyset''$ are relatively 2-random then their coarse degrees do not form a minimal pair.

Proof uses a connection between coarse computability and lowness.

Minimal pairs in the coarse degrees revisited

Thm. TFAE for a c.e. Turing degree **a**:

1. If $A \in \mathbf{a}$ is c.e. and $\rho(A) = \frac{1}{2}$ then $A \leqslant_{\mathbf{c}} \emptyset$.
2. **a** is low.

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Let $X, Y \leqslant_{\mathbf{T}} \emptyset''$ be relatively 2-random.

Relativizing a previous construction, there is an $\emptyset'$-c.e. $B >_{\mathbf{T}} \emptyset'$ s.t. the jump of any coarse description of X or Y computes B.

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Relativizing a previous construction, there is an $\emptyset'$-c.e. $B >_{\mathbf{T}} \emptyset'$ s.t. the jump of any coarse description of X or Y computes B.

Let A be a c.e. set with $A' = B$ s.t. $\rho(A) = \frac{1}{2}$ and $A \not\leqslant_{\mathbf{c}} \emptyset$.

Then any coarse description of X or Y computes a coarse description of A.

ML-covering

Thm (Hirschfeldt, Nies, and Stephan). If $X \ngeqslant_{\mathbf{T}} \emptyset'$ is 1-random and $A \leqslant_{\mathbf{T}} X$ is c.e., then A is K-trivial.

Is every K-trivial computed by an incomplete 1-random?

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Recall: If X, Y are relatively 1-random and $A \leqslant_{\mathrm{T}} X, Y$, then A is K-trivial.

Is every K-trivial computed by a pair of relatively 1-random sets?

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Thm (Day and Miller / Bienvenu, Greenberg, Kučera, Nies, and Turetsky). There is an incomplete 1-random that computes every K-trivial.

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Thm (Bienvenu, Greenberg, Kučera, Nies, and Turetsky). There is a K-trivial that is not computable from any pair of relatively 1-random sets.

X is LR-hard if $\emptyset'$ is low for 1-randomness relative to X, i.e., every set that is X-1-random is 2-random.

Thm (Bienvenu, Greenberg, Kučera, Nies, and Turetsky). There is a K-trivial A s.t. every 1-random that computes A is LR-hard.

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Suppose $A \in X^{\mathfrak{c}}$ for 1-random X.

Let $P_0 = \{2^n : n \in \omega\}$ and $P_1 = \omega \setminus P_0$ partition ω.

Then $A \leqslant_{\mathrm{T}} X_1$, so X_0 is 2-random. Thus:

Cor. There are K-trivials that are not in $X^{\mathfrak{c}}$ for any Δ^0_2 1-random X.

LR-hardness and coarse computability

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Cor. There are K-trivials that are not in $X^{\mathfrak{c}}$ for any Δ^0_2 1-random X.

Open Question. Is every K-trivial in $X^{\mathfrak{c}}$ for some 1-random X?