Families of sets and their degree spectra

Kalimullin I.Sh.

Kazan Federal University
e-mail:Iskander.Kalimullin@ksu.ru

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The degree spectra for families

- A countable family of sets $\mathcal{F} \subseteq 2^\omega$ is (uniformly) $\mathbf{x}$-c.e. if for $X \in \mathbf{x}$ and some computable function f we have

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- The degree spectrum of $\mathcal{F}$ is the collection $\mathbf{Sp}(\mathcal{F})$ of all Turing degrees $\mathbf{x}$ such that $\mathcal{F}$ is $\mathbf{x}$-c.e.

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- There is a family $\mathcal{F}$ such that $\textbf{Sp}\,(\mathcal{F}) =$ the non-K-trivial degrees(*Faizrahmanov*, 2012).

A general lemma

Lemma. Let

$$\mathcal{F} = \{\{n\} \oplus F | F \text{ is finite and } \Phi^{F \oplus \emptyset'}(n) \downarrow\}.$$

Let Y be an X-c.e. set such that for every $Z =^* Y$ we have

$$(\forall n)[\Phi^{Z \oplus \emptyset'}(n) \downarrow].$$

Then $\mathcal{F}$ is X-c.e.

An easy example

Theorem (Wehner). If

$$\mathcal{F} = \{\{n\} \oplus F | F \text{ is finite and } F \neq W_n\}.$$

Then $\mathcal{F}$ is X-c.e $\iff$ X is not computable, i.e.

$$\mathbf{Sp}(\mathcal{F}) = \{\mathbf{x} | \mathbf{x} > \mathbf{0}\}.$$

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Proof ($\Longleftarrow$). If X is not computable then there is a Y such that Y is X-c.e. but Y is not c.e. so that for each $Z =^* Y$ we have

$$(\forall n)[Z \neq W_n].$$

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Proof ($\Longrightarrow$). If X is computable and $\mathcal{F}$ is X-c.e. then for every n we can uniformly enumerate a set $W_{f(n)}$ such that

$$W_{f(n)} \neq W_n.$$

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- For a specific $\{V_n\}_{n\in\omega}$ some weak version of Recursion Theorem holds that allows to prove the reverse implication.

The non-superlow degrees, an easy way

- (Faizrahmanov, 2010) $X' \in \Pi^{-1}_{\omega} \iff X' \in \Delta^{-1}_{\omega}$, so that X is not superlow $\iff$ there is an X-c.e. $Y \notin \Pi^{-1}_{\omega}$.

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- Let $\{V_n\}_{n\in\omega}$ be the Gödel $\emptyset'$-numbering of all Π^{-1}_{ω} sets. Set

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- By the general lemma we have X is non-superlow $\Longrightarrow \mathcal{F}$ is X-c.e.
- Suppose X is superlow and $\mathcal{F}$ is X-c.e. Then for every n we can uniformly enumerate $W^X_{f(n)}$ such that $W^X_{f(n)} \neq V_n$. But since X is superlow we can effectively translate $W^X_{f(n)}$ to $V_{g(n)}$ so that

$$V_{g(n)} \neq V_n.$$

For Gödel numbering this is impossible.

The non-K-trivial degrees

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where σ runs over $2^{<\omega}$.
- By the general lemma we have X is non-K-trivial $\Longrightarrow \mathcal{F}$ is X-c.e.

The reverse direction

Suppose X is K-trivial and the family $\mathcal{F}$ is X-c.e. Let σ_n be the first enumerated string such that

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Then X is low for K so that for some $k \in \omega$ we have

$$K(\sigma) \leq K^X(\sigma) + k - 1.$$

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Using Recursion Theorem we can find an index n such that

$$\Phi_n^X(\tau) = \begin{cases} \sigma_{n+k} \upharpoonright U(\tau), & \text{if } U(\tau) \downarrow \leq |\sigma_{n+k}|; \\ \uparrow & \text{otherwise,} \end{cases}$$

where U is the optimal prefix-free operator

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Suppose X is K-trivial and the family $\mathcal{F}$ is X-c.e. Let σ_n be the first enumerated string such that

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where U is the optimal prefix-free operator, so that for every $m \leq |\sigma_{n+k}|$ we have

$$K^X_{\Phi_n}(\sigma_{n+k} \restriction m) = K(m).$$

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Now it follows

$$K(\sigma_{n+k} \upharpoonright m) \leq K^X(\sigma_{n+k} \upharpoonright m) + k - 1 \leq K^X_{\Phi_n}(\sigma_{n+k} \upharpoonright m) + n + k$$

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$$K(\sigma_{n+k} \upharpoonright m) \leq K^X(\sigma_{n+k} \upharpoonright m) + k - 1 \leq K(m) \qquad + n + k$$

for every $m \leq |\sigma_{n+k}|$.