The Muchnik Lattice and Intuitionistic Logic

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Computational semantics for IPC

Brouwer–Heyting–Kolmogorov interpretation: a proof of $A \to B$ is a construction which transforms any proof of A into a proof of B.

This suggests there should be some computational semantics for IPC. Realisability is one such attempt, today we look at the Medvedev and Muchnik lattices.

Medvedev and Muchnik reducibility

Definition. Let $\mathcal{A}, \mathcal{B} \subseteq \omega^\omega$. We say that $\mathcal{A}$ *Medvedev reduces to* $\mathcal{B}$, denoted by $\mathcal{A} \leq_M \mathcal{B}$, if there exists a Turing functional Ψ such that $\psi(B) \subseteq A$.

Furthermore, we say that $\mathcal{A}$ *Muchnik reduces to* $\mathcal{B}$, denoted by $\mathcal{A} \leq_w \mathcal{B}$, if for every $f \in \mathcal{B}$ there exists $g \in \mathcal{A}$ such that $g \leq_T f$.

Brouwer algebras

Definition. A bounded distributive lattice is a poset with a least element 0, a largest element 1, finite least upper bounds $x \oplus y$ and finite greatest lower bounds $x \otimes y$, where $\oplus$ and $\otimes$ distribute over each other.

Definition. (McKinsey–Tarski) A *Brouwer algebra* is a bounded distributive lattice with a binary implication operator $\rightarrow$ satisfying:

$$x \oplus z \geq y \text{ if and only if } z \geq x \rightarrow y$$

i.e. $x \rightarrow y$ is the least element z satisfying $x \oplus z \geq y$.

The theory of a Brouwer algebra

Let B be a Brouwer algebra and let $\alpha : \mathrm{Var} \to B$ be a valuation. Then α extends to all formulas by interpreting logical disjunction $\vee$ as $\otimes$, logical conjunction $\wedge$ as $\oplus$, logical implication as $\to$ and falsum $\perp$ as 1.

Definition. The propositional theory of a Brouwer algebra B, $\mathrm{Th}(B)$, is defined as

$$\{\phi \mid \alpha(\phi) = 0 \text{ for all valuations } \alpha \text{ of } B\}.$$

Theorem. (McKinsey–Tarski)

$$\bigcap\{\mathrm{Th}(B) \mid B \text{ finite Brouwer algebra}\} = \mathrm{IPC}$$

Medvedev and Muchnik lattices

The equivalence classes of ω^ω under Medvedev equivalence form a Brouwer algebra $\mathcal{M}$, with operations given by:

$$\begin{aligned}
\mathcal{A} \oplus \mathcal{B} &= \{f \oplus g \mid f \in \mathcal{A}, g \in B\}, \\
\mathcal{A} \otimes \mathcal{B} &= \{0 \star f \mid f \in \mathcal{A}\} \cup \{1 \star g \mid g \in \mathcal{B}\}, \\
\mathcal{A} \to \mathcal{B} &= \{n \star f \mid \forall g \in \mathcal{A}. \Psi_n(f \oplus g) \in \mathcal{B}\}.
\end{aligned}$$

The equivalence classes of ω^ω under Muchnik equivalence also form a Brouwer algebra $\mathcal{M}_w$ (in fact, they form a completely distributive lattice), with operations given by:

$$\begin{aligned}
\mathcal{A} \oplus \mathcal{B} &= \{f \oplus g \mid f \in \mathcal{A}, g \in B\}, \\
\mathcal{A} \otimes \mathcal{B} &= \mathcal{A} \cup \mathcal{B}, \\
\mathcal{A} \to \mathcal{B} &= \{f \mid \forall g \in \mathcal{A} \exists h \in \mathcal{B}. f \oplus g \geq_T h\}.
\end{aligned}$$

Theory of $\mathcal{M}$ and $\mathcal{M}_w$

Theorem. (Medvedev, Muchnik, Sorbi)
$\mathrm{Th}(\mathcal{M}) = \mathrm{Th}(\mathcal{M}_w) = \mathrm{IPC} + \neg A \vee \neg\neg A.$

Principal factors of Brouwer algebras

Given an element u of a distributive lattice B, the quotient of B by the principal filter $C(u) = \{x \in B \mid x \geq u\}$ is also a distributive lattice. In fact, if B is a Brouwer algebra, then $B/C(u)$ is also a Brouwer algebra, with implication given by

$$[y] \rightarrow_{B/C(u)} [z] = [(y \otimes u) \rightarrow_B (z \otimes u)].$$

This quotient is isomorphic to $[0, u]_B = \{x \in B \mid x \leq u\}$, where the implication is the implication of B restricted to $[0, u]_B$.

Principal factors of $\mathcal{M}$ and $\mathcal{M}_w$

Theorem. (Skvortsova) *There exists* $\mathcal{A} \in \mathcal{M}$ *such that* $\mathrm{Th}(\mathcal{M}/C(\mathcal{A})) = \mathrm{IPC}$.

Theorem. (Sorbi–Terwijn) *There exists* $\mathcal{A} \in \mathcal{M}_w$ *such that* $\mathrm{Th}(\mathcal{M}_w/\mathcal{A}) = \mathrm{IPC}$.

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Goal: find natural examples of such $\mathcal{A}$.

Splitting classes

Definition. Let $\mathcal{A} \subseteq \omega^\omega$ be a non-empty countable class which is downwards closed under Turing reducibility. We say that $\mathcal{A}$ is a *splitting class* if for every $f \in \mathcal{A}$ and every finite subset $\mathcal{B} \subseteq \{g \in \mathcal{A} \mid g \not\leq_T f\}$ there exist $h_0, h_1 \in \mathcal{A}$ such that $h_0, h_1 \geq_T f$, $h_0 \oplus h_1 \notin \mathcal{A}$ and for all $g \in \mathcal{B}$: $g \oplus h_0, g \oplus h_1 \notin \mathcal{A}$.

Theorem. *Let $\mathcal{A}$ be a splitting class. Then* $\mathrm{Th}(\mathcal{M}_w/\overline{\mathcal{A}}) = \mathrm{IPC}$.

Splitting classes

Definition. Let $\mathcal{A} \subseteq \omega^\omega$ be a non-empty class of cardinality $\aleph_1$ which is downwards closed under Turing reducibility. We say that $\mathcal{A}$ is an $\aleph_1$ *splitting class* if for every $f \in \mathcal{A}$ and every countable subset $\mathcal{B} \subseteq \{g \in \mathcal{A} \mid g \not\leq_T f\}$ there exist $h_0, h_1 \in \mathcal{A}$ such that $h_0, h_1 \geq_T f$, $h_0 \oplus h_1 \notin \mathcal{A}$ and for all $g \in \mathcal{B}$: $g \oplus h_0, g \oplus h_1 \notin \mathcal{A}$.

Theorem. *Let $\mathcal{A}$ be an $\aleph_1$ splitting class. Then $\mathrm{Th}(\mathcal{M}_w/\overline{\mathcal{A}}) = \mathrm{IPC}$.*

Examples

The following are splitting classes:

- $\{f \in \omega^\omega \mid f \text{ low}\}$ (using a modification of Posner–Robinson);
- $\{f \in \omega^\omega \mid f \leq_T \emptyset' \text{ 1-generic degree}\} \cup \{f \in \omega^\omega \mid f \text{ computable}\}$ (also using Posner–Robinson, and using Haught);
- $\{f \in \omega^\omega \mid f \text{ hyperimmune-free and low}_2\}$ (using a Miller–Martin tree construction);
- $\{f \in \omega^\omega \mid f \text{ computably traceable and low}_2\}$.

Assuming the continuum hypothesis, the following are $\aleph_1$ splitting classes:

- $\{f \in \omega^\omega \mid f \text{ hyperimmune-free}\}$;
- $\{f \in \omega^\omega \mid f \text{ computably traceable}\}$.

Why splitting classes yield IPC

Theorem. *The Muchnik lattice $\mathcal{M}_w$ is isomorphic to the lattice of upsets of the Turing degrees.*

Theorem. *For any downwards closed class $\mathcal{A} \subseteq \omega^\omega$, the theory of $\mathcal{M}_w/\overline{\mathcal{A}}$ is equal to the theory of $\mathcal{A}$ as a Kripke frame.*

Why splitting classes yield IPC

Definition. (De Jongh and Troelstra) Let $(X_1, \leq_1), (X_2, \leq_2)$ be Kripke frames. A surjective function $\alpha : X_1 \to X_2$ is called a *p-morphism* if

1. f is an order homomorphism: $x \leq_1 y \to f(x) \leq_2 f(y)$,
2. $\forall x \in X_1 \forall y \in X_2 (f(x) \leq_2 y \to \exists z \in X_1 (x \leq_1 x \wedge f(z) = y))$.

Proposition. *If there exists a p-morphism from $(X_1, \leq_1)$ to $(X_2, \leq_2)$, then $\mathrm{Th}(X_1, \leq_1) \subseteq \mathrm{Th}(X_2, \leq_2)$.*

Theorem. $\mathrm{Th}(2^{<\omega}) = \mathrm{IPC}$.

Therefore: if $\mathcal{A}$ is a downwards closed class such that there exists a p-morphism onto $2^{<\omega}$, then $\mathrm{Th}(\mathcal{M}_w/\overline{\mathcal{A}}) = \mathrm{IPC}$.

Why splitting classes yield IPC

Therefore: if $\mathcal{A}$ is a downwards closed class such that there exists a p-morphism onto $2^{<\omega}$, then $\mathrm{Th}(\mathcal{M}_w/\overline{\mathcal{A}}) = \mathrm{IPC}$.

Our definition of a splitting class exactly allows us to do that.

Definition. Let $\mathcal{A} \subseteq \omega^\omega$ be a non-empty countable class which is downwards closed under Turing reducibility. We say that $\mathcal{A}$ is a *splitting class* if for every $f \in \mathcal{A}$ and every finite subset $\mathcal{B} \subseteq \{g \in \mathcal{A} \mid g \not\leq_T f\}$ there exist $h_0, h_1 \in \mathcal{A}$ such that $h_0, h_1 \geq_T f$, $h_0 \oplus h_1 \notin \mathcal{A}$ and for all $g \in \mathcal{B}$: $g \oplus h_0, g \oplus h_1 \notin \mathcal{A}$.

The idea is to build a p-morphism α step by step. We can use $\mathcal{B}$ to avoid the points on which we already defined α previously, while we can use h_0 and h_1 to split into two branches.

Open questions

- Does $\mathrm{Th}(\mathcal{M}_w/\overline{\{f \mid f \text{ hyperimmune-free}\}}) = \mathrm{IPC}$ follow from ZFC?
- Can something similar be done for the Medvedev lattice?