Counting the Changes of Random Δ^0_2 Sets

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Abstract. Consider a Martin-Löf random Δ^0_2 set Z. We give lower bounds for the number of changes of $Z_s \upharpoonright_n$ for computable approximations of Z. We show that each nonempty Π^0_1 class has a low member Z with a computable approximation that changes only $o(2^n)$ times. We prove that each superlow ML-random set already satisfies a stronger randomness notion called balanced randomness, which implies that for each computable approximation and each constant c, there are infinitely many n such that $Z_s \upharpoonright_n$ changes more than $c2^n$ times.

1 Introduction

A *computable approximation* of a set $Z \subseteq \mathbb{N}$ is a computable sequence $(Z_s)_{s\in\mathbb{N}}$ of finite sets such that $Z(x) = \lim_s Z_s(x)$ for each x. The Shoenfield Limit Lemma states that a set $Z \subseteq \mathbb{N}$ is Δ^0_2 iff Z has a computable approximation.

In Sections 3 to 5 we are interested in the number of changes of $Z_s \upharpoonright_n$ for computable approximations of a Martin-Löf random Δ^0_2 set Z. We give some lower bounds. Next, we obtain a hierarchy theorem saying that allowing more changes yields new ω-c.e. ML-random sets. Thereafter, we prove the "$o(2^n)$ changes" low basis theorem which says that each nonempty Π^0_1 class has a low member Z with a computable approximation such that $Z \upharpoonright_n$, the initial segment of length n, changes only $o(2^n)$ times. We conclude that there is a computable approximation of a low ML-random set which changes only $o(2^n)$ times.

In [1, Sect. 8.6] evidence was presented for the following thesis: Among ML-random sets,

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being computationally less complex is equivalent to being more random.

For instance, a ML-random set forms a minimal pair with $\emptyset'$ iff it is weakly 2-random. In the final section we use the foregoing results to give some more evidence for this when the ML-random set is Δ^0_2.

To specify what we mean by being more random, we consider variants of Demuth randomness, a notion that strengthens ML-randomness but is still compatible with being Δ^0_2. Demuth tests (see [1, Def. 3.6.24]) generalize Martin-Löf tests $(G_m)_{m\in\mathbb{N}}$ in that one can exchange the m-th component a computably bounded number of times. A set $Z \subseteq \mathbb{N}$ passes a Demuth test if Z is in only finitely many final versions of the G_m.

The passing condition that Z is not in at least one of the G_m yields weak Demuth randomness. In this case, we can require as well that $G_m \supseteq G_{m+1}$ for each m, since we can replace G_m by $\bigcap_{i\leq m} G_i$ if necessary. A test with this property will be called *monotonic*. Note that the number of version changes is still computably bounded. Thus Z is weakly Demuth random iff it passes all monotonic Demuth tests (where passing the test can be taken in either sense).

We introduce balanced randomness, an even more restricted form of weak Demuth randomness where the bound on the number of changes of the m-th version is $O(2^m)$. Each balanced random set is ML-random and Turing incomplete.

For evidence of the direction from left to right in the thesis above, we show that a ML-random set that is superlow is already balanced random. Being ω-c.e. tracing is a highness property due to Greenberg and Nies [2] that is incompatible with superlowness (see Sect. 6). In fact we show that a ML-random set that is not ω-c.e. tracing is already balanced random.

Evidence for the direction from right to left in the thesis above is given by the fact that a Demuth random set bounds only generalized low$_1$ sets, and the result of [3] that a c.e. set Turing below a Demuth random set must be strongly jump-traceable. In [3] further evidence for this direction is given by showing that a weakly Demuth random set Z is not superhigh, namely, $Z' \not\geq_{\mathrm{tt}} \emptyset''$. (However, it can be high.) We conjecture that a balanced random set is not *LR*-complete, and prove a result in that direction.

2 Counting Changes of a Δ^0_2 Set

For a computable approximation $(Z_s)_{s\in\mathbb{N}}$, unless otherwise stated, we will assume that $Z_s(x) = 0$ for each $x \geq s$. Given such an approximation, for a number n and a stage number $s > 0$, to say that $Z\upharpoonright_n$ *changes* at stage s means that $Z_s\upharpoonright_n \neq Z_{s-1}\upharpoonright_n$.

When we say that we bound the number of changes for a Δ^0_2 set Z from above, we mean that the changes of *some* approximation can be bounded from above.

Definition 1. Let $f\colon \mathbb{N} \to \mathbb{N}$. We say a set $Z \subseteq \mathbb{N}$ is *f-c.e.* if there is a computable approximation $(Z_s)_{s\in\mathbb{N}}$ of Z such that for each n, $Z_s\upharpoonright_n$ changes at most

$f(n)$ times via this approximation. Terminology such as $O(f)$-c.e. set, $o(f)$-c.e. set and so on has the obvious meaning. For instance, Z is $o(f)$-c.e. if there is a function $g \in o(f)$ such that Z is g-c.e.

Each left-c.e. set is $o(2^n)$-c.e.:

Fact 2. *Let Z be a left-c.e. set as shown by the computable approximation $(Z_s)_{s\in\mathbb{N}}$. Then Z is $o(2^n)$-c.e. via this computable approximation.*

Proof. Given k, let t be the least stage such that $Z_t \upharpoonright_{k+1}$ has the final value. Let $n \geq t + k + 1$. By our convention that $Z_s(x) = 0$ for each $x \geq s$, $Z \upharpoonright_n$ changes at no more than $2^t \leq 2^{n-k-1}$ stages that are $\leq t$. Furthermore, since the approximation cannot return to previous states, $Z \upharpoonright_n$ changes at no more than 2^{n-k-1} stages that are greater than t. Thus $Z \upharpoonright_n$ changes at no more than 2^{n-k} stages. □

Actually, the fact still holds if we require only that the approximation to $Z \upharpoonright_n$ can never return to a previous value.

3 Some Lower Bounds on the Number of Changes of a ML-random Set

In this section we assume that Z is a ML-random Δ^0_2 set with a fixed computable approximation $(Z_s)_{s\in\mathbb{N}}$. We give some lower bounds on the number of times $Z \upharpoonright_n$ can change. We confirm the intuition that the number of changes cannot be far below 2^n.

First we look at computable functions bounding the number of changes of $Z \upharpoonright_n$ for only infinitely many n.

Proposition 3. *Let $q : \mathbb{N} \to \mathbb{Q}^+$ be computable. If $Z \upharpoonright_n$ changes fewer than $\lfloor 2^n q(n) \rfloor$ times for infinitely many n, then $\sum_n q(n) = \infty$.*

Proof. Assume for contradiction that $\sum_n q(n) < \infty$. We define an effective sequence $(\mathcal{S}_i)_{i\in\mathbb{N}}$ of Σ^0_1 classes in the following way. For each n, we put into $\mathcal{S}_n$ the first $\lfloor 2^n q(n) \rfloor$ versions of $[Z \upharpoonright_n]$. Clearly $(\mathcal{S}_i)_{i\in\mathbb{N}}$ is a sequence of uniformly c.e. open sets and $\lambda \mathcal{S}_n \leq q(n)$ for all n, where λ is Lebesgue measure. Thus $(\mathcal{S}_i)_{i\in\mathbb{N}}$ is a Solovay test. By hypothesis $Z \in S_n$ for infinitely many n. This means that Z fails the test $(\mathcal{S}_i)_{i\in\mathbb{N}}$ and therefore is not ML-random. □

Example 4. $Z \upharpoonright_n$ changes at least $2^n n^{-2}$ times for almost every n.

The proof of the foregoing proposition can easily be extended to the case that the function q is effectively approximable from below, that is, $q(n) = \sup_s q_s(n)$ for an effective sequence of rationals that is nondecreasing in s. For instance, we can let $q(n) = 2^{-K(n)}$, where K is prefix-free Kolmogorov complexity. Thus, in the example above, in fact we have a lower bound of $2^{n-K(n)}$.

If for almost every n the number of changes of $Z \upharpoonright_n$ is bounded above by $2^n q(n)$, then the function q is in fact bounded away from 0.

Proposition 5. *Let $q : \mathbb{N} \to \mathbb{Q}^+$ be computable. If $Z \upharpoonright_n$ changes fewer than $\lfloor 2^n q(n) \rfloor$ times for almost every n, then* $\inf_n q(n) > 0$.

Proof. Let n^* be a number such that the bound holds from n^* on. Assume for a contradiction that $\inf_n q(n) = 0$. We show that $\exists^\infty n\, K(Z \upharpoonright_n) \leq^+ n$, contrary to the assumption that Z is ML-random. To do so we build a bounded request (aka Kraft-Chaitin) set L. Let $(n_i)_{i>0}$ be a computable sequence of numbers greater than n^* such that $q(n_i) < 2^{-i}$ for each i. For each s, we put the request

$$\langle n_i, Z_s \upharpoonright_{n_i} \rangle$$

into L. For each $i > 0$, the weight put into L is at most $2^{-n_i} 2^{n_i} q(n_i) \leq 2^{-i}$. Thus L is a bounded request set. Hence by the usual machine existence theorem (aka Kraft-Chaitin Theorem), we have $\exists^\infty n\, K(Z \upharpoonright_n) \leq^+ n$ as required. □

The proof of the foregoing proposition can easily be extended to the case that the function q is effectively approximable from *above.* For each i, we can search for an s and an n_i such that $q_s(n_i) < 2^{-i}$.

We remark that neither Proposition 3 nor Proposition 5 can be extended to reals of positive effective Hausdorff dimension. It is easy to construct a real Z with effective Hausdorff dimension 1, and a computable function q such that $\sum_n q(n) < \infty$, and $Z \upharpoonright_n$ changes fewer than $\lfloor 2^n q(n) \rfloor$ times for almost every n.

It is natural to ask what else we can say about the number of times $Z \upharpoonright_n$ can change for a Δ^0_2 ML-random Z. In particular, we consider strengthening Propositions 3 and 5 simultaneously: whenever $Z \upharpoonright_n$ changes fewer than $\lfloor 2^n q(n) \rfloor$ times for infinitely many n, then $q(n)$ is bounded away from zero on these n. By the following proposition this is true if q is a computable nonincreasing function, but by Corollary 11 this fails in general.

Proposition 6. *Let $q : \mathbb{N} \to \mathbb{Q}^+$ be computable and nonincreasing. If $Z \upharpoonright_n$ changes fewer than $\lfloor 2^n q(n) \rfloor$ times for infinitely many n, then* $\inf_n q(n) > 0$.

Proof. Suppose the contrary, that $\inf_n q(n) = 0$. Let $(n_i)_{i\in\mathbb{N}}$ be a computable sequence of natural numbers such that for every i, n_i is the least number larger than n_{i-1} such that $q(n_i) < 2^{-i-1}$. We build a Solovay test $(\mathcal{S}_i)_{i\in\mathbb{N}}$ by the following. For each i enumerate into $\mathcal{S}_i$ the first $2^{n_i - i}$ different versions of $[Z \upharpoonright_{n_i}]$. Then $\lambda \mathcal{S}_i \leq 2^{-i}$ for every i. Since Z is ML-random and $Z \upharpoonright_n$ changes fewer than $\lfloor 2^n q(n) \rfloor$ times for infinitely many n, we fix $m > n_0$ and $i > 0$ such that $Z \upharpoonright_m$ changes fewer than $\lfloor 2^m q(m) \rfloor$ times, $Z \notin S_i$ and i is the least such that $n_i \geq m$. Since $Z \notin S_i$, there must be at least $2^{n_i - i} + 1$ many distinct elements in the set $\{Z_s \upharpoonright_{n_i} : s \in \mathbb{N}\}$. Now since $n_{i-1} < m$ we have $q(m) \leq q(n_{i-1}) < 2^{-i}$. Hence $Z \upharpoonright_m$ changes fewer than 2^{m-i} times. This is a contradiction. □

4 A Hierarchy Theorem for ML-random ω-c.e. Sets

Using a method of Kučera one can code a given set into a path on a Π^0_1 class of positive measure. The method rests on the following lemma (see [1, Lem. 3.3.1]), where $\lambda(\mathcal{C}|x)$ denotes $2^{|x|}\lambda(\mathcal{C} \cap [x])$.

Lemma 7. *Let $\mathcal{C} \subseteq 2^\omega$ be measurable and $\lambda(\mathcal{C}|x) \geq 2^{-(r+1)}$. Then for every $n \geq |x| + r + 2$ there are distinct strings $y_0, y_1 \succ x$ with $|y_i| = n$ such that $\lambda(\mathcal{C}|y_i) > 2^{-(r+2)}$ for $i = 0, 1$.*

An *order function* is a nondecreasing unbounded computable function.

Theorem 8. *Let b be an order function such that $\forall n\, b(n) \geq \epsilon 2^n$ for some positive real ϵ. Then for each order function s there is a ML-random Z which is $s \cdot b$-c.e. but not b-c.e.*

We can restate Proposition 5 as follows: if the ML-random set Z is b-c.e. for some computable function b, then there is $\epsilon > 0$ such that $\forall n\, b(n) \geq \epsilon 2^n$. This shows that the additional hypothesis $\forall n\, b(n) \geq \epsilon 2^n$ in this hierarchy theorem does not restrict its generality.

Proof (Theorem 8). The idea is the following. To make Z ML-random, we ensure that it belongs to an appropriate Π^0_1-class. To make Z non b-c.e., let $(f_e)_{e \in \mathbb{N}^+}$ be an enumeration of all total computable functions f mapping pairs of natural numbers to strings such that for all n, $\#\{t : f(n,t) \neq f(n,t+1)\} \leq b(n)$, $|f(n,t)| = n$, and $f(n,t) \prec f(n+1,t)$. Each such f is the approximation of some b-c.e. set. Conversely, if a set is b-c.e. then there is some f giving the set in the limit. Thus it suffices to ensure that for every e there is an n such that $\lim_t f_e(n,t) \neq Z \restriction_n$.

Here are the details. Recall that s is the given order function. Choose a computable sequence $(n_e)_{e \in \mathbb{N}^+}$ such that $n_1 = 0$,

$$s(n_e) > e + 1/\epsilon, \quad \text{and} \quad n_{e+1} \geq n_e + e + 2.$$

Let $\mathcal{P}$ be a Π^0_1-class such that $\mathcal{P} \subseteq \mathsf{MLR}$, where MLR is the class of ML-random sets, and $\lambda\mathcal{P} > 1/2$. Let $\hat{\mathcal{P}}$ be the Π^0_1 class of paths through the Π^0_1 tree

$$T = \{y : (\forall i)[n_i \leq |y| \rightarrow \lambda(\mathcal{P}|(y \restriction_{n_i})) \geq 2^{-(i+1)}]\}.$$

Note that $\hat{\mathcal{P}} \subseteq \mathcal{P}$. Since $\lambda\mathcal{P} \geq 1/2$, by Lemma 7, $\hat{\mathcal{P}}$ is nonempty.

We define $z_0 \prec z_1 \prec z_2 \prec \cdots$ in such a way that $|z_e| = n_e$ and $z_e \neq \lim_t f_e(n_e, t)$. We also define $Z = \bigcap_e [z_e]^\prec$. In this way, we ensure that for all $e \geq 1$, $Z \restriction_{n_e} \neq \lim_t f_e(n_e, t)$ and therefore Z is not b-c.e. At the same time, we ensure that $Z \in \hat{\mathcal{P}}$, and hence Z is ML-random.

The definition of z_e proceeds by steps. Let $z_{0,s} = \emptyset$ and for $e > 0$ let

$$z_{e+1,s} = \min\{[z] \subseteq \hat{\mathcal{P}}_s : |z| = n_{e+1} \ \wedge \ z \succ z_{e,s} \ \wedge \ f_e(n_e, s) \neq z\}. \qquad (1)$$

Recall that each Π^0_1 class $\mathcal{P}$ has an effective approximation by descending clopen sets $\mathcal{P}_s$; see [1, Sect. 1.8].

Suppose $z_{e,s}$ has already been defined. By Lemma 7 and the definition of $\hat{\mathcal{P}}$, there are two distinct strings $y_0, y_1 \succ z_{e,s}$ such that $|y_i| = n_e$ and $[y_i] \cap \hat{\mathcal{P}} \neq \emptyset$. Hence $z_{e+1,s}$ is well defined in equation (1).

To show that Z is $s \cdot b$-c.e., define a computable approximation $(Z_i)_{i \in \mathbb{N}^+}$ with $Z_s = z_{s,s}$. Suppose $n_e \leq n < n_{e+1}$.

If $Z_{s+1} \restriction_n \neq Z_s \restriction_n$ then

$$[Z_s \upharpoonright_n] \not\subseteq \hat{\mathcal{P}}_{s+1} \text{ or } \exists i \le e\, f_i(n, s+1) \ne f_i(n, s).$$

The former may occur at most 2^n many times, and the latter at most $e \cdot b(n)$ times. For all $e \ge 1$, the number of changes of $Z \upharpoonright_n$ is at most

$$\begin{aligned} 2^n + e \cdot b(n) &\le b(n)/\epsilon + e \cdot b(n) \\ &\le b(n)(e + 1/\epsilon) \\ &\le b(n) \cdot s(n_e) \le b(n) \cdot s(n). \end{aligned}$$

5 Counting Changes for Sets Given by the (Super)Low Basis Theorem

The low basis theorem of Jockusch and Soare [4] says that every nonempty Π^0_1 class has a member Z that is low, that is, $Z' \le_{\mathrm{T}} \emptyset'$. The proof actually makes Z *superlow*, that is, $Z' \le_{\mathrm{tt}} \emptyset'$. Here we study possible bounds on the number of changes for a low member of the class. Surprisingly, we find that to make the member superlow will in general take more changes, not fewer.

Theorem 9. *Let $\mathcal{P}$ be a nonempty Π^0_1 class. For each order function h, the class $\mathcal{P}$ has a superlow $2^{n+h(n)}$-c.e. member.*

Proof. The idea is to run the proof of the superlow basis theorem with a c.e. operator W^X that codes X' only at a sparse set of positions, and simply copies X for the other bit positions. Let R be the infinite computable set $\{n\colon\ h(n+1) > h(n)\}$. Define the c.e. operator W by

$$W^X(n) = \begin{cases} X(i) & \text{if } n \text{ is the } i\text{-th smallest element in } \mathbb{N} - R \\ X'(j) & \text{if } n \text{ is the } j\text{-th smallest element in } R \end{cases} \tag{2}$$

By the proof of the superlow basis theorem as in [1, Thm. 1.8.38], there is a $Z \in \mathcal{P}$ such that $B = W^Z$ is left-c.e. via some approximation (B_s). Let Z_s be the computable approximation of Z given by $Z_s(i) = B_s(n)$ where n is the i-th smallest element in $\mathbb{N} - R$. If $Z_s \upharpoonright_n$ changes then $B_s \upharpoonright_{n+h(n)}$ changes. Thus $Z_s \upharpoonright_n$ changes at most $2^{n+h(n)}$ times. Furthermore, $Z' \le_{\mathrm{m}} B$. Since B is ω-c.e. we have $B \le_{\mathrm{tt}} \emptyset'$, so Z is superlow. □

Theorem 18 below shows that if $\mathcal{P} \subseteq \mathsf{MLR}$, no superlow member can be $O(2^n)$-c.e. On the other hand, if we merely want a low member, we can actually get away with $o(2^n)$ changes. For the case $\mathcal{P} \subseteq \mathsf{MLR}$, this shows that $o(2^n)$-c.e. ML-random sets can be very different from the Turing complete ML-random set Ω, even though Ω is also $o(2^n)$-c.e. by Fact 2.

Theorem 10. *Each nonempty Π^0_1 class $\mathcal{P}$ contains a low $o(2^n)$-c.e. member.*

Proof. We combine the construction in the proof of Theorem 9 with a dynamic coding of the jump. At each stage we have movable markers γ_k at the positions where $X'(k)$ is currently coded. Thus, the positions where X' is coded become sparser and sparser as the construction proceeds.

Construction. At stage 0 let $\gamma_{0,0} = 1$ and B_0 be the empty set.
Stage $t > 0$.
(i). Let $W^X[t]$ be the c.e. operator such that

$$W^X[t](v) = \begin{cases} X(i) & \text{if } v \text{ is the } i\text{-th smallest element} \\ & \text{not of the form } \gamma_{k,t-1} \\ X'(k) & \text{if } v = \gamma_{k,t-1}\,. \end{cases} \tag{3}$$

We define a sequence of Π^0_1 classes $\mathcal{Q}_n[t]$ $(n \in \mathbb{N})$ according to the proof of the low basis theorem as in [1, Thm. 1.8.38], but at stage t we use the operator $W[t]$ instead of the jump operator.

Let $\mathcal{Q}_0[t] = \mathcal{P}$. If $\mathcal{Q}_n[t]$ has been defined, let

$$\mathcal{Q}_{n+1}[t] = \begin{cases} \mathcal{Q}_n[t] & \text{if for all } X \in \mathcal{Q}_{n,t}[t], \\ & \text{we have } n \in W^X[t] \\ \left\{X \in \mathcal{Q}_n[t] \,:\, n \notin W^X[t]\right\} & \text{otherwise.} \end{cases}$$

In the first case, define $B_t(n) = 1$; in the second case, define $B_t(n) = 0$.

(ii). Let k be least such that $k = t$ or $B_t\restriction_{2k} \neq B_{t-1}\restriction_{2k}$. Define $\gamma_{r,t} = \gamma_{r,t-1}$ for $r < k$, and $\gamma_{r,t} = t + 2r$ for $t \geq r \geq k$.

Verification.

Claim 1. *B is left-c.e. via the computable approximation* $(B_t)_{t\in\mathbb{N}}$.

Suppose i is least such that $B_t(i) \neq B_{t-1}(i)$. Since $\gamma_{r,t-1} > 2r$ for each r, this implies that $\gamma_{r,t} = \gamma_{r,t-1}$ for all r such that $\gamma_{r,t-1} \leq i$. Thus the construction up to $\mathcal{Q}_i[t]$ behaves like the usual construction to prove the low basis theorem, whence we have $B_{t-1}(i) = 0$ and $B_t(i) = 1$.

We conclude that $\gamma_k = \lim_t \gamma_{k,t}$ exists for each k, and therefore $\mathcal{Q}_n = \lim_t \mathcal{Q}_n[t]$ exists as well.

By the compactness of 2^ω there is $Z \in \bigcap_n \mathcal{Q}_n$. Clearly Z is low because $Z'(k) = B(\gamma_k)$ and the expression on the right can be evaluated by $\emptyset'$. It remains to show the following.

Claim 2. *Z is $o(2^n)$-c.e.*

We have a computable approximation to Z given by

$$Z_t(i) = B_t(v) \text{ where } v \text{ is the } i\text{-th smallest number not of the form } \gamma_{k,t}.$$

Given n let k be largest such that $\gamma_k \leq n$. We show that $Z\restriction_n$ changes at most 2^{n-k+1} times.

For $n \geq r \geq k$ let t_r be least stage t such that $\gamma_{r+1,t} > n$. Then $B_t\restriction_{2r}$ is stable for $t_r \leq t < t_{r+1}$. Since $(B_t)_{t\in\mathbb{N}}$ is a computable approximation via which B is left-c.e., $B\restriction_{n+r}$ changes at most 2^{n-r} times for $t \in [t_r, t_{r+1})$. Hence $Z\restriction_n$ changes at most 2^{n-r} times for such t. The total number of changes is therefore bounded by $\sum_{k\leq r\leq n} 2^{n-r} < 2^{n-k+1}$. □

Corollary 11. *There is a ML-random Z and a computable $q : \mathbb{N} \to \mathbb{Q}^+$ such that $Z \upharpoonright_n$ changes fewer than $\lfloor 2^n q(n) \rfloor$ times for infinitely many n, and $\lim_n q(n) = 0$.*

Proof. Follow the proof of Theorem 10, and let $\mathcal{P}$ be a Π^0_1 class containing only ML-randoms. We define $q(m) = 2^{-r+1}$, where r is the least such that $\gamma_{r,m} \geq m$. Then q is computable and $\lim_n q(n) = 0$ because each marker reaches a limit. Also, $q(\gamma_r) = 2^{-r+1}$ for every r. By the proof of Theorem 10, for every r, $Z \upharpoonright_{\gamma_r}$ changes fewer than $2^{\gamma_r - r+1} = \lfloor 2^{\gamma_r} q(\gamma_r) \rfloor$ times. □

6 Balanced Randomness

Basics on balanced randomness. We study a more restricted form of weak Demuth randomness (which was defined in the introduction). The bound on the number of changes of the m-th version is now $O(2^m)$.

Definition 12. A *balanced test* is a sequence of c.e. open sets $(G_m)_{m \in \mathbb{N}}$ such that $\forall m\, \lambda G_m \leq 2^{-m}$; furthermore, there is a function f such that G_m equals the Σ^0_1 class $[W_{f(m)}]^{\prec}$ and $f(m) = \lim_s g(m,s)$ for a computable function g such that the function mapping m to the size of the set $\{s\colon\ g(m,s) \neq g(m, s-1)\}$ is in $O(2^m)$.

A set Z *passes* the test if $Z \notin G_m$ for some m. We call Z *balanced random* if it passes each balanced test.

We denote $[W_{g(m,s)}]^{\prec}$ by $G_m[s]$ and call it the *version* of G_m at stage s.

Example 13. No $O(2^n)$-c.e. set is balanced random.

To see this, simply let $G_m[s] = [Z_s \upharpoonright_m]$; then Z fails the balanced test $(G_m)_{m \in \mathbb{N}}$.

Again, we may monotonize a test and thus assume $G_m \supseteq G_{m+1}$ for each m, because the number of changes of $\bigcap_{i \leq m} G_i[s]$ is also $O(2^m)$.

Let $(\alpha_i)_{i \in \mathbb{N}}$ be a nonincreasing computable sequence of rationals that converges effectively to 0, for instance $\alpha_i = 1/i$. If we build monotonicity into the definition of balanced randomness, we can replace the bound 2^{-m} on the measure of the m-th component by α_m, and bound the number of changes by $O(1/\alpha_m)$. Thus, the important condition is being balanced in the sense that the measure bound times the bound on the number of changes is $O(1)$. We emulate a test $(G_m)_{m \in \mathbb{N}}$ by a test $(H_i)_{i \in \mathbb{N}}$ as in Definition 12 by letting $H_i[s] = G_m[s]$, where m is least such that $2^{-i} \geq \alpha_m > 2^{-i-1}$.

Difference randomness and Turing incompleteness. Franklin and Ng have recently introduced *difference randomness*, where the m-th component of a test is a class of the form $A_m - B_m$ with measure at most 2^{-m}, for uniformly given Σ^0_1 classes A_m, B_m. To pass such a test means not to be in $A_m - B_m$ for some m. (We could replace the individual B_m in each component by $B = \bigcup B_m$. We may also assume that the test is monotonic after replacing $A_m - B_m$ by $\bigcap_{i \leq m} A_i - B$ if necessary.)

Proposition 14. *Each balanced random set is difference random.*

Proof. Given a test $(A_m - B_m)_{m\in\mathbb{N}}$, we may assume that $\lambda(A_{m,t} - B_{m,t}) \leq 2^{-m}$ for each t (these are the clopen sets effectively approximating A_m, B_m). At stage t let i be greatest such that $\lambda B_{m,t} \geq i2^{-m}$, and let $t^* \leq t$ be least such that $\lambda B_{m,t^*} \geq i2^{-m}$. Let $G_m[t] = A_m - B_{m,t^*}$. Then G_m changes at most 2^m times. Clearly $A_m - B_m$ is contained in the last version of G_m. For each t we have $\lambda G_m[t] \leq 2^{-m+1}$, so after omitting the first component we have a balanced test. □

Franklin and Ng [5] show that for ML-random sets, being difference random is equivalent to being Turing incomplete. Nonetheless, it is instructive to give a direct proof of this fact for balanced randomness.

Proposition 15. *Each balanced random set is Turing incomplete.*

Proof. Suppose Z is ML-random and Turing complete. Then $\Omega = \Gamma(Z)$ for some Turing functional Γ. By a result of Miller and Yu (see [1, Prop. 5.1.14]), there is a constant c such that $2^{-m} \geq \lambda\{Z\colon \Omega \upharpoonright_{m+c} \prec \Gamma(Z)\}$ for each m. Now let the version $G_m[t]$ copy $\{Z\colon \Omega_t \upharpoonright_{m+c} \prec \Gamma_t(Z)\}$ as long as the measure does not exceed 2^{-m}. Then Z fails the balanced test $(G_m)_{m\in\mathbb{N}}$. □

Balanced randomness and being ω-c.e. tracing. The following (somewhat weak) highness property was introduced by Greenberg and Nies [2]; it coincides with the class $\mathcal{G}$ in [1, Proof of 8.5.17].

Definition 16. Z is called *ω-c.e. tracing* if each function $f \leq_{\mathrm{wtt}} \emptyset'$ has a Z-c.e. trace T^Z_x such that $|T^Z_x| \leq 2^x$ for each x.

Since we trace only total functions, by a method of Terwijn and Zambella (see [1, Thm. 8.2.3]), the bound 2^x can be replaced by any order function without changing the class. Greenberg and Nies [2] show that there is a single benign cost function such that each c.e. set obeying it is Turing below each ω-c.e. tracing ML-random set. In particular, each strongly jump-traceable, c.e. set is below each ω-c.e. tracing set.

Fact 17. *No superlow set Z is ω-c.e. tracing.*

To prove this, one notes that T^Z_x is truth-table below $\emptyset'$ uniformly in x.

Theorem 18. *Let Z be a ML-random set. If Z is not balanced random then Z is ω-c.e. tracing. In particular, Z is not superlow.*

Proof. Fix a balanced test $(G_m)_{m\in\mathbb{N}}$ such that $Z \in \bigcap_m G_m$. Suppose we are given a function $f \leq_{\mathrm{wtt}} \emptyset'$ with computable use bound h. Thus there is a computable approximation $f(x) = \lim_s f_s(x)$ with at most $h(x)$ changes. Let $(m_i)_{i\in\mathbb{N}}$ be a computable sequence of numbers such that

$$\textstyle\sum_i h(i)2^{-m_i} < \infty,$$

for instance $m_i = \lfloor \log h(i) + 2\log(i+1) \rfloor$.

To obtain the required trace for f, we define an auxiliary Solovay test $\mathcal{S}$ of the form $\bigcup \mathcal{S}_i$. We put $[\sigma]$ into $\mathcal{S}_i$ if there are 2^i versions $G_{m_i}[t]$ such that $[\sigma] \subseteq G_{m_i}[t]$. Clearly $\mathcal{S}_i$ is uniformly Σ^0_1. We show that $\lambda\mathcal{S}_i = O(2^{-i})$ for each i. Let (σ_k) be a prefix free set of strings such that $\bigcup_k [\sigma_k] = \mathcal{S}_i$. Then

$$O(1) \geq \sum_s \lambda G_{m_i}[s] \geq \sum_s \sum_k \lambda(G_{m_i}[s] \cap [\sigma_k]) \geq 2^i \sum_k \lambda[\sigma_k] = 2^i \lambda\mathcal{S}_i.$$

To define the c.e. operators T^Z_i, when Z enters $G_{m_i}[s]$, put $f_s(i)$ into T^Z_i. Since Z passes the Solovay test $\mathcal{S}$, for almost every i we put at most 2^i numbers into T^Z_i.

To show that f is traced, we define a further Solovay test $\mathcal{R}$. When $f_s(i) \neq f_{s-1}(i)$, put the current version $G_{m_i}[s]$ into $\mathcal{R}$. Note that $\mathcal{R}$ is a Solovay test because $\sum_i h(i)2^{-m_i} < \infty$. Since Z passes the test $\mathcal{R}$ but fails $(G_m)_{m\in\mathbb{N}}$, we have $f(i) \in T^Z_i$ for almost every i. For, if $f_s(i) \neq f_{s-1}(i)$ then Z must enter a further version $G_m[t]$ for some $t \geq s$, so we can put the new value $f_s(i)$ into T^Z_i. □

By Theorem 10 we have the following result.

Corollary 19. *There is an ω-c.e. tracing low ML-random set.*

Proof. Applying Theorem 10 to a Π^0_1 class $\mathcal{P} \subseteq \mathsf{MLR}$, we obtain a low ML-random set that is $o(2^n)$-c.e. This set is not balanced random. Then, by Theorem 18 the set is ω-c.e. tracing. □

Recall that any incomplete ML-random set is difference random. So the proof also shows that some difference random set is not balanced random.

We do not know at present whether the converse of Theorem 18 holds: if Z is balanced random, does it fail to be ω-c.e. tracing? Any LR-complete set is ω-c.e. tracing by [1, Thm. 8.4.15]. So this would imply that a balanced random set is not LR-complete; in particular, no K-trivial set can be cupped above $\emptyset'$ via a balanced random set. By the method of [1, Lem. 8.5.18], we have the following somewhat weaker converse of Theorem 18.

Theorem 20. *Suppose Z is an $O(h(n)2^n)$-weak Demuth random set for some order function h. Then Z is not ω-c.e. tracing.*

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