Characterization, Definability and Separation via Saturated Models

Carlos Areces Facundo Carreiro Santiago Figueira

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Abstract

Three important results about the expressivity of a modal logic $\mathfrak{L}$ are the Characterization Theorem (that identifies a modal logic $\mathfrak{L}$ as a fragment of a better known logic), the Definability theorem (that provides conditions under which a class of $\mathfrak{L}$-models can be defined by a formula or a set of formulas of $\mathfrak{L}$), and the Separation Theorem (that provides conditions under which two disjoint classes of $\mathfrak{L}$-models can be separated by a class definable in $\mathfrak{L}$).

We provide general conditions under which these results can be established for a given choice of model class and modal language whose expressivity is below first order logic. Besides some basic constraints that most modal logics easily satisfy, the fundamental condition that we require is that the class of ω-saturated models in question has the Hennessy-Milner property with respect to the notion of observational equivalence under consideration. Given that the Characterization, Definability and Separation theorems are among the cornerstones in the model theory of $\mathfrak{L}$, this property can be seen as a test that identifies the adequate notion of observational equivalence for a particular modal logic.

1 Introduction

Syntactically, modal languages [7] are propositional languages extended with *modal operators*. Indeed, the basic modal language is defined as the extension of the propositional language with the unary operator $\Diamond$. Although these languages have a very simple syntax, they are extremely useful to describe and reason about *relational structures*. A relational structure is a nonempty set together with a family of n-ary relations. Given the generality of this definition it is not surprising that modal logics are used in a wide range of disciplines: mathematics, philosophy, computer science, computational linguistics, etc. For example, in theoretical computer science, labeled transition systems (which are nothing but relational structures) are used to model the execution of a program.

An important observation that might have gone unnoticed in the above paragraphs is that we talk about modal *logics*, in plural. There is, nowadays, a wide variety of modal languages and an extensive menu of modal operators to choose from: *Since* and *Until* [19], universal modality [15], difference modality [11], fix-point operators [21], are some of the possibilities to name only a few. This multiplicity is both a boon and a bane. On the one hand, the variety comes in handy when we need to choose the proper logic to model a particular problem. But it also means that many results have to be established time and again for each new logic that arrives in town. It is here when a solid model theory is useful. With the proper theoretical tools, some results might be established just by verifying certain properties of the class of models defining the logic. In particular, many model theoretical results for a logic $\mathfrak{L}$ rely on the availability of an adequate notion of "indiscernibility" or observational equivalence, i.e., a notion that specifies when two models are indistinguishable by formulas of $\mathfrak{L}$.

We investigate Characterization, Definability, and Separation theorems for modal logics: three model-theoretical results intimately related with the notion of observational equivalence. We pursue a general study of these properties without referring to a particular modal logic. In general, the validity of these theorems is a good indicator that the underlying notion of observational equivalence for a given logic is indeed the correct one.

First-order logic, modal logics and similarity These three notions will play a mayor role and it will be useful to discuss them and their interaction right away. First-order logic (FO) will delineate our framework and we will assume its syntax, semantics and basic properties well known.[1] All the modal logics covered by our results are fragments of FO, and we will make use of some of FO's main model theoretic properties to prove our results. We will introduce the basic (uni)modal logic BML in detail but, in the rest of this paper we will work with an arbitrary modal logic. We will only require it to be *adequately below first-order logic* as per Definition 4. Finally, we will discuss different notions of observational equivalence. They will depend on the particular logic under consideration but, once more, we will abstract away their common aspects in the notion of an *adequate similarity* as per Definition 6.

Let us start by introducing syntax and semantics of BML. Let PROP be a countable, infinite set of propositional symbols. Formulas in BML are generated by the grammar:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Diamond\varphi,$$

where p is a propositional symbol in PROP. BML is interpreted over relational models $\mathcal{M} = (M, R, V)$, where M is a nonempty domain, R is a binary relation on M, and V is a valuation mapping propositional symbols to subsets over M. The pair $\langle \mathcal{M}, w\rangle$ for w an element in M is called a pointed model. We usually drop brackets and write $\mathcal{M}, w$ instead of $\langle \mathcal{M}, w\rangle$. Given a pointed model $\mathcal{M}, w$ we define when a BML-formula φ is true in $\mathcal{M}$ at w (notation $\mathcal{M}, w \Vdash \varphi$) as follows:

$$\begin{array}{rcl}
\mathcal{M}, w \Vdash p & \text{iff} & w \in V(p) \\
\mathcal{M}, w \Vdash \neg\varphi & \text{iff} & \mathcal{M}, w \nVdash \varphi \\
\mathcal{M}, w \Vdash \varphi \wedge \psi & \text{iff} & \mathcal{M}, w \Vdash \varphi \text{ and } \mathcal{M}, w \Vdash \psi \\
\mathcal{M}, w \Vdash \Diamond\varphi & \text{iff} & \mathcal{M}, v \Vdash \varphi \text{ for some } v \in M \text{ such that } wRv.
\end{array}$$

The right notion of observational equivalence for BML is that of a *bisimulation*. A bisimulation between two models $\mathcal{M} = (M, R, V)$ and $\mathcal{M}' = (M', R', V')$ is a nonempty relation $Z \subseteq M \times M'$ satisfying the following conditions:

(i) *Atomic harmony:* if wZw' then w and w' satisfy the same propositional symbols, i.e., $w \in V(p)$ iff $w' \in V'(p)$ for all propositional symbols p;

(ii) *Forth condition*: if wZw' and wRv then there is v' s.t. vZv' and $w'R'v'$;

(iii) *Back condition*: if wZw' and $w'R'v'$ then there is v s.t. vZv' and wRv.

[1] We will use standard notation for first-order models and formulas and, in particular, we will use $\models$ for the satisfiability relation between a first-order model $\mathcal{M}$, an assignment g and a first-order formula φ.

Two pointed models $\mathcal{M}, w$ and $\mathcal{M}', w'$ are called *bisimilar* if there is a bisimulation Z between M and M' such that wZw'. A well known result in basic modal logic states that if $\mathcal{M}, w$ and $\mathcal{M}', w'$ are bisimilar then they are *modally equivalent*, i.e., for any BML-formula φ we have $\mathcal{M}, w \Vdash \varphi$ iff $\mathcal{M}', w' \Vdash \varphi$. The reverse implication is not true in general. A model $\mathcal{M}$ is called modally-saturated if for every state $w \in M$ and every set Σ of formulas, if every finite subset of Σ is satisfiable in some successor of w, then Σ itself is satisfiable in some successors of w. An important result states that if two modally saturated models are modally equivalent then they are bisimilar [7].

We now switch to first-order logic. Notice, first, that a relational model $\mathcal{M} = (M, R, V)$ is essentially a first-order model over the language with a binary relation symbol and unary predicate symbols for the propositional symbols. Second, bisimulations are the modal analogue of the first-order notion of *partial isomorphism.* That is, partial isomorphisms are the right notion of observational equivalence for FO. Given a model $\mathcal{M}$ and $w_1, \ldots, w_n$ elements in M, we write $(\mathcal{M}, w_1, \ldots, w_n)$ for the extension of $\mathcal{M}$ with $w_1, \ldots, w_n$ as new constant symbols (interpreted in the obvious way). A partial isomorphism between two first-order models $\mathcal{M}$ and $\mathcal{M}'$ is a binary relation Z on pairs of finite sequences $\langle w_1, \ldots, w_n \rangle$, $\langle w'_1, \ldots, w'_n \rangle$ of elements of M and M' of the same length such that $\emptyset Z \emptyset$ and

(i) *Atomic harmony:* if $\langle w_1, \ldots, w_n \rangle Z \langle w'_1, \ldots, w'_n \rangle$ then $(\mathcal{M}, w_1, \ldots, w_n)$ and $(\mathcal{M}', w'_1, \ldots, w'_n)$ satisfy the same atomic sentences;

(ii) *Forth condition:* if $\langle w_1, \ldots, w_n \rangle Z \langle w'_1, \ldots, w'_n \rangle$ then for all $v \in \mathcal{M}$ there is $v' \in \mathcal{M}'$ such that $\langle w_1, \ldots, w_n, v \rangle Z \langle w'_1, \ldots, w'_n, v' \rangle$;

(iii) *Back condition:* if $\langle w_1, \ldots, w_n \rangle Z \langle w'_1, \ldots, w'_n \rangle$ then for all $v' \in \mathcal{M}'$ there is $v \in \mathcal{M}$ such that $\langle w_1, \ldots, w_n, v \rangle Z \langle w'_1, \ldots, w'_n, v' \rangle$.

The coincidences between the definitions of bisimulation and partial isomorphism are clear. They also play a similar role: any two partially isomorphic first-order models are elementarily equivalent (i.e., they satisfy the same first-order formulas). The reverse implication is not true in general but it holds for ω-saturated models [9]. In other words, partial isomorphisms and bisimulations are both carefully tuned to their respective logics.

Characterization, definability and separation for the basic modal logic A well known result shows that BML is at most as expressive as FO: there is a truth preserving translation ST_x mapping formulas of BML to formulas of FO (with at most one free variable x). That is, for every BML-formula φ, $\mathcal{M}, w \Vdash \varphi$ iff $\mathcal{M} \models \mathrm{ST}_x(\varphi)(w)$.[2] In fact, BML is strictly less expressive than FO: it coincides with the formulas of FO that are preserved under bisimulations. This is known as the van Benthem characterization theorem [38]:

Theorem 1 (Characterization)**.** *A first-order formula φ is equivalent to the translation of a BML formula iff φ is invariant under bisimulations.*

This theorem can be used, for instance, to prove undefinability results for the basic modal logic. A property that can be expressed with a first-order formula with one free variable but is not invariant under bisimulation (e.g., irreflexivity of a binary relation) cannot be expressed by a formula of BML.

[2] $\mathcal{M} \models \psi(w)$ is the standard notation for $\mathcal{M}, v \models \psi(x)$ for any valuation v such that $v(x) = w$. See for instance [26].

The first-order Definability theorem states that a class of models K is definable by means of a set of formulas if and only if K is closed under ultraproducts and partial isomorphisms, and the complement of K is closed under ultrapowers. K is definable by a single formula if and only if both K and its complement are closed under ultraproducts and partial isomorphisms. It follows, for example, that the class of finite models is not definable in FO because it is not closed under ultraproducts. The Definability theorem for BML was proved by de Rijke [12]:

Theorem 2 (Definability)**.** *A class of relational pointed models K is definable by means of a set of BML-formulas (respectively, a single BML-formula) iff K is closed under bisimulations and ultraproducts and its complement is closed under ultrapowers (respectively, bisimulations and ultraproducts).*

By Theorem 2 the class of reflexive pointed models, for example, is not definable by a single formula or by a set of formulas in BML as it is not closed under bisimulations.

First-order Separation is closely related to Definability: it provides conditions to separate two disjoint classes of models M and N by means of an elementary class (i.e., a class defined by a first-order formula). That is, it is possible to find an elementary class M' such that $\mathsf{M} \subseteq \mathsf{M}'$ and $\mathsf{N} \cap \mathsf{M}' = \emptyset$. The Separation theorem for FO states that M' always exists if M is closed under isomorphisms and ultraproducts, and N is closed under isomorphisms and ultrapowers. The Separation theorem for BML was also proved by de Rijke in [12].

Theorem 3 (Separation)**.** *Let M and N be classes of relational pointed models with $\mathsf{M} \cap \mathsf{N} = \emptyset$. If M is closed under bisimulations and ultraproducts, and N is closed under bisimulations and ultrapowers (respectively, ultraproducts), then there exists a class M' of pointed models definable by means of a set of BML-formulas (respectively a single BML-formula) such that $\mathsf{M} \subseteq \mathsf{M}'$ and $\mathsf{N} \cap \mathsf{M}' = \emptyset$.*

Characterization, Definability and Separation have been investigated for many different modal logics (indeed the literature on these topic is extremely wide, see [7, 18, 22, 25, 29, 37, 13] among others). In each particular case, a different notion of observational equivalence is involved, and an ad-hoc proof is provided. Still, the spirit of the proofs is similar and they rely on first-order model theoretic tools. This is explicitly mentioned by Blackburn, de Rijke and Venema when discussing Separation and Definability [7, p. 109]:

> This close connection to first-order logic may explain why the results of this section seem to generalize to any modal logic that has a standard translation into first-order logic. For example, all the results of this section can also be obtained for basic temporal logic.

Our main goal is to sum up the key ingredients used in these proofs and identify sufficient model theoretical conditions that an *arbitrary* (even sub-boolean) modal logic has to fulfill for these results to hold. These conditions are captured by the notion of *adequate similarity* that we introduce in Definition 6.

This article is an extended version of [8]. We have refined the definition of the general framework, included full proofs of all results, and added two new sections: one concerning Separation results and one with applications of this framework to several concrete logics. The rest of the article is organized as follows. In Section 2 we introduce basic definitions. In Sections 3, 4 and 5 we prove generalizations of the Characterization, Definability and Separation theorems. In Section 6 we show how to use the previous theorems in particular logics providing both new results and alternative proofs to well established ones. In Section 7 we compare our results to related work in the area. Finally in Section 8 we draw our conclusions and propose further lines of research.

2 The general framework

We will introduce basic definitions about syntax, semantics, expressive power and simulation. Together, they will define our general framework.

Syntax We say that a modal logic $\mathfrak{L}$ has an *adequate syntax* if it extends the language defined by the following grammar:

$$\varphi ::= p \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \top \mid \bot,$$

where p is one of countably many propositional symbols. $\textsc{form}(\mathfrak{L})$ denotes the set of all formulas of $\mathfrak{L}$. Note that negation might not be present in the language.

For first-order we assume a signature σ and denote the set of all first-order σ-formulas as $\textsc{form}(\text{FO})$. Given first-order variables $x_1, \ldots, x_n$ we write $\varphi(x_1, \ldots, x_n)$ to say that the free variables of φ are among $x_1, \ldots, x_n$ (but not necessarily all of them). This notation extends to sets $\Gamma(x_1, \ldots, x_n)$ of first-order formulas.

Semantics We say that $\mathfrak{L}$ has *adequate semantics* if the *meaning* or *extension* of each formula in $\mathfrak{L}$ is specified in model theoretic ways. More precisely, we assume that formulas are evaluated over structures we will call $\mathfrak{L}$-models, that each $\mathfrak{L}$-model $\mathcal{M}$ has as domain a non empty set of elements (denoted $|\mathcal{M}|$), and that the extension of each formula is specified over $|\mathcal{M}|$. A *pointed* $\mathfrak{L}$*-model* is a pair $\langle \mathcal{M}, \bar{w} \rangle$ where $\mathcal{M}$ is an $\mathfrak{L}$-model and $\bar{w} \in |\mathcal{M}|^n$ for some fixed n that we will call the *dimension* of the pointed model.[3] Given an $\mathfrak{L}$-formula φ, we use the notation $\mathcal{M}, \bar{w} \Vdash \varphi$ for "φ is true at $\bar{w}$ in $\mathcal{M}$" and $\mathcal{M}, \bar{w} \nVdash \varphi$ for "φ is false at $\bar{w}$ in $\mathcal{M}$". The definition of $\Vdash$ depends on $\mathfrak{L}$. We only impose that $\wedge$, $\vee$, $\top$ and $\bot$ have the usual interpretations:

$$\begin{array}{rll} \mathcal{M}, \bar{w} \Vdash \varphi \wedge \psi & \text{iff} & \mathcal{M}, \bar{w} \Vdash \varphi \text{ and } \mathcal{M}, \bar{w} \Vdash \psi \\ \mathcal{M}, \bar{w} \Vdash \varphi \vee \psi & \text{iff} & \mathcal{M}, \bar{w} \Vdash \varphi \text{ or } \mathcal{M}, \bar{w} \Vdash \psi \\ \mathcal{M}, \bar{w} \Vdash \top & & \text{always} \\ \mathcal{M}, \bar{w} \Vdash \bot & & \text{never.} \end{array}$$

We assume that all pointed models of a given logic $\mathfrak{L}$ have the same dimension. Given $\mathfrak{L}$ we use $\textsc{pmods}(\mathfrak{L})$ to denote the class of pointed $\mathfrak{L}$-models. Given M a class of (modal or first-order) models, we will denote by $\overline{\mathsf{M}}$ the complement of M with respect to a universe class C that will always be clear from context.

Expressive power We define formally what it means for a logic $\mathfrak{L}$ to be at most as expressive as first-order logic. Intuitively, this will be the case if there is a truth-preserving translation of $\mathfrak{L}$-formulas into FO-formulas. Since we have fixed no specific semantics for $\mathfrak{L}$, this translation should be understood modulo a suitable translation of pointed models of $\mathfrak{L}$ to pointed models of FO.

[3]For BML, $n = 1$ and formulas are evaluated at a single point; other modal logics, e.g., multidimensional modal logics like arrow and interval logics [27], are evaluated over tuples.

A *pointed* FO-*model* is a pair $\langle \mathcal{M}, \bar{w} \rangle$ where $\mathcal{M}$ is an FO-model for the signature σ and $\bar{w} \in |\mathcal{M}|^n$ for some fixed n that we call the *dimension* of the pointed model. $\textsc{pmods}_n(\text{FO})$ denotes the class of all pointed FO-models of dimension n. We will assume a fixed dimension and drop the subscript. Given a pointed FO-model $\langle \mathcal{M}, \bar{w} \rangle$ for $\bar{w} = w_1, \ldots, w_n$ of dimension n, we will also consider $\bar{w}$ as a finite valuation $g : \{x_1, \ldots, x_n\} \to |\mathcal{M}|$ interpreting variables $x_1, \ldots, x_n$ as $w_1, \ldots, w_n$, respectively.

Given an ultrafilter D over an index I, the ultraproduct of pointed FO-models $\{\mathcal{G}_i, g_i\}_{i \in I}$ (notation: $\prod_D \mathcal{G}_i, g_i$) is a pointed FO-model $\langle \mathcal{G}, g \rangle$ where $\mathcal{G} = \prod_D \mathcal{G}_i$ is the ultraproduct of $\{\mathcal{G}_i\}_{i \in I}$ modulo D and g is the ultralimit of $\{g_i\}_{i \in I}$ modulo D. Equivalently one may interpret any σ-pointed model $\langle \mathcal{G}, g \rangle$ as a σ'-model $\mathcal{G}'$ where $\sigma' = \sigma \cup \{c_1, \ldots, c_n\}$ for new constant symbols $c_i \notin \sigma$ and where the interpretation of c_i in $\mathcal{G}'$ is $g(x_i)$. Then the ultraproduct $\prod_D \mathcal{G}_i, g_i$ is a pointed model $\langle \mathcal{G}, g \rangle$ where $\mathcal{G}$ is the reduct of $\mathcal{G}' = \prod_D \mathcal{G}'_i$ to σ, and $\bar{g}(x_i)$ is the interpretation of c_i in $\mathcal{G}'$.

Let $\langle \mathcal{M}, \bar{w} \rangle, \langle \mathcal{N}, \bar{v} \rangle \in \textsc{pmods}(\mathfrak{L})$. We write $\mathcal{M}, \bar{w} \Rrightarrow_{\mathfrak{L}} \mathcal{N}, \bar{v}$ if for every $\mathfrak{L}$-formula φ, $\mathcal{M}, \bar{w} \Vdash \varphi$ implies $\mathcal{N}, \bar{v} \Vdash \varphi$. We write $\mathcal{M}, \bar{w} \equiv_{\mathfrak{L}} \mathcal{N}, \bar{v}$ when $\mathcal{M}, \bar{w} \Rrightarrow_{\mathfrak{L}} \mathcal{N}, \bar{v}$ and $\mathcal{N}, \bar{v} \Rrightarrow_{\mathfrak{L}} \mathcal{M}, \bar{w}$. We will use a similar notation for pointed FO-models and we drop subscripts when the logic involved is clear from context.

Definition 4 (Adequately below first-order)**.** *We say that a logic $\mathfrak{L}$ with adequate syntax and semantics* is below first-order *if models in* $\textsc{pmods}(\mathfrak{L})$ *have dimension n and there is a class of models* $\mathsf{K} \subseteq \textsc{pmods}_n(\text{FO})$ *for a fixed first-order signature σ, a bijective map* $\mathsf{Tm} : \textsc{pmods}(\mathfrak{L}) \to \mathsf{K}$ *(called a* model translation *of $\mathfrak{L}$ into* K*) and a map* $\mathsf{Tf} : \textsc{form}(\mathfrak{L}) \to \textsc{form}(\text{FO})$ *(called a* formula translation *of $\mathfrak{L}$) satisfying the following conditions:*

(i) If $\varphi \in \textsc{form}(\mathfrak{L})$ and $\langle \mathcal{M}, \bar{w} \rangle \in \textsc{pmods}(\mathfrak{L})$ then

$$\mathcal{M}, \bar{w} \Vdash \varphi \quad \textit{iff} \quad \mathsf{Tm}(\mathcal{M}, \bar{w}) \models \mathsf{Tf}(\varphi);$$

(ii) The range $\mathsf{Ran}(\mathsf{Tf})$ *of* Tf *is closed under conjunction and disjunction, up to semantic equivalence: if $\alpha, \beta \in \mathsf{Ran}(\mathsf{Tf})$ then there exists $\varphi_{\alpha \wedge \beta} \in \mathsf{Ran}(\mathsf{Tf})$ (resp. $\varphi_{\alpha \vee \beta} \in \mathsf{Ran}(\mathsf{Tf})$) such that $\varphi_{\alpha \wedge \beta} \equiv_{\text{FO}} \alpha \wedge \beta$ (resp. $\varphi_{\alpha \vee \beta} \equiv_{\text{FO}} \alpha \vee \beta$).*

$\mathfrak{L}$ *is* adequately below first-order *if, additionally,* K *is closed under isomorphisms and ultraproducts.*

For notational simplicity, and without loss of generality, we will read (ii) as saying that $\mathsf{Tf}(\varphi \wedge \psi) = \mathsf{Tf}(\varphi) \wedge \mathsf{Tf}(\psi)$ and $\mathsf{Tf}(\varphi \vee \psi) = \mathsf{Tf}(\varphi) \vee \mathsf{Tf}(\psi)$.

Observe that a number of useful properties obtain when a given logic $\mathfrak{L}$ is adequately below first order. For example, as K is closed under ultraproducts it can be shown that $\mathfrak{L}$ is compact.

Similarity As we have seen, the notions of observational equivalence for different logics can greatly vary in shape and form. For instance, a bisimulation (the right notion for BML) is a relation between the domains of two fixed pointed models. On the other hand, a partial isomorphism (the right notion for FO) is a relation between sequences of elements of two fixed models. Still, they both imply that the two models are indistinguishable by the formulas of the logic.

As we want to consider arbitrary modal logics, we cannot possibly know the particular shape of the notion of observational equivalence involved. Therefore, we will abstract their main property in the more general notion of *similarity*.

Definition 5 ($\mathfrak{L}$-similarity)**.** *A relation* $\rightrightarrows_{\mathfrak{L}} \subseteq \textsc{pmods}(\mathfrak{L}) \times \textsc{pmods}(\mathfrak{L})$ *is an* $\mathfrak{L}$-similarity *if* $\mathcal{M}, \bar{w} \rightrightarrows_{\mathfrak{L}} \mathcal{N}, \bar{v}$ *implies* $\mathcal{M}, \bar{w} \Rrightarrow_{\mathfrak{L}} \mathcal{N}, \bar{v}$. *We drop the subscript when the logic is clear from context.*

Observe that the relation $\rightrightarrows_{\mathfrak{L}}$ is not necessarily symmetric, since we contemplate in our framework logics without negation. In these cases, an asymmetric relation is the right notion of similarity (see, e.g. [24]).

It is important to understand the difference between the notion of $\mathfrak{L}$-similarity and the notion of observational equivalence. Observe that an $\mathfrak{L}$-similarity is a relation linking *pointed models* (i.e., it is defined over a class of pointed models), while an observational equivalence relation is a relation defined between two given models linking observationally equivalent states.

Consider, for example, two fixed BML models $\mathcal{M}$, $\mathcal{N}$. A bisimulation is a relation $Z \subseteq |\mathcal{M}| \times |\mathcal{N}|$ satisfying the harmony, back and forth constraints (cf. p. 2). The associated notion of BML-similarity is *induced* by this notion of bisimulation. Namely, define $\mathcal{M}, w \rightrightarrows_{\mathrm{BML}} \mathcal{N}, v$ if and only if there exists a bisimulation $Z \subseteq |\mathcal{M}| \times |\mathcal{N}|$ such that wZv. In the case of FO, the same idea can be used to induce a notion of FO-similarity by requiring the existence of a partial isomorphism.

Given a signature σ, recall that a first-order σ-structure $\mathcal{M}$ *realizes* a set of σ-formulas $\Gamma(\bar{x})$ if there exists $\bar{w} \in |\mathcal{M}|^n$ such that $\mathcal{M} \models \varphi(\bar{w})$ for every $\varphi \in \Gamma(\bar{x})$. Given a subset $A \subseteq |\mathcal{M}|$, the expanded model $(\mathcal{M}, A)$ where each element in A is considered a new constant symbols with the obvious interpretation will be denoted by $\mathcal{M}_A$ and its signature by σ_A. The *theory of* $\mathcal{M}$ is the set of all σ-sentences that hold in $\mathcal{M}$. A model $\mathcal{M}$ is said to be ω*-saturated* if the following property holds: Let A be a finite subset of $|\mathcal{M}|$, then every set of formulas $\Gamma(\bar{x})$ over σ_A consistent with the theory of $\mathcal{M}_A$ is realized in $\mathcal{M}_A$.

Definition 6 (Adequate $\mathfrak{L}$-similarity)**.** *Let* $\mathfrak{L}$ *be adequately below first-order over a class of models* K. *We say that an* $\mathfrak{L}$*-similarity* $\rightrightarrows_{\mathfrak{L}}$ *is* adequate *for* $\mathfrak{L}$ *if the class of* ω*-saturated models in* K *has the Hennessy-Milner property with respect to* $\rightrightarrows_{\mathfrak{L}}$. *That is, if* $\mathsf{Tm}(\mathcal{M}, w)$ *and* $\mathsf{Tm}(\mathcal{N}, v)$ *are* ω*-saturated then* $\mathcal{M}, w \Rrightarrow_{\mathfrak{L}} \mathcal{N}, v$ *implies* $\mathcal{M}, w \rightrightarrows_{\mathfrak{L}} \mathcal{N}, v$.

Many modal logics fit withing the proposed framework and we will discuss examples in detail in Section 6. In particular, we will show that the Characterization, Definability and Separation theorems hold for the basic temporal logic, the hybrid logics $\mathcal{HL}$ and $\mathcal{HL}(@)$ and different memory logics.

The above definitions outline our general framework: we will show in the next sections that any logic $\mathfrak{L}$ adequately below first-order, with an adequate $\mathfrak{L}$-similarity will satisfy the theorems of Definability, Characterization and Separation. In the next sections we will assume fixed a modal logic $\mathfrak{L}$ which is adequately below first-order and has an adequate notion of $\mathfrak{L}$-similarity $\rightrightarrows_{\mathfrak{L}}$ for its class $\textsc{pmods}(\mathfrak{L})$ of pointed models. We also assume an arbitrary but fixed dimension n for the models and therefore, unless otherwise stated, vectors such as $\bar{x}$ and $\bar{w}$ will be of size n.

3 Characterization

One of the central notions of the Characterization theorem stated in Section 1 was that of bisimulation invariance. In the following definition we cast this notion in the context of our framework.

Definition 7 (Invariance for $\mathfrak{L}$-similarity)**.** *A formula* $\alpha(\bar{x}) \in \text{FORM}(\text{FO})$ *is* invariant for $\mathfrak{L}$-similarity *if for all* $\langle \mathcal{M}, \bar{w}\rangle$, $\langle \mathcal{N}, \bar{v}\rangle \in \text{PMODS}(\mathfrak{L})$ *such that* $\mathcal{M}, \bar{w} \rightrightarrows_{\mathfrak{L}} \mathcal{N}, \bar{v}$*, we have that* $\mathsf{Tm}(\mathcal{M}, \bar{w}) \models \alpha(\bar{x})$ *implies* $\mathsf{Tm}(\mathcal{N}, \bar{v}) \models \alpha(\bar{x})$*.*

Notice that the property is defined for first-order formulas, but the $\mathfrak{L}$-similarity relation is defined between pointed $\mathfrak{L}$-models. By our definition, a first-order formula $\alpha(\bar{x})$ is 'invariant for $\mathfrak{L}$-similarity' if, for every two pointed $\mathfrak{L}$-models $\mathcal{M}, \bar{w}$ and $\mathcal{N}, \bar{v}$ such that $\mathcal{M}, \bar{w} \rightrightarrows_{\mathfrak{L}} \mathcal{N}, \bar{v}$ whenever $\alpha(\bar{x})$ is true in $\mathsf{Tm}(\mathcal{M}, \bar{w})$ then it is also true in $\mathsf{Tm}(\mathcal{N}, \bar{v})$. That is, we check simulation in $\mathfrak{L}$ and satisfaction in FO.[4]

We are almost ready to prove the Characterization theorem. In the proof we will need the following results. The first is a fairly straightforward generalization of first-order compactness to a class of pointed FO-models and we provide only a brief proof. The second result, instead, is the core of the Characterization theorem and shows that if the $\mathfrak{L}$-theory of $\mathcal{M}, \bar{w}$ is included in that of $\mathcal{N}, \bar{v}$ then all $\mathfrak{L}$-simulation K-invariant first-order formulas satisfied by $\mathsf{Tm}(\mathcal{M}, \bar{w})$ are also satisfied by $\mathsf{Tm}(\mathcal{N}, \bar{v})$.

Proposition 8 (Relativized first-order compactness)**.** *Let* C *be a class of pointed* FO*-models which is closed under ultraproducts and let* $\Sigma(\bar{x})$ *be a set of first-order formulas. If every finite set* $\Delta(\bar{x}) \subseteq \Sigma(\bar{x})$ *has a model in* C*, then* $\Sigma(\bar{x})$ *has a model in* C*.*

Proof. Let $\langle \mathcal{G}_i, \bar{w}_i\rangle \in \mathsf{C}$ be a model for each finite subset $\Delta_i(\bar{x}) \subseteq \Sigma(\bar{x})$. Algebraic proofs of the compactness theorem (cf. [20, Theorem 4.3]) show that the ultraproduct of the models $\mathcal{M}, \bar{w} := \prod_D \mathcal{G}_i, \bar{w}_i$ (for a suitable ultrafilter D) satisfies $\mathcal{M}, \bar{w} \models \Sigma(\bar{x})$. As C is closed under ultraproducts we conclude that $\mathcal{M}, \bar{w} \in \mathsf{C}$. Although [20, Theorem 4.3] is proved for sets of first-order *sentences*, our result for pointed models can be obtained by extending the first-order language with new constants and working with sentences for the extended language. $\square$ $\square$

Proposition 9. *Let* $\mathsf{M}_1, \mathsf{M}_2 \subseteq \text{PMODS}(\mathfrak{L})$ *be such that* $\mathsf{Tm}(\mathsf{M}_1)$ *and* $\mathsf{Tm}(\mathsf{M}_2)$ *are closed under ultrapowers. Let* $\langle \mathcal{M}, \bar{w}\rangle \in \mathsf{M}_1$ *and* $\langle \mathcal{N}, \bar{v}\rangle \in \mathsf{M}_2$ *be such that* $\mathcal{N}, \bar{v} \Rrightarrow_{\mathfrak{L}} \mathcal{M}, \bar{w}$*. Then there exist models* $\langle \mathcal{M}^*, \bar{w}^*\rangle \in \mathsf{M}_1$ *and* $\langle \mathcal{N}^*, \bar{v}^*\rangle \in \mathsf{M}_2$ *that satisfy the following:*

(i) Their translations are pairwise elementarily equivalent: $\mathsf{Tm}(\mathcal{M}, \bar{w}) \equiv_{\text{FO}} \mathsf{Tm}(\mathcal{M}^*, \bar{w}^*)$ *and* $\mathsf{Tm}(\mathcal{N}, \bar{v}) \equiv_{\text{FO}} \mathsf{Tm}(\mathcal{N}^*, \bar{v}^*)$*.*

(ii) They are pairwise $\mathfrak{L}$*-equivalent:* $\mathcal{M}, \bar{w} \equiv_{\mathfrak{L}} \mathcal{M}^*, \bar{w}^*$ *and* $\mathcal{N}, \bar{v} \equiv_{\mathfrak{L}} \mathcal{N}^*, \bar{v}^*$*.*

(iii) $\mathcal{N}^*, \bar{v}^* \rightrightarrows_{\mathfrak{L}} \mathcal{M}^*, \bar{w}^*$*.*

Proof. Let $\mathcal{M}_{\text{FO}}, g_{\bar{w}} = \mathsf{Tm}(\mathcal{M}, \bar{w})$ and $\mathcal{N}_{\text{FO}}, g_v = \mathsf{Tm}(\mathcal{N}, \bar{v})$. Take $\mathcal{M}^+_{\text{FO}}, \mathcal{N}^+_{\text{FO}}$ to be ω-saturated ultrapowers of $\mathcal{M}_{\text{FO}}$ and $\mathcal{N}_{\text{FO}}$. As the classes are closed under ultrapowers, the saturated models lay in the same class as the original models.

By [9, Corollary 4.1.13] we have an elementary embedding $d : |\mathcal{M}_{\text{FO}}| \to |\mathcal{M}^+_{\text{FO}}|$. Let $g^+_{\bar{w}}$ be an assignment for $\mathcal{M}^+_{\text{FO}}$ with $g^+_{\bar{w}}(x) = d(g_{\bar{w}}(x))$. Take the modal preimage of $\mathcal{M}^+_{\text{FO}}, g^+_{\bar{w}}$ and call it $\mathcal{M}^*, \bar{w}^* = \mathsf{Tm}^{-1}(\mathcal{M}^+_{\text{FO}}, g^+_{\bar{w}})$. We repeat the same process and assign similar names to models derived from $\mathcal{N}$.

(i) As a consequence of [9, Corollary 4.1.13], since there is an elementary embedding, we have that $\mathcal{M}_{\text{FO}}, g_{\bar{w}} \equiv_{\text{FO}} \mathcal{M}^+_{\text{FO}}, g^+_{\bar{w}}$. The same argument works with $\mathcal{N}_{\text{FO}}$ and $\mathcal{N}^+_{\text{FO}}$.

[4] An alternative, as done in [16], would be to define similarity for FO-models. We could then say that a first-order formula $\alpha(\bar{x})$ is invariant for $\mathfrak{L}$-similarity if, for every two FO-models $\mathcal{M}, \bar{w}$ and $\mathcal{N}, \bar{v}$ such that $\mathcal{M}, \bar{w} \rightrightarrows_{\text{FO}} \mathcal{N}, \bar{v}$, $\alpha(\bar{x})$ true in $\mathcal{M}, \bar{w}$, implies it is also true in $\mathcal{N}, \bar{v}$.

$$
\begin{tikzcd}
 & \mathcal{N}_{\mathrm{FO}}, g_{\bar{v}} \arrow[r, leftrightarrow, "\equiv_{\mathrm{FO}}"] & \mathcal{N}^{+}_{\mathrm{FO}}, g^{+}_{\bar{v}} \arrow[dr, dotted, leftrightarrow, "\mathsf{Tm}"] & \\
\mathcal{N}, \bar{v} \arrow[ur, dotted, leftrightarrow, "\mathsf{Tm}"] \arrow[rrr, leftrightarrow, "\equiv_{\mathfrak{L}}"] \arrow[dd, "\Rrightarrow_{\mathfrak{L}}"'] & & & \mathcal{N}^{*}, \bar{v}^{*} \arrow[dd, "\rightrightarrows_{\mathfrak{L}}"] \\
 & \mathcal{M}_{\mathrm{FO}}, g_{\bar{w}} \arrow[r, leftrightarrow, "\equiv_{\mathrm{FO}}"] & \mathcal{M}^{+}_{\mathrm{FO}}, g^{+}_{\bar{w}} \arrow[dr, dotted, leftrightarrow, "\mathsf{Tm}"] & \\
\mathcal{M}, \bar{w} \arrow[ur, dotted, leftrightarrow, "\mathsf{Tm}"] \arrow[rrr, leftrightarrow, "\equiv_{\mathfrak{L}}"] & & & \mathcal{M}^{*}, \bar{w}^{*}
\end{tikzcd}
$$

Figure 1: Directions for the detour

(ii) Thanks to the truth-preserving translations, $\mathcal{M}, \bar{w} \equiv_{\mathfrak{L}} \mathcal{M}^*, \bar{w}^*$ and similarly for $\mathcal{N}, \bar{v}$ and $\mathcal{N}^*, \bar{v}^*$. Hence, $\mathcal{N}^*, \bar{v}^* \Rrightarrow_{\mathfrak{L}} \mathcal{M}^*, \bar{w}^*$.

(iii) As both $\mathcal{M}^{+}_{\mathrm{FO}}, g^{+}_{\bar{w}}$ and $\mathcal{N}^{+}_{\mathrm{FO}}, g^{+}_{\bar{v}}$ are ω-saturated, and $\mathcal{N}^*, \bar{v}^* \Rrightarrow_{\mathfrak{L}} \mathcal{M}^*, \bar{w}^*$, by adequacy of $\mathfrak{L}$-similarity we conclude that $\mathcal{N}^*, \bar{v}^* \rightrightarrows_{\mathfrak{L}} \mathcal{M}^*, \bar{w}^*$. □

□

Corollary 10 (Detour). *Let $\alpha(\bar{x}) \in \textsc{form}(\mathrm{FO})$ be invariant for $\mathfrak{L}$-similarity. If $\mathcal{N}, \bar{v} \Rrightarrow_{\mathfrak{L}} \mathcal{M}, \bar{w}$ and $\mathsf{Tm}(\mathcal{N}, \bar{v}) \models \alpha(\bar{x})$ then $\mathsf{Tm}(\mathcal{M}, \bar{w}) \models \alpha(\bar{x})$.*

Proof. Let the models and notation be as in Proposition 9 (with $\mathsf{M}_1 = \mathsf{M}_2 = \textsc{pmods}(\mathfrak{L})$). Figure 1 helps illustrate the situation along with the relationship among the various models. Think of it as a cube, the front face represents the $\mathfrak{L}$-models and the back face has the FO-models.

We have to prove that $\mathcal{N}, \bar{v} \Rrightarrow_{\mathfrak{L}} \mathcal{M}, \bar{w}$ and $\mathcal{N}_{\mathrm{FO}}, g_{\bar{v}} \models \alpha(\bar{x})$ imply $\mathcal{M}_{\mathrm{FO}}, g_{\bar{w}} \models \alpha(\bar{x})$. As $\mathcal{N}_{\mathrm{FO}}, g_{\bar{v}} \models \alpha(\bar{x})$ and $\mathcal{N}^{+}_{\mathrm{FO}}, g^{+}_{\bar{v}}$ is elementarily equivalent to $\mathcal{N}_{\mathrm{FO}}, g_{\bar{v}}$, then $\mathcal{N}^{+}_{\mathrm{FO}}, g^{+}_{\bar{v}} \models \alpha(\bar{x})$. Because $\alpha(\bar{x})$ is invariant under $\mathfrak{L}$-similarity and $\mathcal{N}^*, \bar{v}^* \rightrightarrows_{\mathfrak{L}} \mathcal{M}^*, \bar{w}^*$ we know that $\mathcal{M}^{+}_{\mathrm{FO}}, g^{+}_{\bar{w}} \models \alpha(\bar{x})$. Again by elementary equivalence we conclude that $\mathcal{M}_{\mathrm{FO}}, g_{\bar{w}} \models \alpha(\bar{x})$. □ □

We are ready to state the characterization theorem. Given a set $\Gamma(\bar{x}) \cup \{\varphi(\bar{x})\}$ of FO-formulas we use $\Gamma(\bar{x}) \models_{\mathsf{K}} \varphi(\bar{x})$ to mean that the entailment holds if we only consider models of K. That is, for all $\langle \mathcal{G}, g \rangle \in \mathsf{K}$, if $\mathcal{G}, g \models \Gamma(\bar{x})$ then $\mathcal{G}, g \models \varphi(\bar{x})$. We say that $\varphi(\bar{x}), \psi(\bar{x})$ are K-equivalent if $\models_{\mathsf{K}} \varphi(\bar{x}) \leftrightarrow \psi(\bar{x})$.

Theorem 11 (Characterization). *A formula $\alpha(\bar{x}) \in \textsc{form}(\mathrm{FO})$ is K-equivalent to the translation of an $\mathfrak{L}$-formula iff $\alpha(\bar{x})$ is invariant for $\mathfrak{L}$-similarity.*

Proof. The left to right implication is a consequence of invariance of $\mathfrak{L}$-formulas for $\mathfrak{L}$-similarity. For the other implication, suppose that $\alpha(\bar{x})$ is invariant for $\mathfrak{L}$-similarity. We need to prove that it is K-equivalent to the translation of an $\mathfrak{L}$-formula. Consider the set of local $\mathfrak{L}$-consequences of α:

$$\mathrm{SLC}(\alpha) = \{\mathsf{Tf}(\varphi) : \varphi \text{ is an } \mathfrak{L}\text{-formula and } \alpha(\bar{x}) \models_{\mathsf{K}} \mathsf{Tf}(\varphi)\}.$$

The following claim shows that it suffices to prove that $\mathrm{SLC}(\alpha) \models_{\mathsf{K}} \alpha(\bar{x})$.

Claim 1. If $\mathrm{SLC}(\alpha) \models_{\mathsf{K}} \alpha(\bar{x})$ then $\alpha(\bar{x})$ is K-equivalent to the translation of an $\mathfrak{L}$-formula.

Proof. Suppose $\mathrm{SLC}(\alpha) \models_{\mathsf{K}} \alpha(\bar{x})$. By compactness (Proposition 8) there is a finite set $\Delta(\bar{x}) \subseteq \mathrm{SLC}(\alpha)$ such that $\Delta(\bar{x}) \models_{\mathsf{K}} \alpha(\bar{x})$, therefore we have $\models_{\mathsf{K}} \bigwedge \Delta(\bar{x}) \to \alpha(\bar{x})$. By definition $\models_{\mathsf{K}} \alpha(\bar{x}) \to \bigwedge \Delta(\bar{x})$, so we conclude $\models_{\mathsf{K}} \alpha(\bar{x}) \leftrightarrow \bigwedge \Delta(\bar{x})$. As every $\beta(\bar{x}) \in \Delta(\bar{x})$ is the translation of an $\mathfrak{L}$-formula and the formula translation preserves conjunctions (cf. Definition 4) then $\bigwedge \Delta(\bar{x})$ is also the translation of some modal formula. $\dashv$ $\square$

To prove $\mathrm{SLC}(\alpha) \models_{\mathsf{K}} \alpha(\bar{x})$, assume $\mathsf{Tm}(\mathcal{M}, \bar{w}) \models \mathrm{SLC}(\alpha)$. We show that $\mathsf{Tm}(\mathcal{M}, \bar{w}) \models \alpha(\bar{x})$. Define the negative theory of $\bar{w}$ as

$$\mathsf{NTh}^{\bar{w}}(\bar{x}) = \{\neg \mathsf{Tf}(\varphi) : \varphi \text{ is an } \mathfrak{L}\text{-formula and } \mathcal{M}, \bar{w} \not\Vdash \varphi\}.$$

Observe that if $\mathfrak{L}$ has negation then $\mathsf{NTh}^{\bar{w}}(\bar{x})$ will be the translation of the modal theory of $\bar{w}$, and every model of $\mathsf{NTh}^{\bar{w}}(\bar{x})$ will be modally equivalent to $\mathcal{M}, \bar{w}$. If $\mathfrak{L}$ does not have negation we will only preserve formulas that are false in $\mathcal{M}, \bar{w}$. The above definition works in both cases.

Let $\Sigma(\bar{x}) = \{\alpha(\bar{x})\} \cup \mathsf{NTh}^{\bar{w}}(\bar{x})$. We will show that

Claim 2. $\Sigma(\bar{x})$ has a model in K.

Proof. Suppose there is no model in K for $\Sigma(\bar{x})$ and use the contrapositive of Proposition 8. Then there is a finite set $\Delta(\bar{x}) \subseteq \Sigma(\bar{x})$ with no model in K. Notice that $\Delta(\bar{x})$ should be of the form $\{\alpha(\bar{x}), \neg\delta_1(\bar{x}), \ldots, \neg\delta_n(\bar{x})\}$ with $\neg\delta_i \in \mathsf{NTh}^{\bar{w}}(\bar{x})$, otherwise it would have a model, namely $\mathsf{Tm}(\mathcal{M}, \bar{w})$.

This means that for every model $\langle \mathcal{A}_{\mathrm{FO}}, g\rangle \in \mathsf{K}$ we have $\langle \mathcal{A}_{\mathrm{FO}}, g\rangle \not\models \Delta(\bar{x})$ and therefore $\langle \mathcal{A}_{\mathrm{FO}}, g\rangle \models \alpha(\bar{x}) \to \neg \bigwedge_i \neg\delta_i$. We can conclude that $\alpha(\bar{x}) \to \bigvee_i \delta_i$ is valid in K, therefore $\alpha(\bar{x}) \models_{\mathsf{K}} \bigvee_i \delta_i$. If $\bigvee_i \delta_i$ is a K-consequence of $\alpha(\bar{x})$ then, as the formula translation preserves disjunction (cf. Definition 4) we have $\bigvee_i \delta_i \in \mathrm{SLC}(\alpha)$. But, as $\mathsf{Tm}(\mathcal{M}, \bar{w}) \models \mathrm{SLC}(\alpha)$ then $\mathsf{Tm}(\mathcal{M}, \bar{w}) \models \bigvee_i \delta_i$. This is a contradiction, since $\mathsf{Tm}(\mathcal{M}, \bar{w}) \not\models \delta_i(\bar{x})$ for every i. $\dashv$ $\square$

As $\Sigma(\bar{x})$ is satisfiable in K we have a model $\langle \mathcal{N}, \bar{v}\rangle \in \textsc{pmods}(\mathfrak{L})$ such that $\mathsf{Tm}(\mathcal{N}, \bar{v}) \models \Sigma(\bar{x})$. We make the following claim.

Claim 3. $\mathcal{N}, \bar{v} \Rrightarrow_{\mathfrak{L}} \mathcal{M}, \bar{w}$.

Proof. Take the contrapositive. Suppose that $\mathcal{M}, \bar{w} \not\Vdash \varphi$ then $\neg\mathsf{Tf}(\varphi) \in \mathsf{NTh}^{\bar{w}}(\bar{x})$ and because $\mathsf{NTh}^{\bar{w}}(\bar{x}) \subseteq \Sigma(\bar{x})$ we have $\mathsf{Tm}(\mathcal{N}, \bar{v}) \models \neg\mathsf{Tf}(\varphi)$ which implies that $\mathsf{Tm}(\mathcal{N}, \bar{v}) \not\models \mathsf{Tf}(\varphi)$. By truth-preservation of the translations we get $\mathcal{N}, \bar{v} \not\Vdash \varphi$. $\dashv$ $\square$

Now we link $\mathsf{Tm}(\mathcal{N}, \bar{v})$ and $\mathsf{Tm}(\mathcal{M}, \bar{w})$ using the Detour (Corollary 10) which lets us transfer validity of $\alpha(\bar{x})$ from the first model to the second. Because $\alpha(\bar{x}) \in \Sigma(\bar{x})$ and $\mathsf{Tm}(\mathcal{N}, \bar{v}) \models \Sigma(\bar{x})$, applying Corollary 10 to $\mathcal{M}, \bar{w}$ and $\mathcal{N}, \bar{v}$ yields $\mathsf{Tm}(\mathcal{M}, \bar{w}) \models \alpha(\bar{x})$. $\square$ $\square$

We have proved that an FO-formula $\alpha(\bar{x})$ is K-equivalent to the translation of an $\mathfrak{L}$-formula if and only if $\alpha(\bar{x})$ is invariant for $\mathfrak{L}$-similarity. In the proof we show that $\alpha(\bar{x})$ was equivalent to the translation of the modal consequences of $\alpha(\bar{x})$. This was accomplished by taking a detour through the class of ω-saturated first-order models (Corollary 10). The use of ω-saturated models has been isolated in Proposition 9. The requirements imposed by our framework (the different *adequacy* conditions) were used in the following steps: closure under ultraproducts was used for compactness in Proposition 8 and the Hennessy-Milner property of ω-saturated models was critically used in Proposition 9. In the proof of Theorem 11 we also used that $\mathfrak{L}$ has conjunctions and disjunctions which are preserved during their translation to first-order logic. Notice, though, that $\mathfrak{L}$ does not need to have negation.

4 Definability

Definability theorems address the question of which properties of models are definable by means of formulas of a given logic. We begin with definability by a set of $\mathfrak{L}$-formulas.

Theorem 12 (Definability by a set)**.** *A class* M *of pointed* $\mathfrak{L}$*-models is definable by a set of* $\mathfrak{L}$*-formulas iff* M *is closed under* $\mathfrak{L}$*-similarity,* $\mathsf{Tm}(\mathsf{M})$ *is closed under ultraproducts and* $\mathsf{Tm}(\overline{\mathsf{M}})$ *is closed under ultrapowers.*

Proof. From left to right, suppose that M is defined by the set Γ of $\mathfrak{L}$-formulas and there is a model $\langle \mathcal{M}, \bar{w}\rangle \in \mathsf{M}$ such that $\mathcal{M}, \bar{w} \sqsupseteq \mathcal{N}, \bar{v}$ for some pointed model $\mathcal{N}, \bar{v}$. Since $\langle \mathcal{M}, \bar{w}\rangle \in \mathsf{M}$, we have $\mathcal{M}, \bar{w} \models \Gamma$. By definition of $\mathfrak{L}$-similarity we have $\mathcal{N}, \bar{v} \models \Gamma$ and therefore $\langle \mathcal{N}, \bar{v}\rangle \in \mathsf{M}$. Hence M is closed under $\mathfrak{L}$-similarity.

It is easy to verify that for every pointed FO-model $\langle \mathcal{G}, g\rangle \in \mathsf{K}$,

$$\langle \mathcal{G}, g\rangle \in \mathsf{Tm}(\mathsf{M}) \qquad \text{iff} \qquad \mathcal{G}, g \models \mathsf{Tf}(\Gamma). \tag{1}$$

To verify that $\mathsf{Tm}(\mathsf{M})$ is closed under ultraproducts, take models $\langle \mathcal{G}_i, g_i\rangle \in \mathsf{Tm}(\mathsf{M})$. Given that for all i, $\mathcal{G}_i, g_i \models \mathsf{Tf}(\Gamma)$, by the fundamental theorem of ultraproducts [9, Theorem 4.1.9], $\prod_D \mathcal{G}_i, g_i \models \mathsf{Tf}(\Gamma)$. Since K is closed under ultraproducts, $\prod_D \mathcal{G}_i, g_i \in \mathsf{K}$ and from (1) we conclude that $\prod_D \mathcal{G}_i, g_i \in \mathsf{Tm}(\mathsf{M})$.

We now verify that $\mathsf{Tm}(\overline{\mathsf{M}})$ is closed under ultrapowers. Take $\langle \mathcal{G}, g\rangle \in \mathsf{Tm}(\overline{\mathsf{M}})$. By (1), $\mathcal{G}, g \not\models \mathsf{Tf}(\Gamma)$. Let $\prod_D \mathcal{G}, g$ be an ultrapower of $\langle \mathcal{G}, g\rangle$ where D is an ultrafilter. Again by the fundamental theorem of ultrapowers (cf. [9, Corollary 4.1.10]), $\prod_D \mathcal{G}, g$ is elementarily equivalent to $\langle \mathcal{G}, g\rangle$. Hence $\prod_D \mathcal{G}, g \not\models \mathsf{Tf}(\Gamma)$. Since $\prod_D \mathcal{G}, g \in \mathsf{K}$, we conclude by (1) that $\prod_D \mathcal{G}, g \in \mathsf{Tm}(\overline{\mathsf{M}})$.

For the right to left direction proceed as follows. Let $\Gamma = \mathsf{Th}(\mathsf{M})$, the set of all $\mathfrak{L}$-formulas which are valid in M. It remains to show that if $\mathcal{M}, \bar{w} \Vdash \Gamma$ then $\langle \mathcal{M}, \bar{w}\rangle \in \mathsf{M}$. Let $\mathcal{M}, \bar{w}$ be a pointed $\mathfrak{L}$-model such that $\mathcal{M}, \bar{w} \Vdash \Gamma$. Define

$$\mathsf{NTh}^{\bar{w}}(\bar{x}) = \{\neg \mathsf{Tf}(\varphi) : \varphi \text{ is an } \mathfrak{L}\text{-formula and } \mathcal{M}, \bar{w} \nVdash \varphi\}.$$

We show that $\mathsf{NTh}^{\bar{w}}(\bar{x})$ is satisfiable in $\mathsf{Tm}(\mathsf{M})$. For contradiction, suppose this is not the case. By Proposition 8, there is a finite $\Delta(\bar{x}) \subseteq \mathsf{NTh}^{w}(\bar{x})$ not satisfiable in $\mathsf{Tm}(\mathsf{M})$. Let $\Delta(\bar{x}) = \{\neg\mathsf{Tf}(\varphi_1), \ldots, \neg\mathsf{Tf}(\varphi_n)\}$ for $\mathfrak{L}$-formulas φ_i such that $\mathcal{M}, \bar{w} \nVdash \varphi_i$ for $i = 1, \ldots, n$. Then $\bigwedge_i \neg\mathsf{Tf}(\varphi_i)$ is unsatisfiable in $\mathsf{Tm}(\mathsf{M})$ and so $\bigvee_i \mathsf{Tf}(\varphi_i)$ is valid in $\mathsf{Tm}(\mathsf{M})$. By the properties of our formula translation, $\bigvee_i \mathsf{Tf}(\varphi_i) = \mathsf{Tf}(\bigvee_i \varphi_i)$, and then $\mathsf{Tf}(\bigvee_i \varphi_i)$ is valid in $\mathsf{Tm}(\mathsf{M})$. Therefore $\bigvee_i \varphi_i$ is valid in M and hence $\bigvee_i \varphi_i \in \Gamma$. This is a contradiction, because $\mathcal{M}, \bar{w} \Vdash \Gamma$, and so $\mathcal{M}, \bar{w} \Vdash \bigvee_i \varphi_i$, but $\mathcal{M}, \bar{w} \nVdash \varphi_i$ for $i = 1, \ldots, n$.

We conclude that there is a model $\langle \mathcal{N}, \bar{v}\rangle \in \mathsf{M}$ with $\mathsf{Tm}(\mathcal{N}, \bar{v}) \models \mathsf{NTh}^{\bar{w}}(x)$. Observe that $\mathcal{N}, \bar{v} \Rrightarrow \mathcal{M}, \bar{w}$. By Proposition 9 (with $\mathsf{M}_1 = \overline{\mathsf{M}}, \mathsf{M}_2 = \mathsf{M}$) there exist ω-saturated extensions $\langle \mathcal{N}^*, \bar{v}^*\rangle \in \mathsf{M}$ and $\langle \mathcal{M}^*, \bar{w}^*\rangle \in \overline{\mathsf{M}}$ such that $\mathcal{N}^*, \bar{v}^* \sqsupseteq \mathcal{M}^*, \bar{w}^*$. As M is closed under $\mathfrak{L}$-similarity, $\langle \mathcal{M}, \bar{w}\rangle \in \mathsf{M}$. $\square$ $\square$

The above result gives necessary and sufficient conditions for a class of $\mathfrak{L}$-models to be definable by a set of $\mathfrak{L}$-formulas. Most of the work is done on the first-order side and is therefore detached from $\mathfrak{L}$. In the last part of the theorem we use Proposition 9 which connects both logics through the class of ω-saturated models. This gives us another hint that this property isolates the very core of characterization, definability and separation results.

Our second result considers classes of models definable by a *single* formula. To prove the result we first need the following lemmas and definitions.

Definition 13 (C-closure, C-elementary class). *Let* $\langle \mathcal{M}, \bar{w}\rangle$ *and* $\langle \mathcal{N}, \bar{v}\rangle \in$ PMODS(FO) *we write* $\mathcal{M}, \bar{w} \cong_p \mathcal{N}, \bar{v}$ *to mean that there exists a partial isomorphism* Z *between* $\mathcal{M}$ *and* $\mathcal{N}$ *such that* $\bar{w} Z \bar{v}$. *Let* $\mathsf{L} \subseteq \mathsf{C} \subseteq$ PMODS(FO),

(i) L *is* C-closed under partial isomorphisms *if for all* $\langle \mathcal{M}, \bar{w}\rangle \in \mathsf{L}$ *and* $\langle \mathcal{N}, \bar{v}\rangle \in \mathsf{C}$ *such that* $\mathcal{M}, \bar{w} \cong_p \mathcal{N}, \bar{v}$ *we have that* $\langle \mathcal{N}, \bar{w}\rangle \in \mathsf{L}$.

(ii) L *is* C-elementary *if there exists a set of first-order formulas* $\Gamma(\bar{x})$ *such that for all* $\langle \mathcal{G}, g\rangle \in \mathsf{C}$ *we have that* $\mathcal{G}, g \models \Gamma(\bar{x})$ *iff* $\langle \mathcal{G}, g\rangle \in \mathsf{L}$.

(iii) L *is* basic C-elementary *if there exists a first-order formula* $\varphi(\bar{x})$ *such that for all* $\langle \mathcal{G}, g\rangle \in \mathsf{C}$ *we have that* $\mathcal{G}, g \models \varphi(\bar{x})$ *iff* $\langle \mathcal{G}, g\rangle \in \mathsf{L}$.

Lemma 14. *Let* $\mathsf{M} \subseteq$ PMODS($\mathfrak{L}$). *If* M *is closed under* $\mathfrak{L}$*-similarity and both* $\mathsf{Tm}(\mathsf{M})$ *and* $\mathsf{Tm}(\overline{\mathsf{M}})$ *are closed under ultrapowers then* $\mathsf{Tm}(\mathsf{M})$ *and* $\mathsf{Tm}(\overline{\mathsf{M}})$ *are* K*-closed under partial isomorphisms.*

Proof. Suppose that $\mathsf{Tm}(\mathsf{M})$ is not K-closed under partial isomorphisms. This means that there exist first-order models $\langle \mathcal{G}, g\rangle \in \mathsf{Tm}(\mathsf{M})$ and $\langle \mathcal{H}, \bar{h}\rangle \in \mathsf{Tm}(\overline{\mathsf{M}})$ such that $\mathcal{G}, g \cong_p \mathcal{H}, \bar{h}$. Let $\mathcal{M}, \bar{w}$ and $\mathcal{N}, \bar{v}$ in PMODS($\mathfrak{L}$) be such that $\mathsf{Tm}(\mathcal{M}, \bar{w}) = \langle \mathcal{G}, g\rangle$ and $\mathsf{Tm}(\mathcal{N}, \bar{v}) = \langle \mathcal{H}, \bar{h}\rangle$. We have $\langle \mathcal{M}, \bar{w}\rangle \in \mathsf{M}$ and $\langle \mathcal{N}, \bar{v}\rangle \notin \mathsf{M}$.

As $\mathcal{G}, g \cong_p \mathcal{H}, \bar{h}$ we know (cf. [9, Proposition 2.4.4]) that $\mathcal{G}, g \models \varphi(\bar{x})$ if and only if $\mathcal{H}, \bar{h} \models \varphi(\bar{x})$. In particular they have the same $\mathfrak{L}$-theory, i.e., $\mathcal{M}, \bar{w} \equiv \mathcal{N}, \bar{v}$. As this implies that $\mathcal{M}, \bar{w} \Rrightarrow \mathcal{N}, \bar{v}$ we can use Proposition 9 and find models $\langle \mathcal{M}^*, \bar{w}^*\rangle \in \mathsf{M}$ and $\langle \mathcal{N}^*, \bar{v}^*\rangle \in \overline{\mathsf{M}}$ such that $\mathcal{M}^*, \bar{w}^* \rightrightarrows \mathcal{N}^*, \bar{v}^*$. But as M is closed under similarity we conclude that $\langle \mathcal{N}^*, \bar{v}^*\rangle \in \mathsf{M}$, a contradiction.

We prove now that $\mathsf{Tm}(\overline{\mathsf{M}})$ is K-closed under partial isomorphisms. Assume, for contradiction, that there exist $\langle \mathcal{G}, g\rangle \in \mathsf{Tm}(\overline{\mathsf{M}})$ and $\langle \mathcal{H}, \bar{h}\rangle \in \mathsf{K} \setminus \mathsf{Tm}(\overline{\mathsf{M}})$ such that $\mathcal{G}, g \cong_p \mathcal{H}, \bar{h}$. As $\langle \mathcal{H}, \bar{h}\rangle \in \mathsf{K} \setminus \mathsf{Tm}(\overline{\mathsf{M}})$ this means that $\langle \mathcal{H}, \bar{h}\rangle \in \mathsf{Tm}(\mathsf{M})$. We have just proved that $\mathsf{Tm}(\mathsf{M})$ is K-closed under partial isomorphism then, as $\mathcal{G}, g \cong_p \mathcal{H}, \bar{h}$ (and because of the symmetry of the partial isomorphism relation), we conclude that $\langle \mathcal{G}, g\rangle \in \mathsf{Tm}(\mathsf{M})$. □ □

Lemma 15 (First-order relativized definability). *Let* C *be a class of pointed* FO*-models which is closed under ultraproducts and let* $\mathsf{L} \subseteq \mathsf{C}$.

(i) L *is a* C*-elementary class iff* L *is closed under ultraproducts,* L *is* C*-closed under partial isomorphisms and* $\overline{\mathsf{L}} \cap \mathsf{C}$ *is closed under ultrapowers.*

(ii) L *is a basic* C*-elementary class iff both* L *and* $\overline{\mathsf{L}} \cap \mathsf{C}$ *are closed under ultraproducts and* C*-closed under partial isomorphisms.*

Proof. Left to right directions are simple. For the right-to-left directions proceed as follows.

(i) Let $\Gamma(\bar{x}) = \{\varphi(\bar{x}) : \models_{\mathsf{L}} \varphi(\bar{x})\}$. We show that $\Gamma(\bar{x})$ defines L. Obviously, for $\langle \mathcal{G}, g\rangle \in \mathsf{L}$, $\mathcal{G}, g \models \Gamma(\bar{x})$. Now let $\langle \mathcal{G}, g\rangle \in \mathsf{C}$ be such that $\mathcal{G}, g \models \Gamma(\bar{x})$. Define

$$\Sigma(\bar{x}) = \{\varphi(\bar{x}) : \mathcal{G}, g \models \varphi(\bar{x})\}.$$

We prove that $\Sigma(\bar{x})$ is satisfiable in L. Suppose not, by Proposition 8 there is a finite subset $\Delta_0(\bar{x}) = \{\varphi_1(\bar{x}), \ldots, \varphi_n(\bar{x})\}$ of $\Sigma(\bar{x})$ which is unsatisfiable in L. Hence, $\models_{\mathsf{L}} \neg \bigwedge_i \varphi_i(\bar{x})$ which means that $\neg \bigwedge_i \varphi_i(\bar{x}) \in \Gamma(\bar{x})$. As $\mathcal{G}, g \models \Gamma(\bar{x})$ we arrive to a contradiction.

By [9, Theorem 6.1.15], $\mathcal{G}, g \equiv_{\mathrm{FO}} \mathcal{H}, h$ if and only if there exist ultrapowers $\mathcal{G}_*, g_*$ and $\mathcal{H}_*, h_*$ such that $\mathcal{G}_*, g_* \cong_p \mathcal{H}_*, h_*$. Because L is closed under ultraproducts, in particular it is closed under ultrapowers, therefore, $\langle \mathcal{H}_*, h_*\rangle \in \mathsf{L}$. As L and C are closed under ultrapowers, $\mathcal{G}, g$ and $\mathcal{G}_*, g_*$ belong to the same class. As L is C-closed under partial isomorphisms and $\mathcal{G}_*, g_* \cong_p \mathcal{H}_*, h_*$ then $\langle \mathcal{G}_*, g_*\rangle \in \mathsf{L}$. Hence $\langle \mathcal{G}, g\rangle \in \mathsf{L}$.

(ii) By (i) we know there are sets $\Gamma(\bar{x}), \Gamma_c(\bar{x})$ defining L and $\overline{\mathsf{L}} \cap \mathsf{C}$ respectively. Observe that the union $\Gamma(\bar{x}) \cup \Gamma_c(\bar{x})$ is not satisfiable in C. By Proposition 8 there exists a finite subset $\Sigma_0(\bar{x}) \subseteq \Gamma(\bar{x}) \cup \Gamma_c(\bar{x})$ which is unsatisfiable in C. Call $\Sigma_0(\bar{x}) = \{\alpha_1(\bar{x}), \ldots, \alpha_n(\bar{x}), \beta_1(\bar{x}), \ldots, \beta_m(\bar{x})\}$ with $\alpha_i(\bar{x}) \in \Gamma(\bar{x})$ and $\beta_j(\bar{x}) \in \Gamma_c(\bar{x})$. As $\Sigma_0(\bar{x})$ is unsatisfiable in C we have $\models_{\mathsf{C}} \bigwedge_i \alpha_i(\bar{x}) \to \neg \bigwedge_i \beta_i(\bar{x})$. Let us see that $\varphi(\bar{x}) = \bigwedge_i \alpha_i(\bar{x})$ defines L.

Let $\langle \mathcal{G}, g\rangle \in \mathsf{C}$. If $\langle \mathcal{G}, g\rangle \in \mathsf{L}$ then $\mathcal{G}, g \models \varphi(\bar{x})$. Suppose $\mathcal{G}, g \models \varphi(\bar{x})$ then $\mathcal{G}, g \not\models \bigwedge_i \beta_i(\bar{x})$ therefore $\mathcal{G}, g \not\models \Gamma_c(\bar{x})$ and $\langle \mathcal{G}, g\rangle \notin \overline{\mathsf{L}} \cap \mathsf{C}$. Hence $\langle \mathcal{G}, g\rangle \in \mathsf{L}$. □ □

Lemma 16. *Let* $\mathsf{M} \subseteq \textsc{pmods}(\mathfrak{L})$*. If* M *is closed under* $\mathfrak{L}$*-similarity and both* $\mathsf{Tm}(\mathsf{M})$ *and* $\mathsf{Tm}(\overline{\mathsf{M}})$ *are closed under ultraproducts then there exists a first-order formula* $\alpha(\bar{x})$ *such that for all* $\langle \mathcal{G}, g\rangle \in \mathsf{K}$ *we have* $\mathcal{G}, g \models \alpha(\bar{x})$ *iff* $\langle \mathcal{G}, g\rangle \in \mathsf{Tm}(\mathsf{M})$*.*

Proof. The proof is a corollary of Lemmas 14 and 15. □ □

Theorem 17 (Definability by a formula)**.** *A class* M *of pointed* $\mathfrak{L}$*-models is definable by a single* $\mathfrak{L}$*-formula iff* M *is closed under* $\mathfrak{L}$*-similarity and both* $\mathsf{Tm}(\mathsf{M})$ *and* $\mathsf{Tm}(\overline{\mathsf{M}})$ *are closed under ultraproducts.*

Proof. From left to right, suppose M is definable by a single $\mathfrak{L}$-formula φ. Using Theorem 12 we conclude that M is closed under $\mathfrak{L}$-similarity and $\mathsf{Tm}(\mathsf{M})$ is closed under ultraproducts.

We prove that $\mathsf{Tm}(\overline{\mathsf{M}})$ is closed under ultraproducts. Observe that $\mathsf{Tm}(\overline{\mathsf{M}}) = \{\langle \mathcal{G}, \bar{g}\rangle : \mathcal{G}, g \models \neg\mathsf{Tf}(\varphi)\} \cap \mathsf{K}$. Now, both intersecting classes are closed under ultraproducts (the former because is definable by a single first-order formula, the latter by hypothesis). Hence $\mathsf{Tm}(\overline{\mathsf{M}})$ is closed under ultraproducts.

For the right to left direction, given that M is closed under $\mathfrak{L}$-similarity and both $\mathsf{Tm}(\mathsf{M})$ and $\mathsf{Tm}(\overline{\mathsf{M}})$ are closed under ultraproducts, by Lemma 16, there is a first-order formula $\alpha(\bar{x})$ such that for every $\langle \mathcal{G}, g\rangle \in \mathsf{K}$ we have that $\mathcal{G}, g \models \alpha(\bar{x})$ iff $\langle \mathcal{G}, g\rangle \in \mathsf{Tm}(\mathsf{M})$. As M is closed under $\mathfrak{L}$-similarity, $\alpha(\bar{x})$ is invariant for $\mathfrak{L}$-similarity. By Theorem 11 $\alpha(x)$ is K-equivalent to the translation of an $\mathfrak{L}$-formula φ, which defines M. □ □

We give necessary and sufficient conditions for a class of $\mathfrak{L}$-models to be definable by a single $\mathfrak{L}$-formula. The right to left direction is the most interesting where we use Lemma 16. For this step, standard proofs such as those found in [7, 24, 22] use *structural* properties of the notion of $\mathfrak{L}$-simulation, e.g., symmetry in the case of BML-bisimulation, and that $\leftrightarroweq\ \subseteq\ \rightrightarrows$

when $\mathfrak{L}$ is the negation free basic modal logic and $\leftrightarrows$ is the BML-bisimulation relation. As a corollary of Lemma 14, in our setting $\cong_p \subseteq \rightrightarrows_{\mathfrak{L}}$ for any $\mathfrak{L}$-similarity regardless of its structural definition. Using this fact, the proof goes smoothly.

5 Separation

We will now prove two separation results. As we discussed before, this kind of theorems provides conditions under which two disjoint classes of pointed models can be separated by a class definable in $\mathfrak{L}$. In what follows, remember that we have required $\mathsf{K} = \mathsf{Tm}(\text{PMODS}(\mathfrak{L}))$ to be closed under ultraproducts.

Theorem 18 (Separation by a set of formulas)**.** *Let* $\mathsf{M}, \mathsf{N} \subseteq \text{PMODS}(\mathfrak{L})$ *be such that:*

1. $\mathsf{M} \cap \mathsf{N} = \emptyset$, M *is closed under* $\mathfrak{L}$*-similarity,*
2. $\mathsf{Tm}(\mathsf{M})$ *is closed under ultraproducts, and*
3. $\mathsf{Tm}(\mathsf{N})$ *is closed under ultrapowers,*

then there is a class $\mathsf{M}' \subseteq \text{PMODS}(\mathfrak{L})$ *definable by a set of* $\mathfrak{L}$*-formulas such that* $\mathsf{M} \subseteq \mathsf{M}'$ *and* $\mathsf{M}' \cap \mathsf{N} = \emptyset$*.*

Proof. Let M' be the $\Rrightarrow$-closure of M, i.e.,

$$\mathsf{M}' = \{\langle \mathcal{M}', \bar{w}' \rangle : \text{There is } \langle \mathcal{M}, \bar{w} \rangle \in \mathsf{M} \text{ such that } \mathcal{M}, \bar{w} \Rrightarrow \mathcal{M}', \bar{w}'\}.$$

It is clear that $\mathsf{M} \subseteq \mathsf{M}'$. We show $\mathsf{M}' \cap \mathsf{N} = \emptyset$. Suppose $\langle \mathcal{N}, \bar{v} \rangle \in \mathsf{M}' \cap \mathsf{N}$, then there is $\langle \mathcal{M}, \bar{w} \rangle \in \mathsf{M}$ such that $\mathcal{M}, \bar{w} \Rrightarrow \mathcal{N}, \bar{v}$. By Proposition 9 there exist $\mathcal{N}^*, \bar{v}^* \in \mathsf{M}$ and $\mathcal{M}^*, \bar{w}^* \in \mathsf{N}$ such that $\mathcal{M}^*, \bar{w}^* \rightrightarrows \mathcal{N}^*, \bar{v}^*$. As M is closed under $\mathfrak{L}$-similarity $\langle \mathcal{N}^*, \bar{v}^* \rangle \in \mathsf{M} \cap \mathsf{N} = \emptyset$, a contradiction.

It remains to prove that M' is definable by a set of $\mathfrak{L}$-formulas. By Theorem 12 it suffices to show the following:

i. M' is closed under $\mathfrak{L}$-similarity

ii. $\mathsf{Tm}(\mathsf{M}')$ is closed under ultraproducts

iii. $\mathsf{Tm}(\overline{\mathsf{M}'})$ is closed under ultrapowers

For (i), M' is closed under $\mathfrak{L}$-similarity because $\mathfrak{L}$-similarity $\rightrightarrows$ implies $\Rrightarrow$.

For (ii), let $\{\mathcal{M}'_i, \bar{w}'_i\}_{i \in I}$ be a family of pointed models in M', and D an ultrafilter over I. Define $\langle \mathcal{G}'_*, g'_* \rangle$ as the ultraproduct of the translation of each $\mathcal{M}'_i, \bar{w}'_i$; i.e., $\mathcal{G}'_*, g'_* = \prod_D \mathsf{Tm}(\mathcal{M}'_i, \bar{w}'_i)$. By definition of M', for each $\mathcal{M}'_i, \bar{w}'_i$ there exists $\langle \mathcal{M}_i, \bar{w}_i \rangle \in \mathsf{M}$ such that $\mathcal{M}_i, \bar{w}_i \Rrightarrow \mathcal{M}'_i, \bar{w}'_i$. Now let $\mathcal{G}_*, g_* = \prod_D \mathsf{Tm}(\mathcal{M}_i, \bar{w}_i)$. As $\mathsf{Tm}(\mathsf{M})$ is closed under ultraproducts then $\langle \mathcal{G}_*, g_* \rangle \in \mathsf{Tm}(\mathsf{M})$. Since M' is closed under $\Rrightarrow$, to show that $\langle \mathcal{G}'_*, g'_* \rangle \in \mathsf{Tm}(\mathsf{M}')$ it suffices to show that $\mathcal{G}_*, g_* \Rrightarrow_{\mathfrak{L}} \mathcal{G}'_*, g'_*$. Let $\varphi \in \mathfrak{L}$, if $\mathcal{G}'_*, g'_* \not\models \mathsf{Tf}(\varphi)$ then by the fundamental theorem of ultraproducts, $\{j \in I : \mathsf{Tm}(\mathcal{M}'_j, \bar{w}'_j) \not\models \mathsf{Tf}(\varphi)\} \in D$, and therefore $\{j \in I : \mathsf{Tm}(\mathcal{M}_j, \bar{w}_j) \not\models \mathsf{Tf}(\varphi)\} \in D$. Again by the fundamental theorem of ultraproducts $\mathcal{G}_*, g_* \not\models \mathsf{Tf}(\varphi)$.

For (iii), let $\mathcal{G}'_*, g'_*$ be an ultrapower of $\mathsf{Tm}(\mathcal{M}', \bar{w}')$, for $\langle \mathcal{M}', \bar{w}' \rangle \in \overline{\mathsf{M}'}$. Notice that $\langle \mathcal{G}'_*, g'_* \rangle \in \mathsf{K}$ as K is closed under ultraproducts. Hence either *(i)* $\langle \mathcal{G}'_*, g'_* \rangle \in \mathsf{K} \cap \mathsf{Tm}(\mathsf{M}')$ or *(ii)* $\langle \mathcal{G}'_*, g'_* \rangle \in \mathsf{K} \cap \overline{\mathsf{Tm}(\mathsf{M}')} = \mathsf{Tm}(\overline{\mathsf{M}'})$. We prove that *(i)* leads to contradiction, finishing the proof:

$\mathcal{G}'_*, g'_*$ is elementarily equivalent to $\mathsf{Tm}(\mathcal{M}', \bar{w}')$. Since $\mathsf{Tm}(\mathsf{M}')$ is closed under elementary equivalence then $\mathsf{Tm}(\mathcal{M}', \bar{w}') \in \mathsf{Tm}(\mathsf{M}')$. By bijectivity of Tm, this implies $\langle \mathcal{M}', \bar{w}' \rangle \in \mathsf{M}'$. □ □

Theorem 19 (Separation by a formula)**.** *Let* $\mathsf{M}, \mathsf{N} \subseteq$ PMODS$(\mathfrak{L})$ *such that* $\mathsf{M} \cap \mathsf{N} = \emptyset$. *If* M *and* N *are closed under* $\mathfrak{L}$*-similarity and both* $\mathsf{Tm}(\mathsf{M})$ *and* $\mathsf{Tm}(\mathsf{N})$ *are closed under ultraproducts, then there exists* $\mathsf{M}' \subseteq$ PMODS$(\mathfrak{L})$ *that is definable by means of a single* $\mathfrak{L}$*-formula and such that* $\mathsf{M} \subseteq \mathsf{M}'$ *and* $\mathsf{M}' \cap \mathsf{N} = \emptyset$.

Proof. Using Theorem 18 first on M and then on N we get a class $\mathsf{M}'' \supseteq \mathsf{M}$ definable by a set Γ_1 of $\mathfrak{L}$-formulas such that $\mathsf{M}'' \cap \mathsf{N} = \emptyset$ and a class $\mathsf{N}'' \supseteq \mathsf{N}$ definable by a set Γ_2 of $\mathfrak{L}$-formulas such that $\mathsf{N}'' \cap \mathsf{M} = \emptyset$.

First observe that $\mathsf{N}'' \cap \mathsf{M}'' = \emptyset$. Indeed, if that is not the case, using $\Rrightarrow$-closure of M'' and N'', we would have that $\mathsf{N} \cap \mathsf{M} \neq \emptyset$. Hence, $\Gamma_1 \cup \Gamma_2$ is unsatisfiable and by compactness, there are $\alpha_1, \ldots, \alpha_n \in \Gamma_1$ and $\beta_1, \ldots, \beta_m \in \Gamma_2$ such that $\gamma = (\bigwedge_i \alpha_i) \wedge (\bigwedge_j \beta_j)$ is unsatisfiable. Let M' be the class defined by $\alpha = \bigwedge_i \alpha_i$. Clearly, $\mathsf{M} \subseteq \mathsf{M}'' \subseteq \mathsf{M}'$ and $\mathsf{M}' \cap \mathsf{N} = \emptyset$ because γ is unsatisfiable. □ □

6 Concrete Results

The general framework we presented can be used to give new and unifying proofs of Characterization, Definability and Separation for logics where these theorems have already been proved, e.g., hybrid logics or temporal logics. It is worth noting that it can even be used to prove results for non-classical modal logics such as monotonic neighbourhood logics [16] where models are *not* Kripke models. [16, Section 5.2] explains how to encode monotone neighbourhood structures as first-order structures (in an extended language). This process would of course fail in the case of *non-monotonic* structures.

The framework can also be used for logics whose model theory has not been fully developed so far (e.g., Memory Logics [2, 3]). In all cases, we only need to check that the requirements of the framework are met.

6.1 Basic Tense Logic

The Basic Tense Logic (BTL) [32, 33] is a modal logic whose syntax is given by the following grammar:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid F\varphi \mid P\varphi.$$

Semantics is defined in terms of Kripke models $\langle W, R, V \rangle$ (together with particular conditions over the accessibility conditions intended to capture time properties like linearity, irreflexivity, etc.). Boolean operators are defined as in BML, the modalities F and P are interpreted as follows:

$$\begin{array}{lll} \mathcal{M}, w \Vdash F\varphi & \text{iff} & \text{there is a } v \in W \text{ such that } wRv \text{ and } \mathcal{M}, v \Vdash \varphi \\ \mathcal{M}, w \Vdash P\varphi & \text{iff} & \text{there is a } v \in W \text{ such that } vRw \text{ and } \mathcal{M}, v \Vdash \varphi. \end{array}$$

In other words, the F modality is a standard $\Diamond$-modality, while P is defined as a $\Diamond$ over the inverse of the accessibility relation. Indeed, the P operator is called the *inverse* operator in description logics [6].

An alternative, but equivalent, formulation interprets BTL over models with two relations R_1, R_2 where the class of relational models is restricted to those satisfying $R_2 = R_1^{-1}$. In this case, both modalities are interpreted as $\Diamond$-modalities over R_1 and R_2 respectively. Hence, we can use the standard formula translation and the usual model translation to see these models as the class of first-order models K where $R_2 = R_1^{-1}$. As K is definable by the first-order formula $\varphi = \forall x.\forall y.R_1(x,y) \leftrightarrow R_2(y,x)$, K is closed under ultraproducts. Hence BTL has adequate syntax and semantics, and it is adequately below first-order. The last thing to check is the existence of an adequate notion of similarity. But the usual notion of bisimulation for the basic multi-modal logic will do. We can conclude Characterization, Definability and Separation theorems for BTL.

To add 'temporal' constraints to the class of models, such as linearity and transitivity notice that there is a bijection from such models to the class K of first order models with the same properties. If this class is FO-definable it will be closed under ultraproducts. Arguing as above we obtain the desired results.

6.2 Hybrid Logics

Hybrid logics augment modal logics with machinery for reasoning about constants and identity. The basic hybrid logic introduces 'nominals' which are propositional variables with the particularity of being true *at a unique* point in the model (and, hence, work as 'names' for certain elements in the domain). Special operators are then added allowing, for example, to indicate that two nominals name the same element.

We show how our framework can be used to show Characterization, Definability and Separation theorems for some hybrid logics. We will consider the basic hybrid logic $\mathcal{HL}$ which extends BML with nominals, and the logic $\mathcal{HL}(@)$ which also adds the @-operator.[5]

To define formulas of $\mathcal{HL}$ and $\mathcal{HL}(@)$, the signature of BML is extended with a set $\textsc{nom} = \{i_1, i_2, \dots\}$ of nominals, disjoint from the set $\textsc{prop}$ of propositional symbols. Formulas of $\mathcal{HL}(@)$ are then defined by the following grammar

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Diamond\varphi \mid i \mid @_i\varphi,$$

where p is a propositional symbol and $i \in \textsc{nom}$. $\mathcal{HL}$ is the fragment of $\mathcal{HL}(@)$ without occurrences of @.

A *hybrid model* is a tuple $\mathcal{M} = \langle W, R, V, G\rangle$ where $\langle W, R, V\rangle$ is a BML-model, and $G : \textsc{nom} \to W$ is an assignment for the nominals. Given a hybrid model $\mathcal{M} = \langle W, R, V, G\rangle$ and $w \in W$, we extend the semantics clauses for BML with the following rules:

$$\begin{array}{rcl} \mathcal{M}, w \Vdash i & \text{iff} & w = G(i), \text{ for } i \in \textsc{nom} \\ \mathcal{M}, w \Vdash @_i\varphi & \text{iff} & \mathcal{M}, G(i) \Vdash \varphi. \end{array}$$

The first-order correspondence language σ for $\mathcal{HL}(@)$ has countably many unary predicate symbols P_i, a binary relation symbol R, equality, and countably many constant symbols c_i. A formula translation Tf of $\mathcal{HL}(@)$ into FO that meets our requirements is given in [7]. In particular, formulas in the image of Tf have at most one free variable x_1. To translate hybrid models to first-order models do as follows. Let K be the class of all first-order σ-structures. Let $\mathcal{M} = \langle W, R, V, G\rangle$ be a hybrid model. Define the model translation

[5] Characterization, Definability and Separation theorems for these and other hybrid logics (e.g., including the $\downarrow$ binder) are investigated in [5].

$\mathsf{Tm}(\mathcal{M}, w) = \langle W, R, (P_i)_{i\in\mathbb{N}}, (c_i)_{i\in\mathbb{N}}, w\rangle$ where $P_i = V(p_i)$ for p_i a propositional symbol and $c_i = G(i)$ for i a nominal.

For $\mathcal{M}_1 = \langle M_1, R_1, V_1, G_1\rangle$ and $\mathcal{M}_2 = \langle M_2, R_2, V_2, G_2\rangle$ two hybrid models, the notion of $\mathcal{HL}(@)$-bisimulation extends the conditions of BML-bisimulation with the following constraints:

(i) *Nominal Harmony:* If mZn, then $G_1(i) = m$ iff $G_2(i) = n$ for all $i \in \text{NOM}$.

(ii) *@:* If $G_1(i) = m$ and $G_2(i) = n$ for some $i \in \text{NOM}$ then mZn.

$\mathcal{HL}$-bisimulation is defined without the @-constraint. It is easily proved (see [5]) that for $\mathfrak{L} \in \{\mathcal{HL}, \mathcal{HL}(@)\}$, if $\mathcal{M}, w \leftrightarroweq_{\mathfrak{L}} \mathcal{N}, v$ then $\mathcal{M}, w \equiv_{\mathfrak{L}} \mathcal{N}, v$. It only remains to prove that the induced notion of $\mathfrak{L}$-bisimilarity is adequate.

Theorem 20. *For $\mathfrak{L} \in \{\mathcal{HL}, \mathcal{HL}(@)\}$, $\mathfrak{L}$-bisimilarity is adequate.*

Proof. Let $\mathcal{M}, m$ and $\mathcal{N}, n$ be two pointed hybrid models such that $\mathsf{Tm}(\mathcal{M}, m)$ and $\mathsf{Tm}(\mathcal{N}, n)$ are ω-saturated and $\mathcal{M}, m \equiv_{\mathfrak{L}} \mathcal{N}, n$. It suffices to give an $\mathfrak{L}$-bisimulation between them. Consider $\sim$ on $|\mathcal{M}| \times |\mathcal{N}|$ defined as

$$w \sim v \ \text{ iff } \ \mathcal{M}, w \equiv_{\mathfrak{L}} \mathcal{N}, v.$$

Proving that $\sim$ is a BML-bisimulation is easy (see [7]). We prove the restrictions for nominals and the @ operator.

Restrictions for nominals: The proof is straightforward when $w \sim v$. $G^{\mathcal{M}}(i) = w$ if and only if $\mathcal{M}, w \Vdash i$. Hence $\mathcal{N}, v \Vdash i$ and $G^{\mathcal{N}}(i) = v$.

Restrictions for @: Let $G_1(i) = w$ and $G_2(i) = v$. As $\sim$ is non-empty there is $(a, b) \in |\mathcal{M}| \times |\mathcal{N}|$ such that $a \sim b$. Then, $\mathcal{M}, a \Vdash \varphi$ iff $\mathcal{N}, b \Vdash \varphi$ for all φ. Given an arbitrary formula ψ we can instantiate $\varphi = @_i\psi$ thus obtaining $\mathcal{M}, a \Vdash @_i\psi$ iff $\mathcal{N}, b \Vdash @_i\psi$ which by definition means that $\mathcal{M}, G_1(i) \Vdash \psi$ iff $\mathcal{N}, G_2(i) \Vdash \psi$. By hypothesis we can replace $G_1(i)$ and $G_2(i)$ and get $\mathcal{M}, w \Vdash \psi$ iff $\mathcal{N}, v \Vdash \psi$ therefore $\mathcal{M}, w \equiv_{\mathfrak{L}} \mathcal{N}, v$ and by definition $\mathcal{M}, w \sim \mathcal{N}, v$. □ □

Hence, the Characterization, Definability and Separation theorems hold for both $\mathcal{HL}$ and $\mathcal{HL}(@)$.

6.3 Memory Logics

Memory logics, introduced in [1] and further investigated in, e.g., [2, 4, 3], allow modeling dynamic behavior through explicit memory operators that *change* the structure where evaluation takes place. Memory logics extends the syntax and semantics of BML with operators that store and retrieve elements of the domain into a *memory* – a subset of the domains of the model. Different operators have been investigated: 'known' ⓚ, 'remember' ⓡφ, 'erase' ⓔφ, 'forget' ⓕφ and 'double diamond' $\langle\!\langle\ \rangle\!\rangle\varphi$. Each memory logic extends BML with at least ⓡ and ⓚ and may have any combination of the other operators.

A *memory model* is a tuple $\mathcal{M} = \langle W, R, V, S\rangle$ where $\langle W, R, V\rangle$ is a BML-model and $S \subseteq W$ is the memory of the model. Given a memory model $\mathcal{M} = \langle W, R, V, S\rangle$ and $w \in W$, the satisfaction conditions for the different memory operators are:

$$\begin{array}{rll}
\mathcal{M}, w \Vdash \text{ⓡ}\varphi & \text{iff} & \langle W, R, V, S \cup \{w\}\rangle, w \Vdash \varphi \\
\mathcal{M}, w \Vdash \text{ⓚ} & \text{iff} & w \in S \\
\mathcal{M}, w \Vdash \text{ⓕ}\varphi & \text{iff} & \langle W, R, V, S \setminus \{w\}\rangle, w \Vdash \varphi \\
\mathcal{M}, w \Vdash \text{ⓔ}\varphi & \text{iff} & \langle W, R, V, \emptyset\rangle, w \Vdash \varphi \\
\mathcal{M}, w \Vdash \langle\!\langle\ \rangle\!\rangle\varphi & \text{iff} & \exists w' \in W, wRw' \text{ and } \langle W, R, V, S \cup \{w\}\rangle, w' \Vdash \varphi.
\end{array}$$

As we can see, ⓡ stores the current point of evaluation into memory, while ⓚ verifies whether the current point of evaluation has been previously memorized. ⓕ removes the evaluation point from the memory while ⓔ completely wipes out the memory. $\langle\!\langle\ \rangle\!\rangle$ is a controlled version of ⓡ, which stores the current point of evaluation but forces a move to an accessible point.

The first-order correspondence language σ for memory logics has countably many unary predicate symbols P_i, a unary predicate symbol S, a binary relation symbol R and equality. An adequate translation from any memory logic obtained by combination of the operators described above can be defined by composition of the translation from memory logics to $\mathcal{HL}(\downarrow)$ found in [2] with the translation from $\mathcal{HL}(\downarrow)$ to FO given in [5]. Model translation is also fairly straightforward. Let K be the class of all first-order models for the signature σ. Let $\mathcal{M} = \langle W, R, V, S\rangle$ and $w \in W$, then $\mathsf{Tm}(\mathcal{M}, w) = \langle W, R, (P_i)_{i\in\mathbb{N}}, S, w\rangle$ where $P_i = V(p_i)$.

Defining bisimulations for memory logics is more involved. Because memory operators can modify the memory, it is not sufficient for a bisimulation to link evaluation points, we also need to keep track of the current memory state. Let $\mathcal{M}$ and $\mathcal{N}$ be memory models, and $Z \subseteq \mathcal{P}(|\mathcal{M}|) \times |\mathcal{M}| \times \mathcal{P}(|\mathcal{N}|) \times |\mathcal{N}|$. That is, bisimulations do not link pairs of states in the related models, but a set of states (which describes the current memory) and a state in one model, with a similar pair in the other model. A *memory bisimulation* for a memory logic $\mathfrak{L}$ can be defined imposing restrictions on Z depending on the operators that $\mathfrak{L}$ has. We summarize the restrictions associated with each operator in the next table. We write R^1 to denote a relation in $\mathcal{M}$ and R^2 is used to denote a relation in $\mathcal{N}$.

always	(nontriv)	Z is not empty.
always	(agree)	If $(A, m)Z(B, n)$, then $m \in V(p)$ iff $n \in V(p)$ for all $p \in$ PROP.
ⓚ	(kagree)	If $(A, m)Z(B, n)$, then $m \in A$ if and only if $n \in B$.
ⓡ	(remember)	If $(A, m)Z(B, n)$, then $(A \cup \{m\}, m)Z(B \cup \{n\}, n)$.
ⓕ	(forget)	If $(A, m)Z(B, n)$, then $(A \setminus \{m\}, m)Z(B \setminus \{n\}, n)$.
ⓔ	(erase)	If $(A, m)Z(B, n)$, then $(\emptyset, m)Z(\emptyset, n)$.
$\Diamond$	(forth)	If $(A, m)Z(B, n)$ and $R^1(m, m')$, then $\exists n' \in |\mathcal{N}|$ such that $R^2(n, n')$ and $(A, m')Z(B, n')$.
	(back)	If $(A, m)Z(B, n)$ and $R^2(n, n')$, then $\exists m' \in |\mathcal{M}|$ such that $R^1(m, m')$ and $(A, m')Z(B, n')$.
$\langle\!\langle\ \rangle\!\rangle$	(mforth)	If $(A, m)Z(B, n)$ and $R^1(m, m')$, then $\exists n' \in |\mathcal{N}|$ such that $R^2(n, n')$ and $(A \cup \{m\}, m')Z(B \cup \{n\}, n')$.
	(mback)	If $(A, m)Z(B, n)$ and $R^2(n, n')$, then $\exists m' \in |\mathcal{M}|$ such that $R^1(m, m')$ and $(A \cup \{m\}, m')Z(B \cup \{n\}, n')$.

Given a memory logic $\mathfrak{L}$ we will refer to the bisimulation defined by the corresponding conditions from the table above for each of the operators present in $\mathfrak{L}$ as 'the bisimulation for $\mathfrak{L}$'. It is shown in [3] that if $\mathfrak{L}$ is any of these memory logics, then $\mathfrak{L}$-bisimulation implies $\mathfrak{L}$-equivalence. That is, if $\langle\mathcal{M}, w\rangle$ and $\langle\mathcal{N}, v\rangle$ are two memory models, then $\mathcal{M}, w \leftrightarroweq_{\mathfrak{L}} \mathcal{N}, v$ implies $\mathcal{M}, w \equiv_{\mathfrak{L}} \mathcal{N}, v$.

Theorem 21. *For any a memory logic $\mathfrak{L}$, $\mathfrak{L}$-bisimilarity is adequate.*

Proof. Remember that an $\mathfrak{L}$-similarity $\rightrightarrows_{\mathfrak{L}}$ is adequate for $\mathfrak{L}$ if the class of ω-saturated models in K has the Hennessy-Milner property with respect to $\rightrightarrows_{\mathfrak{L}}$.

Let $\mathcal{M} = \langle W, R, V, S\rangle$ and $\mathcal{N} = \langle W', R', V', S'\rangle$ be two arbitrary $\mathfrak{L}$-models. Consider the pointed models $\mathcal{M}, m$ and $\mathcal{N}, n$ such that $\mathsf{Tm}(\mathcal{M}, m)$ and $\mathsf{Tm}(\mathcal{N}, n)$ are ω-saturated and $\mathcal{M}, m \equiv_{\mathfrak{L}} \mathcal{N}, n$. Again, it suffices to give an $\mathfrak{L}$-bisimulation between them. We propose the binary relation $\sim$ defined as

$$(A, w) \sim (B, v) \ \text{ iff } \ \mathcal{M}', w \equiv_{\mathfrak{L}} \mathcal{N}', v$$

as a candidate for a bisimulation where $\mathcal{M}' = \langle W, R, V, A\rangle$, $\mathcal{N}' = \langle W', R', V', B\rangle$ and $A \cup \{w\} \subseteq W$, $B \cup \{v\} \subseteq W'$. Suppose that $(A, w) \sim (B, v)$. $\sim$ satisfies *(nontriv)* and *(agree)* by definition. We now prove that the other restrictions also holds if the language contains the corresponding operator. In what follows, for $\mathcal{M} = \langle W, R, V, S\rangle$ let $\mathcal{M}[w] = \langle W, R, V, S \cup \{w\}\rangle$.

Restrictions for ⓚ $w \in A$ iff $\mathcal{M}', w \models$ ⓚ iff $\mathcal{N}', v \models$ ⓚ iff $v \in B$.

Restrictions for ⓡ $(A, w) \sim (B, v)$ implies that for every φ, $\mathcal{M}', w \models \varphi$ iff $\mathcal{N}', v \models \varphi$. In particular, $\mathcal{M}', w \models$ ⓡψ iff $\mathcal{N}', v \models$ ⓡψ, which by the definition of satisfaction, holds precisely when $\mathcal{M}'[w], w \models \psi$ iff $\mathcal{N}'[v], v \models \psi$ and hence $(A \cup \{w\}, w) \sim (B \cup \{v\}, v)$.

The restrictions for ⓕ and ⓔ are established similarly, while the restrictions for $\Diamond$ is proved as for BML (see [7, Proposition 2.54]).

Restrictions for $\langle\!\langle\ \rangle\!\rangle$ Since $(A, w) \sim (B, v)$, we have already seen in the ⓡ case that $\mathcal{M}'[w], w \models \psi$ iff $\mathcal{N}'[v], v \models \psi$.[6] This implies that $\mathcal{M}'[w], w \equiv_{\mathfrak{L}} \mathcal{N}'[v], v$.

We claim without proof (see [28]) that memorization preserves ω-saturation, i.e., if $\mathcal{M}$ is ω-saturated, then this also holds for $\mathcal{M}[w]$. Suppose now that w' is a successor of w. Let Σ be the set of all formulas true at $\mathcal{M}'[w], w'$. For every finite subset $\Delta \subseteq \Sigma$ we have $\mathcal{M}'[w], w' \models \bigwedge \Delta$, hence $\mathcal{M}'[w], w \models \langle\!\langle\ \rangle\!\rangle \bigwedge \Delta$. By $\mathfrak{L}$-equivalence we have $\mathcal{N}'[v], v \models \langle\!\langle\ \rangle\!\rangle \bigwedge \Delta$ which means that for every Δ we have a v-successor satisfying it. By preservation of ω-saturation under memorization, we can conclude that there exists v', a successor of v, so that $\mathcal{N}'[v], v' \models \Sigma$. As $\mathcal{M}'[w], w'$ and $\mathcal{N}'[v], v'$ make the same formulas true, they are $\mathfrak{L}$-equivalent and by definition they will be related by the bisimulation. This establishes *(mforth)* as $(A \cup \{w\}, w') \sim (B \cup \{v\}, v')$. The proof for *(mback)* is similar.

We have shown that in all cases $\mathfrak{L}$-simulation satisfy the required constraints. This proves that given two ω-saturated $\mathfrak{L}$-equivalent models we are able to construct an $\mathfrak{L}$-simulation between them. This suffices to show that the class of ω-saturated models has the Hennessy-Milner property with respect to the induced notion of $\mathfrak{L}$-bisimilarity. □ □

As a result of the above theorem, our framework guarantees the Characterization, Definability and Separation theorems for memory logics. Notice that Separation had not been previously investigated for this family of logics.

[6]We can use ⓡ here because it is in the language of any memory logic.

7 Related work

Other generalizations Generalizations such as the ones we investigated here have been pursued in the work of Hollenberg. In [17] he shows a relativized and extended version of van Benthem's characterization theorem for so-called *normal* first-order definable modalities. These modalities are defined by Σ_1^0-formulas and, therefore, they cannot define the *Since* and *Until* operators of temporal logics or the operators of graded modal logic. Also, these results do not apply to sub-boolean logics and only discuss characterization, not considering definability and separation. Our proofs work for any modality with a first order translation and both boolean and sub-boolean logics.

Finite models Although our framework cover many logics, it cannot be used to prove results for the class of *finite* models. This is an important class which lies beyond our reach since it is not closed under ultraproducts. Many characterization and definability theorems are known to hold in the class of finite models. For example, Rosen [34] shows that van Benthem's characterization theorem holds for the class of finite models for the basic modal logic. Otto [30] gives an elementary proof which works in both the finite and infinite cases.

Game-based techniques Our framework is not applicable to finite models mainly because it is based on a compactness argument. An alternative technique involves a game-based analysis (see [14] for a comprehensive introduction). This technique is closer to the methods used in finite model theory and applies in a very uniform way to both classes of finite and infinite models. Following these approach, Dawar and Otto [10] give several characterization theorems which cover a broad range of classes of frames and models (such as transitive, finite and rooted frames). Their results, however, are mostly geared towards BML (with bisimulation as observational equivalence) and a few extensions like BML with the global modality. Our framework is constrained to ultraproduct-closed classes of models and applies to a fairly wide class of modal logics and their corresponding notions of observational equivalence.

Coalgebraic correspondence theory Coalgebraic modal logic [31] seeks to uniformly study modal logic over different types of structures. For that purpose, the models are taken to be coalgebras for an endofunctor in the category of sets. Various kinds of structures can be seen as coalgebras, for instance: streams, Kripke models, Markov chains and neighborhood frames, among many others. Also, several kinds of modalities over these structures can be captured as appropriate *predicate liftings*. In [35] the authors prove a van Benthem-Rosen result (i.e., a characterization result for both the class of finite and infinite models) for different kinds of coalgebras. The result is very general but it only applies to rank-1 axiomatizable logics [36]. Moreover, the modalities definable in this framework are limited by the naturality constraint of predicate liftings. The generalization obtained using a coalgebraic framework is orthogonal to our approach: while their results apply to many widely different types of structures, our results apply to many different logics and similarity notions.

8 Conclusions and further work

When doing first-order model theory, partial isomorphisms and many other well-known tools are at hand. But sometimes one is satisfied with languages which are less expressive than first-order but have a better computational behavior. This is where modal logics enter the scene. If one aims to develop a model theory for some given modal logic $\mathfrak{L}$, one probably needs to devise a *reasonable* notion of simulation. What does *reasonable* mean here? How do we calibrate such a notion? It is clear that at least the following should hold:

$$\text{If } \mathcal{M}, w \rightrightarrows_{\mathfrak{L}} \mathcal{N}, v \text{ then } \mathcal{M}, w \Rrightarrow_{\mathfrak{L}} \mathcal{N}, v \tag{2}$$

but this is not enough. In the process of finding the right simulation notion, candidates are often checked against finite models, or against image finite models. In those cases, one expects to be able to prove the converse of (2). The results of this article indicate that guaranteeing the converse of (2) for the class of ω-saturated models is enough to develop a basic model theory for $\mathfrak{L}$. This key condition together with some other reasonable restrictions related to syntax, semantics and expressive power of $\mathfrak{L}$ are presented in Section 2 and define our general framework. Our main results, which are in Sections 3, 4 and 5, state that any modal logic that fits our framework has a form of Characterization, Definability and Separation. In all of them, the notion of *similarity* plays a central role.

One point to be observed is that our framework requires K, the image of the model translation, to be closed under ultraproducts and isomorphism. In particular, classes of models which are definable by a set of first-order formulas (i.e., elementary classes) satisfy this condition. However, that is not the only case. The framework can also be used with classes defined by a set of Σ^1_1-formulas [9, Corollary 4.1.14] (i.e., existential second-order logic) and, more generally, with pseudo-elementary classes [9, Exercise 4.1.17].

Notice also, that the general framework we introduced does not stipulate anything particularly *modal* in the logic under investigation (it is actually difficult to provide a good definition of what a modal logic is). As such, it would be interesting to investigate whether the approach applies to logics which are not usually considered modal such as, for example, linear logic.

Another natural extension of our framework is to include logics without *disjunction* in their language. Several description logics that do not include the disjunction operator in their language, for example, are known to satisfy preservation theorems [23]. A third possible line of work is to extend our results to model classes which are not closed under ultraproducts, such as the class of finite models. One last interesting question is to investigate the relation between Separation and Interpolation. It is well know that in some logics, separation implies a strong form of Craig Interpolation [9, 22], but interpolation *fails* for many logics covered by our framework (e.g., memory logics). A careful study of which additional conditions are required to obtain interpolation would surely be interesting.

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