Randomness and Halting Probabilities

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Abstract

We consider the question of randomness of the probability $\Omega_U[X]$ that an optimal Turing machine U halts and outputs a string in a fixed set X. The main results are as follows:

- $\Omega_U[X]$ is random whenever X is Σ^0_n-complete or Π^0_n-complete for some $n \geq 2$.
- However, for $n \geq 2$, $\Omega_U[X]$ is not n-random when X is Σ^0_n or Π^0_n. Nevertheless, there exists Δ^0_{n+1} sets such that $\Omega_U[X]$ is n-random.
- There are Δ^0_2 sets X such that $\Omega_U[X]$ is rational. Also, for every $n \geq 1$, there exists a set X which is Δ^0_{n+1} and Σ^0_n-hard such that $\Omega_U[X]$ is not random.

We also look at the range of Ω_U as an operator. We prove that the set $\{\Omega_U[X] : X \subseteq 2^{<\omega}\}$ is a finite union of closed intervals. It follows that for any optimal machine U and any sufficiently small real r, there is a set $X \subseteq 2^{<\omega}$ recursive in $\emptyset' \oplus r$, such that $\Omega_U[X] = r$.

The same questions are also considered in the context of infinite computations, and lead to similar results.

1 Introduction

The first example of a random real was Chaitin's Ω [4], which represents the probability that an optimal prefix Turing machine halts on an arbitrary input. In fact there is no single Ω, but a whole class of Ω numbers, one Ω_U for each optimal machine U. It happens that such reals are exactly the left computably enumerable (left-c.e.) random reals (Cf. [10]). Seeking for other examples of *significative* random reals, possibly not left-c.e. nor right-c.e., we consider the probability $\Omega_U[X]$ that an optimal prefix Turing machine U halts and outputs a string in a fixed non-empty set X. Grigorieff, 2002, conjectured that such reals are random when $X \neq \emptyset$, and that the harder the set X, the more random is $\Omega_U[X]$. As stated, the conjecture is false; we study related positive and negative results in §2 and §3.

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In §4 and §5 we consider the following question related to the converse of the Conjecture: given a real $r \in [0,1]$, is there some optimal machine U and a set $X \subseteq 2^{<\omega}$ such that $r = \Omega_U[X]$? And if so, what are such pairs (U, X)?

Theorem 4.2 proves that for any optimal machine U, the range $\{\Omega_U[X] : X \subseteq 2^{<\omega}\}$ is a finite union of closed intervals. It also proves that for any sufficiently small real r, there is a set $X \subseteq 2^{<\omega}$ recursive in $\emptyset' \oplus r$, such that $\Omega_U[X] = r$. In particular, this result asserts that for any optimal machine U there are Δ^0_2 sets X such that $\Omega_U[X]$ is a rational number, which implies $\Omega_U[X]$ is computable and hence not random.

Theorem 5.1 shows that for any given c.e. random real r, and for any given recursively enumerable set X, there is an optimal machine U such that $r = \Omega_U[X]$.

§6 is devoted to the notion of optimal machine and introduces some particularizations relevant to positive instances of the conjecture.

Finally in §7 we study the version of Conjecture 1.2 for infinite computations on monotone machines, a landscape where more positive instances have been obtained.

1.1 Notations

We denote by $2^{<\omega}$ the set of all finite words on the alphabet $\{0,1\}$ and by $2^{\leq n}$ the set of all words up to size n. The empty word is denoted by λ and the length of a word σ by $|\sigma|$. We denote by $\#X$ the number of elements of the finite set X. $P(X)$ denotes the power set of X and $P_{<\omega}(X)$ is the set of all finite subsets of X. We use $\mu(\mathcal{X})$ to denote the Lebesgue measure of a subset $\mathcal{X}$ of the Cantor space 2^ω of all infinite binary words of length ω.

As usual, we commit to prefix Turing machines, which are exactly the partial recursive functions with prefix-free domain. We assume Martin-Löf's definition of randomness (or its equivalent counterpart in terms of program-size complexity). For $n \geq 1$, n-randomness is randomness relative to oracle $\emptyset^{(n-1)}$. 1-randomness will be denoted *randomness*.

If M is a prefix Turing machine, we define $K_M(x)$ as the length of the shortest description of x using machine M, i.e. $K_M(x) = \min\{|p| : M(p) = x\}$ in case $x \in \text{range}(M)$ and $K_M(x) = +\infty$ in case $x \notin \text{range}(M)$.

1.2 A conjecture on randomness

Definition 1.1. Let $U : 2^{<\omega} \to 2^{<\omega}$ denote any prefix Turing machine. For $X \subseteq 2^{<\omega}$, let $U^{-1}(X) = \{p \in 2^{<\omega} : U(p) \in X\}$ and define

$$\Omega_U[X] = \sum_{p \in U^{-1}(X)} 2^{-|p|} = \mu(U^{-1}(X)2^\omega).$$

The third author has put forward the following conjecture on randomness, in the spirit of Rice's theorem for computability. It involves the notion of optimal prefix Turing machine as defined in the theory of program-size complexity (Cf. Definition 6.1).

Conjecture 1.2. For any nonempty $X \subseteq 2^{<\omega}$, the probability $\Omega_U[X]$ that an optimal prefix Turing machine U on an arbitrary input halts and gives an output in X is random. Moreover, if X is Σ^0_n-hard then this probability is n-random.

It turn out that *the conjecture is false as stated.* The following two theorems gather known negative and positive results about the conjecture with some of the main results of this paper.

Theorem 1.3 (Negative results)**.**

1. *There are optimal machines U for which*

   i. *$\Omega_U[X]$ is rational (hence not random) for any finite set X.*

   ii. *$\Omega_U[X]$ is not Borel normal (hence not random) for some infinite Π^0_1 set X.*

   Cf. Proposition 2.1 and also [8], Corollary 2.2 and Remark 2.3.

2. *For any optimal machine U,*

   i. *There is a Δ^0_2 set X such that $\Omega_U[X]$ is rational.* Cf. Theorems 2.7 and 4.2.

   ii. *(There are hard sets not inducing randomness). For any $A \subseteq \mathbb{N}$, there is a $\Delta^{0,A}_2$ set X which is $\Sigma^{0,A}_1$-hard and such that $\Omega_U[X]$ is not normal (hence not random). In particular, if $n \geq 1$ then there is a Δ^0_{n+1} set which is Σ^0_n-hard and such that $\Omega_U[X]$ is not random.* Cf. Theorem 2.8.

3. *(For $n \geq 2$, no Σ^0_n set gives n-randomness). For any optimal machine U and any $A \subseteq \mathbb{N}$ such that $\emptyset' \leq_T A$, if X is $\Sigma^{0,A}_1$ or $\Pi^{0,A}_1$ then $\Omega_U[X]$ is not random in A. In particular, if $n \geq 2$ and X is Σ^0_n or Π^0_n then $\Omega_U[X]$ is not n-random.* Cf. Theorem 2.9.

Nevertheless, the conjecture holds under some particular or some stronger hypotheses. The first result supporting the conjecture is Chaitin's [4] random real Ω, and corresponds to the case $\Omega_U[X]$ where $X = 2^{<\omega}$, and U is any optimal prefix machine.

Theorem 1.4 (Positive results)**.**

1. *Let U be any optimal machine. If $X \subseteq 2^{<\omega}$ is infinite and Σ^0_1 then $\Omega_U[X]$ is random.* Cf. [5][1].

2. *If U is optimal by adjunction (see Def. 6.1) and X is finite not empty then $\Omega_U[X]$ is random.* Cf. [2].

3. *Let U be any optimal machine. If $A \subseteq \mathbb{N}$ is such that $\emptyset' \leq_T A$ and X is $\Sigma^{0,A}_1$-complete or $\Pi^{0,A}_1$-complete then $\Omega_U[X]$ is random. In particular, if $n \geq 2$ and X is Σ^0_n-complete or Π^0_n-complete then $\Omega_U[X]$ is random.* Cf. Theorem 3.2.

4. *Let U be any optimal machine. If $A \subseteq \mathbb{N}$ is such that $\emptyset' \leq_T A$ then there is a $\Delta^{0,A}_2$ set X such that $\Omega_U[X]$ is random in A. In particular, if $n \geq 1$ then there is a Δ^0_{n+1} set X such that $\Omega_U[X]$ is n-random.* Cf. Corollary 4.4.

[1]Stated without proof in [5], last assertion of Note p. 141.

1.3 Open questions

Is $\Omega_U[X]$ random when X is co-r.e.? This rather simple question remained unsolved. We know that the answer is *no* for finite sets, but we might analyze what happens when X is infinite. In [8, Theorem 5] there is a partial negative answer to this question when we can fix an optimal machine. We do not know what happens when U is given (for example, for optimal by adjunction machines). Is there always an infinite co-r.e. set X for which $\Omega_U[X]$ is not random, regardless the underlying U?

Is it true that for $n \geq 2$, if X is Σ^0_n-complete then $\Omega_U[X]$ is m-random, for some $m \in \{2, \ldots, n-1\}$? Putting together Part 3 of Theorems 1.3 and 1.4 we just know that for any $n \geq 2$, if X is Σ^0_n-complete then $\Omega_U[X]$ is random but not n-random. This leaves open the possibility that there is a shift in the second part of Conjecture 1.2.

2 Negative results about the Conjecture

2.1 Failure for finite sets with particular optimal machines

Proposition 2.1. *Every prefix Turing machine M has a restriction M' to some recursively enumerable set such that $K_M = K_{M'}$ (hence M' is optimal whenever M is) and $\Omega_{M'}[X]$ is rational (hence not random), for every finite set $X \subseteq 2^{<\omega}$.*

Proof. Let $(p_i, y_i)_{i\in\mathbb{N}}$ be a recursive enumeration of the graph of M. Define a total recursive function $f : \mathbb{N} \to \mathbb{N}$ such that $f(i)$ is the smallest $j \leq i$ satisfying

$$y_j = y_i \ , \ |p_j| = \min\{|p_k| : k \leq i, y_k = y_i\}.$$

Let M' be the prefix machine with graph $\{(p_{f(i)}, y_{f(i)}) : i \in \mathbb{N}\}$. Clearly, M' is a restriction of M to some recursively enumerable set. Also, for every $x \in 2^{<\omega}$, if j is least such that $x = y_j$ and $|p_j| = K_M(x)$ then $f(i) = j$ for all $i \geq j$ such that $y_i = x$. Therefore, ${M'}^{-1}(\{x\})$ is finite, hence $\Omega_{M'}[\{x\}] = \sum_{q\in {M'}^{-1}(x)} 2^{-|q|}$ is a finite sum of rational numbers, hence is rational. The same is true for finite sets $X \subseteq 2^{<\omega}$. $\square$

Applying the above Proposition to an optimal machine U, we get the following straightforward corollary, first obtained in [8] with a different proof.

Corollary 2.2. *There is an optimal Turing machine U such that for every finite set $X \subseteq 2^{<\omega}$ the real $\Omega_U[X]$ is rational, hence not random.*

Remark 2.3. Using Proposition 2.1 it is easy to construct an infinite Π^0_1 set X such that $\Omega_U[X]$ is not normal, hence not random. In fact, in [8] it was proven that there is an infinite Π^0_1 set X such that $\Omega_U[X]$ is neither c.e. nor random.

2.2 Failure for Δ^0_2 sets with all optimal machines

We recall some results of [4] which will be used in the proofs.

Lemma 2.4. *Let U be optimal.*

1. *Coding Theorem:* $\exists c_1\ \forall\sigma\ 2^{-K_U(\sigma)} \leq \Omega_U[\{\sigma\}] \leq 2^{-K_U(\sigma)+c_1}$
2. *Maximal complexity of finite strings:*

$$\exists c_2\ \forall\sigma\ K_U(\sigma) < |\sigma| + K_U(|\sigma|) + c_2;$$

$$\exists c_3\ \#\{\sigma \in 2^m : K_U(\sigma) < m + K_U(m) - k\} \leq 2^{m-k+c_3}.$$

The next lemma can be found in unpublished work of Solovay [12, IV-20]. We include the proof because Solovay's notes are not universally available.

Lemma 2.5. *If U is optimal then $\exists c_4\ \forall n\ \exists m \leq n\ (n \leq m + K_U(m) \leq n + c_4)$.*

Proof. Choose $c_4 \in \mathbb{N}$ such that $c_4 > K_U(0)$ and $K_U(m+1) \leq K_U(m) + c_4 - 1$, for all $m \in \mathbb{N}$. Given $n \in \mathbb{N}$, let $m \in \mathbb{N}$ be the least number satisfying $n \leq m + K_U(m)$, which clearly holds for some $m \leq n$. We claim that $m + K_U(m) < n + c_4$. This holds because $0 + K_U(0) < c_4 \leq n + c_4$ and, since $m - 1 + K_U(m-1) < n$, then $m + K_U(m) \leq m - 1 + K_U(m-1) + c_4 < n + c_4$. □

Putting these two lemmas together, we get the following result.

Lemma 2.6. *If U is optimal then $\exists d\ \forall n\ \exists\sigma\ (2^{-n-d} \leq \Omega_U[\{\sigma\}] \leq 2^{-n+d})$. In fact, for some constant d' there are at least $2^n/(d'\,n^2)$ strings $\sigma \in 2^{<\omega}$ satisfying the inequalities.*

Proof. Let c_1, c_2, c_3, c_4 be constants as in Lemma 2.4 and Lemma 2.5. Then $\#\{\sigma \in 2^m : K_U(\sigma) < m + K_U(m) - (c_3+1)\} \leq 2^{m-1}$, for all $m \in \mathbb{N}$. For $n + c_3 + 1$, there is an $m \leq n + c_3 + 1$ such that $n + c_3 + 1 \leq m + K_U(m) \leq n + c_3 + 1 + c_4$. In particular, all strings $\sigma \in 2^m$ satisfy $K_U(\sigma) \leq m + K_U(m) + c_2 \leq n + c_2 + c_3 + c_4 + 1$.

Now, there are at least 2^{m-1} strings $\sigma \in 2^m$ such that $K_U(\sigma) \geq m + K_U(m) - (c_3+1)$ hence such that $K_U(\sigma) \geq n + c_3 + 1 - (c_3+1) = n$. For such strings, we then have $n \leq K_U(\sigma) \leq n + c_2 + c_3 + c_4 + 1$. Therefore, for $d = \max(c_1, c_2 + c_3 + c_4 + 1)$, there are at least 2^{m-1} strings σ such that $2^{-n-d} \leq \Omega_U[\{\sigma\}] \leq 2^{-n+d}$. Finally, note that

$$m - 1 \geq n + c_3 - K_U(m) \geq n - 2\log(m) - \mathcal{O}(1).$$

Therefore, at least $\mathcal{O}(1)2^n/n^2$ strings $\sigma \in 2^{<\omega}$ satisfy $2^{-n-d} \leq \Omega_U[\{\sigma\}] \leq 2^{-n+d}$. □

With this lemma we can prove that Conjecture 1.2 fails for Δ^0_2 sets.

Theorem 2.7. *For every optimal U there is a Δ^0_2 set $X \subseteq 2^{<\omega}$ such that $\Omega_U[X]$ is not random.*

Proof. Let $d, d' \in \mathbb{N}$ be the constants from Lemma 2.6 and let k be such that $i < 2^i/(d'\, i^2)$ for $i \geq k$. Letting $c = k + d$, Lemma 2.6 insures the existence of a sequence $(\sigma_i)_{i\in\mathbb{N}}$ of distinct strings such that $2^{-i-c-1} < \Omega_U[\{\sigma_i\}] \leq 2^{-i+c}$, for all $i \in \mathbb{N}$. Note that $\emptyset'$ can compute such a sequence (and even compute the set of strings in the sequence). Indeed, denoting by U_s the computable approximation of U obtained with s computation steps, $\Omega_{U_s}[\{\tau\}] = \sum_{U_s(p)=\tau} 2^{-|p|}$ is nondecreasing in s and tends to $\Omega_U[\{\tau\}]$ when $s \to \infty$. Thus, for any rational r, $\Omega_U[\{\tau\}] > r$ iff $\exists s\ \Omega_{U_s}[\{\tau\}] > r$. Hence it is decidable in $\emptyset'$ whether $\Omega_U[\{\tau\}] > r$ or not.

We build a Δ^0_2 set X in stages $\{X_s\}_{s\in\mathbb{N}}$. At stage $s+1$ we decide whether or not σ_s is in X in order to insure that the block of bits of $\Omega_U[X]$ from $s-c$ to $s+c+1$ is not all zeros.

Stage 0. Let $X_0 = \emptyset$.

Stage $s+1$. If $s < c$ then $X_{s+1} = X_s$. Else, using $\emptyset'$, decide if the $2c+2$ bits of $\Omega_U[X_s]$ from $s-c$ to $s+c+1$ are all zero. If these bits are all zero, let $X_{s+1} = X_s \cup \{\sigma_s\}$. Otherwise, let $X_{s+1} = X_s$. Consider the first case. Because $\Omega_U[\{\sigma_s\}] > 2^{-s-c-1}$ there exists $j \leq s+c+1$ such that the j-th bit of $\Omega_U[\{\sigma_s\}]$ is 1. On the other hand, because $\Omega_U[\{\sigma_s\}] \leq 2^{-s+c}$, we have $\Omega_U[\{\sigma_s\}] \restriction s-c-1 = 0^{s-c-1}$. Then there is $s-c \leq j \leq s+c+1$ such that the j-th bit of $\Omega_U[\{\sigma_s\}]$ is 1. Notice that if bit $s-c$ is 1 then all the bits of positions greater than $s-c$ are 0. Hence, $\Omega_U[X_{s+1}] \restriction s-c-1 = \Omega_U[X_s] \restriction s-c-1$. Therefore, the work of earlier stages has been preserved and also $\Omega_U[X_{s+1}]$ is not all zeros on the block of bits from $s-c$ to $s+c+1$.

It follows inductively that, for every $s \geq c$, the block of bits of $\Omega_U[X]$ from $s-c$ to $s+c+1$ is not all zeros. Therefore, $\Omega_U[X]$ is not normal and hence not random.

Notice that if $\Omega_U[X_s]$ is a dyadic rational, then the construction can get hung up at stage s. However, in this case $\Omega_U[X_s]$ is not random for the finite set X_s, so the theorem holds. Notice also that this construction works independently of the optimal machine chosen. $\square$

The above result can be improved: Theorem 4.2 (Cf. §4) shows that there are Δ^0_2 sets X such that $\Omega_U[X]$ is a rational number. Another improvement shows that hardness is not enough to get randomness.

Theorem 2.8. *For every optimal U and any $A \subseteq \mathbb{N}$, there is a $\Delta^{0,A}_2$ set $X \subseteq 2^{<\omega}$ which is $\Sigma^{0,A}_1$-hard and such that $\Omega_U[X]$ is not random. In particular, if $n \geq 1$ there is a Δ^0_{n+1} set $X \subseteq 2^{<\omega}$ which is Σ^0_n-hard and such that $\Omega_U[X]$ is not random.*

Proof. First observe that there is a constant b such that

$$2^{-b}\Omega_U[\{0\sigma\}] \leq \Omega_U[\{\sigma\}] \leq 2^{b}\Omega_U[\{0\sigma\}].$$

As in the proof of Theorem 2.7, let $(\sigma_i)_{i\in\mathbb{N}}$ be the sequence of distinct strings such that

$$2^{-i-c-1} < \Omega_U[\{\sigma_i\}] \leq 2^{-i+c}$$

for an appropriate constant c and such that all σ_i start with a 0. Let $Y = \{10^e : e \in A'\}$ be a set which codes A' with all strings starting with 1. So no σ_i belongs to Y.

The construction of $\widetilde{X}$ is similar to the construction of X in the proof of Theorem 2.7, but now it is relative to A': at stage $s+1$ we may or may not put the string σ_s depending whether the block of bits of $\Omega_U[Y \cup \widetilde{X_s}]$ from $s-c$ to $s+c+1$ are all zeroes. Observe that $\Omega_U[Y \cup \widetilde{X_s}]$ is A'-computable because

$$\Omega_U[Y \cup \widetilde{X_s}] = \Omega_U[Y] + \Omega_U[\widetilde{X_s}]$$

is a left-c.e. real relative to A. Define $X = \tilde{X} \cup Y$ and note that by construction $X \leq_T A'$ and A' many-one reduces to X. Thus, X is $\Sigma_1^{0,A}$ hard. $\square$

2.3 Failure of n-randomness for Σ_n^0 and Π_n^0 sets

Theorem 2.9. *Let $A \subset \mathbb{N}$ be such that $\emptyset' \leq_T A$ (where $\leq_T$ is Turing reducibility). If U is any optimal machine and $X \subseteq 2^{<\omega}$ is $\Sigma_1^{0,A}$ or $\Pi_1^{0,A}$ then $\Omega_U[X]$ is not random in A. In particular, if $n \geq 2$ and X is Σ_n^0 or Π_n^0 then $\Omega_U[X]$ is not n-random.*

Proof. The case X is finite is trivial since then $\Omega_U[X]$ is Δ_2^0 hence computable in $\emptyset'$.

Case X is infinite $\Sigma_1^{0,A}$. Fix $m \in \mathbb{N}$. With oracle $\emptyset'$, we can (uniformly in m) find a finite subset $Z \subset 2^{<\omega}$ such that $\Omega_U[Z] > \Omega_U - 2^{-m-1}$ and compute $\varepsilon > 0$ such that $\varepsilon < \min\{\Omega_U[\{z\}] : z \in Z\}$. Then

$$\sum_{\sigma:\Omega_U[\{\sigma\}]\leq\varepsilon} \Omega_U[\{\sigma\}] < 2^{-m-1}. \tag{1}$$

Let $(x_s)_{s\in\mathbb{N}}$ be an injective A-computable enumeration of X and set $X_s = \{x_t : t < s\}$. We build an A-Martin-Löf test $(T_m)_{m\in\mathbb{N}}$ for $\Omega_U[X]$. The idea is to define a $\Sigma_1^{0,A}$ class T_m by laying down successive intervals to the right of $\Omega_U[X_s]$. Set $T_m = \bigcup_{s\in\mathbb{N}} I_{m,s}$ where $I_{m,s} = (\Omega_U[X_s], \Omega_U[X_s] + \delta)$ and $\delta = 2^{-m-1}/(1 + 1/\varepsilon)$. Observe that

$$\{\sigma \in 2^{<\omega} : \Omega_U[X_s] < .\sigma < \Omega_U[X_s] + \delta\}$$

is $\Sigma_1^{0,\emptyset'}$ (with index computable from A) because $\Omega_U[X_s]$ is computable in $\emptyset'$. Therefore $I_{m,s}$ and T_m are $\Sigma_1^{0,A}$. For s big enough, $\Omega_U[X_s] < \Omega_U[X] < \Omega_U[X_s] + \delta$, so that $\Omega_U[X] \in I_{m,s}$. Thus, $\Omega_U[X] \in T_m$.

Since $\Omega_U[X_{s+1}] = \Omega_U[X_s] + \Omega_U[\{x_s\}]$, we have

$$\begin{aligned}\Omega_U[\{x_s\}] \geq \delta \ &\Rightarrow\ I_{m,s} \text{ and } I_{m,s+1} \text{ are disjoint}\\ &\Rightarrow\ \mu(\bigcup_{t\leq s+1} I_{m,t}) = \mu(\bigcup_{t\leq s} I_{m,t}) + \delta.\end{aligned}$$

Now, for all s, $\mu(\bigcup_{t\leq s+1} I_{m,t}) \leq \mu(\bigcup_{t\leq s} I_{m,t}) + \Omega_U[\{x_s\}]$. Since $\varepsilon \geq \delta$, the above properties yield

$$\begin{aligned}\mu(T_m) \ &\leq\ (\sum_{s:\Omega_U[\{x_s\}]\leq\varepsilon} \Omega_U[\{x_s\}]) + \delta\ (\#\{s : \Omega_U[\{x_s\}] > \varepsilon\} + 1)\\ &<\ 2^{-m-1} + \delta(1 + 1/\varepsilon) = 2^{-m}.\end{aligned}$$

(use (1) and the fact that $\#\{\sigma : \Omega_U[\{\sigma\}] \geq \varepsilon\} \leq \Omega_U/\varepsilon \leq 1/\varepsilon$).

Thus, we have constructed an A-Martin-Löf test $(T_m)_{m\in\mathbb{N}}$ such that $\Omega_U[X] \in \bigcap_{m\in\mathbb{N}} T_m$, proving that $\Omega_U[X]$ is not random in A.

Case X is $\Pi_1^{0,A}$. Since $\Omega_U[X] = \Omega_U - \Omega_U[2^{<\omega} \setminus X]$, use the above case and the fact that Ω_U is A-computable. □

3 Positive results about the Conjecture

In this section we give positive instances of Conjecture 1.2; in particular, the random numbers yielded by Theorems 3.2 and 3.3 are not necessarily computably enumerable. The proof method broadens the known proof techniques, which relied on the property that the numbers be computably enumerable in their degree of randomness.

3.1 Completeness and computable choice

To prove randomness in Theorems 3.2, 3.3, we use the following technical Lemma 3.1, which insures that some computable reductions associated to complete sets can be used as computable choice functions in a highly noncomputable environment.

Lemma 3.1. *Let $A \subset \mathbb{N}$ be such that $\emptyset' \leq_T A$. Suppose $X \subseteq \mathbb{N}$ is $\Sigma_1^{0,A}$-complete and $\mathcal{R} \subseteq 2^{<\omega} \times P_{<\omega}(\mathbb{N})$ is $\Sigma_1^{0,A}$ and satisfies*

$$\forall Z \in P_{<\omega}(\mathbb{N}) \quad \{\sigma : \mathcal{R}(\sigma, Z)\} \textit{ has at least } \#(Z) + 1 \textit{ elements} \tag{2}$$

(in particular, this is the case if $\{\sigma : \mathcal{R}(\sigma, Z)\}$ is infinite for all Z).

Then there exists $f : 2^{<\omega} \to \mathbb{N}$ injective total computable *such that*

$$\forall \sigma \in 2^{<\omega} \ [(\exists Z \subset X \ \mathcal{R}(\sigma, Z)) \ \Rightarrow \ \exists Z \subset X \ (\mathcal{R}(\sigma, Z) \wedge f(\sigma) \in X \setminus Z)].$$

Moreover, for such an f one can take some computable reduction of $\{\sigma : \exists Z \in P_{<\omega}(X) \ \mathcal{R}(\sigma, Z)\}$ to X.

Proof. 1. Let $W^A \subset \mathbb{N}^2$ be universal for $\Sigma_1^{0,A}$ subsets of $\mathbb{N}$, i.e. W is $\Sigma_1^{0,A}$ and every $\Sigma_1^{0,A}$ subset of $\mathbb{N}$ is a section $W_e^A = \{n : (e, n) \in W^A\}$ of W^A for some e. Since X is $\Sigma_1^{0,A}$-complete, there exists a total computable injective reduction $F : \mathbb{N}^2 \to \mathbb{N}$ of W^A to X, i.e. $W^A = F^{-1}(X)$. Then, for every e, the map $F_e : \mathbb{N} \to \mathbb{N}$ such that $F_e(n) = F(e, n)$ is a total computable injective reduction of W_e^A to X.

2. Let $S = \{\sigma : \exists Z \in P_{<\omega}(X) \ \mathcal{R}(\sigma, Z)\}$. Clearly, S is $\Sigma_1^{0,A}$. Property (2) insures that S is infinite.

Letting e be some integer (to be fixed by the recursion theorem such that $W_e^A = \mathrm{range}(\theta_e) = S$), uniformly in e, we inductively define an injective total A-computable map $\theta_e : \mathbb{N} \to S$ (to be an enumerations of S).

Since F_e is computable, its range is computable with oracle $\emptyset'$, so that the set $X \setminus \mathrm{range}(F_e)$ is $\Sigma_1^{0,A}$. Fix some A-computable enumeration ρ of $\mathcal{R}$.

Stage s. Let (σ, Z) be the least pair (relative to ρ) such that

$$\sigma \notin \{\theta_e(t) : t < s\} \ \wedge \ Z \subseteq \{F_e(\theta_e(t)) : t < s\} \cup (X \setminus \mathrm{range}(F_e)) \tag{3}$$

Property (2) insures that there is always such a σ. Set $\theta_e(s) = \sigma$.

3. Let $\xi : \mathbb{N} \to \mathbb{N}$ be total computable such that $\mathrm{range}(\theta_e) = W^A_{\xi(e)}$. The recursion theorem insures that there exists e so that $W^A_e = W^A_{\xi(e)}$.

Since F_e is an injective total computable reduction of W^A_e to X, the last equality insures that F_e is a reduction of $\mathrm{range}(\theta_e)$ to X. In particular,

$$\mathrm{range}(F_e \circ \theta_e) = F_e(\mathrm{range}(\theta_e)) = F_e(W^A_{\xi(e)}) = F_e(W^A_e) = \mathrm{range}(F_e) \cap X.$$

Hence $\mathrm{range}(F_e \circ \theta_e) \cup (X \setminus \mathrm{range}(F_e)) = X$. Using (3), this insures that $\theta_e(s) \in S$ for all s. This also yields that every finite subset of X is included in $\{F_e(\theta_e(t)) : t < s\} \cup (X \setminus \mathrm{range}(F_e))$ for s large enough. Using (3) again, we see that every $\sigma \in S$ is in the range of θ_e. Thus, $S = \mathrm{range}(\theta_e)$.

4. Let $f = F_e$. Then f is injective total computable. Also, if $\sigma = \theta_e(s)$ and Z is as in property (3), then $\mathcal{R}(\sigma, Z)$ holds and, since $F_e \circ \theta_e$ is injective, $f(\sigma) = F_e(\theta_e(s)) \notin \{F_e(\theta_e(t)) : t < s\}$, hence $f(\sigma) \notin Z$. □

3.2 Randomness of $\Omega_U[X]$ when X is Σ^0_n or Π^0_n complete, $n \geq 2$

The above Lemma 3.1 allows us to extend to $\Omega_U[X]$ Chaitin's argument to prove the randomness of Ω_U.

Theorem 3.2. *Let U be optimal. If $X \subseteq 2^{<\omega}$ is $\Sigma^{0,A}_1$-complete for some $A \subset \mathbb{N}$ such that $\emptyset' \leq_T A$ then $\Omega_U[X]$ is random. In particular, if $n \geq 2$ and X is Σ^0_n complete then $\Omega_U[X]$ is random.*

Proof. 1. *The relation $\mathcal{R} \subset 2^{<\omega} \times P_{<\omega}(\mathbb{N})$.* In order to apply Lemma 3.1, we set

$$\mathcal{R} = \{(\lambda, \emptyset)\} \ \cup \ \{(\sigma, Z) : \sigma \in \mathrm{dom}(U) \ \wedge \ \Omega_U[Z] > U(\sigma)\}$$

where $U(\sigma)$ is identified with a dyadic rational number. Observe that digits of $\Omega_U[Z]$ can be computed from the finite set Z using oracle $\emptyset'$, and the strict inequality $\Omega_U[Z] > U(\sigma)$ can be decided using oracle $\emptyset'$. Since $\emptyset' \leq_T A$, this insures that $\mathcal{R}$ is A-computable. One easily checks that $\mathcal{R}$ satisfies property (2) of Lemma 3.1 (in fact $\{\sigma : \mathcal{R}(\sigma, Z)\}$ is even infinite when $Z \neq \emptyset$).

2. *A constant from the invariance theorem.*

Let f be given by Lemma 3.1. Consider the restriction of f to $\mathrm{dom}(U)$. This is a partial computable function with prefix-free domain. Hence there exists a constant c such that, for all $\sigma \in \mathrm{dom}(U)$,

$$K_U(f(\sigma)) \leq K_{f \restriction \mathrm{dom}(U)}(f(\sigma)) + c \leq |\sigma| + c.$$

3. *Chaitin's argument pushed up to* $\Omega_U[X]$.

Consider the infinite binary expansion of $\Omega_U[X]$ which, in case it is dyadic (which is not the case, in fact), does end with 1^ω. For $m \in \mathbb{N}$, let σ be such that $U(\sigma) = \Omega_U[X] \upharpoonright m$. Since $\Omega_U[X] > \Omega_U[X] \upharpoonright m$, we see that there exists a finite subset Z of X such that $\Omega_U[Z] > \Omega_U[X] \upharpoonright m$, i.e. such that $\mathcal{R}(\sigma, Z)$.

Clearly, Z must contain all elements $a \in X$ such that $\Omega_U[\{a\}] > 2^{-m}$.

Using Lemma 3.1, we see that $f(\sigma) \in X \setminus Z$. Thus, $\Omega_U[\{f(\sigma)\}] \leq 2^{-m}$. In particular, $K_U(f(\sigma)) \geq m$. Now, since $\sigma \in \mathrm{dom}(U)$, Point 2 yields $K_U(f(\sigma)) \leq |\sigma| + c$. Hence $|\sigma| \geq m - c$.

Thus, every program σ such that $U(\sigma) = \Omega_U[X] \upharpoonright m$ has length $\geq m - c$. This proves that $K_U(\Omega_U[X] \upharpoonright m) \geq m - c$ and hence that $\Omega_U[X]$ is random. $\square$

The case of $\Pi_1^{0,A}$-complete sets X is obtained with a similar argument.

Theorem 3.3. *Let $A \subset \mathbb{N}$ be such that $\emptyset' \leq_T A$. If X is $\Pi_1^{0,A}$-complete then $\Omega_U[X]$ is random. In particular, if $n \geq 2$ and X is Π_n^0 complete then $\Omega_U[X]$ is random.*

Proof. 1. *The relation $\mathcal{R}$.* We now let

$$\mathcal{R} = \{(\sigma, Z) : \sigma \in \mathrm{dom}(U) \ \wedge \ \Omega_U - \Omega_U[Z] < U(\sigma) + 2^{-|U(\sigma)|+1}\}.$$

Now, $\mathcal{R}$ is $\Sigma_1^{0,A}$ (express $\Omega_U - \Omega_U[Z] < ...$ as $\exists m\ \Omega_U \upharpoonright m - \Omega_U[Z] \upharpoonright m + 2^{-m+1} < (...) \upharpoonright m$) and satisfies property (2) from Lemma 3.1.

2. *Chaitin's argument pushed up to* $\Omega_U[X]$.

For $m \in \mathbb{N}$, let σ be such that $U(\sigma) = \Omega_U[X] \upharpoonright m$. Observe that,

$$\Omega - \Omega_U[\mathbb{N} \setminus X] = \Omega_U[X] < \Omega_U[X] \upharpoonright m + 2^{-m+1}$$

so that there exists a finite subset Z of $\mathbb{N} \setminus X$ such that

$$\Omega - \Omega_U[Z] < \Omega_U[X] \upharpoonright m + 2^{-m+1}$$

i.e. such that $\mathcal{R}(\sigma, Z)$.

Observe that if $z \in \mathbb{N} \setminus (Z \cup X)$ then

$$\begin{array}{rcl} \Omega & \geq & \Omega_U[Z] + \Omega_U[X] + \Omega_U[\{z\}] \\ \Omega_U[\{z\}] & \leq & \Omega - \Omega_U[Z] - \Omega_U[X] \\ & \leq & \Omega_U[X] \upharpoonright m + 2^{-m+1} - \Omega_U[X] \\ & \leq & 2^{-m+1} \qquad \text{since } \Omega_U[X] \upharpoonright m - \Omega_U[X] \leq 0. \end{array}$$

Let f be as in Lemma 3.1. Then $f(\sigma) \in (\mathbb{N} \setminus X) \setminus Z$. Therefore $\Omega_U[\{f(\sigma)\}] \leq 2^{-m+1}$. In particular, $K_U(f(\sigma)) \geq m - 1$. Now, since $\sigma \in \mathrm{dom}(U)$, Point 2 of the proof of Theorem 3.2 yields $K_U(f(\sigma)) \leq |\sigma| + c$. Hence $|\sigma| \geq m - 1 - c$.

Thus, every program σ such that $U(\sigma) = \Omega_U[X] \upharpoonright m$ has length $\geq m - 1 - c$. This proves that $\Omega_U[X]$ is random. $\square$

4 The set $\{\Omega_U[X] : X \subseteq 2^{<\omega}\}$

4.1 A lemma about sums of subseries

Lemma 4.1. *Let $(a_i)_{i\in\mathbb{N}}$ be a sequence of strictly positive real numbers satisfying*

1. $\lim_{i\to+\infty} a_i = 0$;
2. $a_i \leq \sum_{j>i} a_j$ *for all* i.

Let $\alpha = \sum_{i\in\mathbb{N}} a_i$ (which may be $+\infty$). Then

$$\{\sum_{i\in I} a_i : I \subseteq \mathbb{N}\} = [0,\alpha].$$

Furthermore, for every $r \in [0,\alpha]$ there exists $I(r) \subseteq \mathbb{N}$ such that $\sum_{i\in I(r)} a_i = r$ and which is computable (non-uniformly) from r and $(a_i)_{i\in\mathbb{N}}$.

Proof. Take $r \in [0,\alpha]$. We define a monotone increasing sequence $(I_t(r))_{t\in\mathbb{N}}$ of finite subsets of $\mathbb{N}$ by the following induction:

$$I_0(r) = \emptyset \quad , \quad I_{t+1}(r) = \begin{cases} I_t(r) \cup \{t\} & \text{if } a_t + \sum_{i\in I_t(r)} a_i \leq r; \\ I_t(r) & \text{otherwise.} \end{cases}$$

Let $I(r) = \bigcup_{t\in\mathbb{N}} I_t(r)$. Since inequality $\sum_{i\in I_t(r)} a_i \leq r$ is true for all t, we get $\sum_{i\in I(r)} a_i \leq r$. We show that $r = \sum_{i\in I(r)} a_i$.

Case $r = \alpha$. Then $I(r) = \mathbb{N}$ and the equality is trivial.

Case $r < \alpha$ and there are only infinitely many t's such that $I_{t+1}(r) = I_t(r)$. For such ts we have $\sum_{i\in I_t(r)} a_i \leq r < a_t + \sum_{i\in I_t(r)} a_i$. Taking limits over such t's and using condition 1, we get equality $\sum_{i\in I(r)} a_i = r$.

Case $r < \alpha$ and there are finitely many t's such that $I_{t+1}(r) = I_t(r)$. We show that this case does not occur. Since $r < \alpha$ we have $I(r) \neq \mathbb{N}$ so that there is at least one t such that $I_{t+1}(r) = I_t(r)$. Let u be the largest such t. Then, $\sum_{i\in I_u(r)} a_i \leq r < a_u + \sum_{i\in I_u(r)} a_i$ and, for all $v > u$, $I_{v+1} = I_v \cup \{v\}$. Therefore, $I(r) = I_u(r) \cup \{i : i > u\}$. Since condition 2 insures $a_u \leq \sum_{i>u} a_i$, we get $r < \sum_{i>u} a_i + \sum_{i\in I_u(r)} a_i = \sum_{i\in I(r)} a_i$, which contradicts inequality $\sum_{i\in I(r)} a_i \leq r$.

The last assertion of the Lemma about the relative computability of $I(r)$ is trivial if $I(r)$ is finite. Since the a_t's are strictly positive, if $I(r)$ is infinite then $r \neq a_t + \sum_{i\in I_t(r)} a_i$ for all t. Thus, enumerating the digits of r and $a_t + \sum_{i\in I_t(r)} a_i$, we get at some finite time either $r < a_t + \sum_{i\in I_t(r)} a_i$ or $r > a_t + \sum_{i\in I_t(r)} a_i$, which proves that the test in the definition of $I_{t+1}(r)$ can be done recursively in r and $(a_i)_{i\in\mathbb{N}}$. $\square$

4.2 $\{\Omega_U[X] : X \subseteq 2^{<\omega}\}$ is a finite union of closed intervals

Point 2 of the following theorem gives an alternative proof of Theorem 2.7 above.

Theorem 4.2. *Let U be optimal.*

1. The set $\{\Omega_U[X] : X \subseteq 2^{<\omega}\}$ is the union of finitely many pairwise disjoint closed intervals with positive lengths, i.e.

$$\{\Omega_U[X] : X \subseteq 2^{<\omega}\} = [a_1, b_1] \cup \ [a_2, b_2] \cup ... \cup \ [a_n, b_n],$$

where $0 = a_1 < b_1 < ... < a_n < b_n = \Omega_U$.

2. Every real $s \in \{\Omega_U[X] : X \subseteq 2^{<\omega}\}$ is of the form $\Omega_U[Y]$ for some Y which is recursive in $s \oplus \emptyset'$. In particular, there exists some Δ^0_2 set $X \subseteq 2^{<\omega}$ such that $\Omega_U[X]$ is rational, hence not random.

Proof. 1i. First, we get $\alpha > 0$ such that $\{\Omega_U[X] : X \subseteq 2^{<\omega}\} \supseteq [0, \alpha]$.

Let $d, d' \in \mathbb{N}$ be the constants of Lemma 2.6 and let k be such that $2^{2d+1}(i+1) \leq 2^i/(d'i^2)$ for $i \geq k$. Using this inequality and Lemma 2.6, one can inductively define a sequence of *pairwise disjoint* sets of strings $(S_i)_{i\geq k}$ such that $\#S_i = 2^{2d+1}$ and $2^{-i-d-1} < \Omega_U[\{\sigma\}] \leq 2^{-i+d}$ for every $\sigma \in S_i$. Notice that, as in Theorem 2.7, the sequence $(S_i)_{i\geq k}$ is computable in $\emptyset'$.

We define an enumeration ψ of $S = \bigcup_{i\geq k} S_i$: for $j, m \in \mathbb{N}$ and $m < 2^{2d+1}$, let $\psi(2^{2d+1}j + m)$ be the m-th element of S_{k+j}. Set $a_i = \Omega_U[\{\psi(i)\}]$, it is clearly positive and $\lim_{i\to+\infty} a_i = 0$. Observe that for any $m \in [0, 2^{2d+1})$, and any $j \geq 0$, $2^{-(k+j)-d-1} < a_{2^{2d+1}j+m} \leq 2^{-(k+j)+d}$ and it is computable in $\emptyset'$. Then, for any such m and j we have

$$\begin{aligned}
\sum_{l>2^{2d+1}j+m} a_l &\geq \sum_{l>j} \sum_{s<2^{2d+1}} a_{2^{2d+1}l+s} \\
&> \sum_{l>j} 2^{2d+1} 2^{-(k+l)-d-1} \\
&= 2^{-(k+j)+d} \\
&\geq a_{2^{2d+1}j+m}.
\end{aligned}$$

Thus, the conditions of Lemma 4.1 are satisfied: $\{\Omega_U[Y] : Y \subseteq S\} = [0, \alpha]$ where $\sum_{i\in\mathbb{N}} a_i = \alpha > 0$.

1ii. Now,

$$\begin{aligned}
\{\Omega_U[X] : X \subseteq 2^{<\omega}\} &= \{\Omega_U[Y] + \Omega_U[Z] : Y \subseteq S,\ Z \cap S = \emptyset\} \\
&= [0, \alpha] + \{\Omega_U[Z] : Z \cap S = \emptyset\} \\
&= \bigcup_{r\in\mathcal{R}} [r, r + \alpha].
\end{aligned}$$

where $\mathcal{R} = \{\Omega_U[Z] : Z \cap S = \emptyset\}$ and $0 \in \mathcal{R}$.

Let $\mathcal{R}_i = \mathcal{R} \cap [i\alpha, (i+1)\alpha[$. Observe that if $r, r' \in \mathcal{R}_i$ then $[r, r+\alpha]$ and $[r', r'+\alpha]$ have non-empty intersection. Hence the union $\bigcup_{r \in \mathcal{R}_i} [r, r+\alpha]$ is an interval J_i (a priori not necessarily closed). Since $\mathcal{R}_i = \emptyset$ for $i\alpha > 1$, we see that $\mathcal{R} = \mathcal{R}_1 \cup ... \cup \mathcal{R}_\ell$ where $\ell \leq \lceil \frac{1}{\alpha} \rceil$. Thus, $\{\Omega_U[X] : X \subseteq 2^{<\omega}\} = J_1 \cup ... \cup J_\ell$. Grouping successive intervals J_is having non-empty intersection, we get the representation $\{\Omega_U[X] : X \subseteq 2^{<\omega}\} = I_1 \cup ... \cup I_n$ where the I_is are pairwise disjoint intervals in $[0, 1]$.

1iii. Since the map $X \mapsto \Omega_U[X]$ is continuous from the compact space $P(2^{<\omega})$ (with the Cantor topology) to $[0, 1]$, its range $\{\Omega_U[X] : X \subseteq 2^{<\omega}\}$ is compact. In particular, the intervals I_is may be taken closed. This proves Point 1 of the Theorem.

2. First, observe that if $I \subseteq \mathbb{N}$ is recursive in $\emptyset'$ then so is $\{\psi(n) \colon n \in I\}$. Given $\sigma \in 2^{<\omega}$, using $\emptyset'$, one can check whether $2^{-j} < \Omega_U[\{\sigma\}]$. Hence one can compute i and m such that σ is the m-th element of S_{k+i}, i.e. such that $\sigma = \psi(2^{2d+1}i + m)$, if such i and m exists (this can also be determined with $\emptyset'$). In case there is no such i and m then $\sigma \notin \{\psi(n) \colon n \in I\}$. Else $\sigma \in \{\psi(n) \colon n \in I\}$ if and only if $2^{2d+1}i + m \in I$.

Case $s \in [0, \alpha]$. Lemma 4.1 insures that there is a set $I(s) \subseteq \mathbb{N}$, computable from $s \oplus \emptyset'$, such that $\sum_{i \in I(s)} a_i = s$. Let $X = \{\psi(n) \colon n \in I(s)\}$. Then X is computable from $s \oplus \emptyset'$ and $\Omega_U[X] = s$.

Case $s \in [r, r+\alpha)$ *for some* $r \in \mathcal{R}$. Let $s = \Omega_U[Z] + \beta$ where $r = \Omega_U[Z]$ and $Z \cap S = \emptyset$ and $\beta < \alpha$. Let Z' be a finite subset of Z such that $\Omega_U[Z \setminus Z'] < \alpha - \beta$. Then the real $\Omega_U[Z']$ is computable in $\emptyset'$ and $\Omega_U[Z \setminus Z'] + \beta = s - \Omega_U[Z']$ is computable in $s \oplus \emptyset'$. Since $\Omega_U[Z \setminus Z'] + \beta < \alpha$, Lemma 4.1 yields $X \subseteq S$ which is computable in $s \oplus \emptyset'$ such that $\Omega_U[Z \setminus Z'] + \beta = \Omega_U[X]$. Since Z' is finite, we see that $X \cup Z'$ is computable in $s \oplus \emptyset'$. Finally, $s = \Omega_U[X \cup Z']$.

Case $s \in [a_j, b_j)$ *with* $1 \leq j \leq n$. Observe that $\bigcup_{r \in \mathcal{R}_i} [r, r+\alpha)$ is equal to J_i with the right endpoint removed. Suppose $I_j = J_i \cup ... \cup J_{i+m}$. Then

$$[a_j, b_j) = \bigcup_{i \leq p \leq i+m} \bigcup_{r \in \mathcal{R}_p} [r, r+\alpha).$$

Thus, $s \in [r, r+\alpha)$ for some $r \in \mathcal{R}$ and the previous case applies.

Case $s = b_j$ *with* $1 \leq j \leq n$. Let $b_j = \Omega_U[X]$. If $\sigma \notin X$ then $\Omega_U[X \cup \{\sigma\}] > b_j$ hence $\Omega_U[X \cup \{\sigma\}] \geq a_{j+1}$. In particular, $\Omega_U[\{\sigma\}] \geq a_{j+1} - b_j$. Which proves that the complement of X contains at most $\lceil \frac{1}{a_{j+1} - b_j} \rceil$ elements. Thus, X is cofinite, hence recursive. □

In relation with Theorem 4.2, we consider the following question: how disconnected is $\{\Omega_U[X] : X \subseteq 2^{<\omega}\}$?

Proposition 4.3. *Let U be optimal. For each $n \geq 1$, there exists a finite modification V of U which is still optimal and such that the set $\{\Omega_V[X] : X \subseteq 2^{<\omega}\}$ is not the union of less than n intervals.*

Proof. Let $(p_i)_{i\in\mathbb{N}}$ be an enumeration of $\mathrm{dom}(U)$ and inductively define integers $i_0 < i_1 < \ldots < i_n$ such that $i_0 = 0$ and, for $k = 0, \ldots, n-1$, letting $H_k = \sum_{i_k \le i < i_{k+1}} 2^{-|p_i|}$ and $T_k = \sum_{i \ge i_k} 2^{-|p_i|}$,

$$H_k \quad > \quad \frac{T_k}{2}. \tag{4}$$

We define a first finite modification $\widehat{V}$ of U as follows:

$$\widehat{V}(p_i) = \begin{cases} 0^k & \text{if } i_k \le i < i_{k+1} \text{ and } 0 \le k < n; \\ U(p_i) & \text{if } i \ge i_n. \end{cases}$$

Clearly, for $0 \le k < n$,

$$\{\Omega_{\widehat{V}}[X] : 0^k \in X \ \wedge \ \forall \ell < k \ 0^\ell \notin X\}; \quad \subseteq \quad [H_k, T_k]; \tag{5}$$

$$\{\Omega_{\widehat{V}}[X] : \forall \ell < n \ 0^\ell \notin X\} \quad \subseteq \quad [0, T_n]. \tag{6}$$

Now, $\Omega_U = T_0$ and inequalities (4) insure that $T_{k+1} = T_k - H_k < H_k$ for $0 \le k < n$. Thus, the intervals $[0, T_n]$, $[H_{n-1}, T_{n-1}]$,..., $[H_0, T_0]$ are pairwise disjoint. Since the sets on the left in (5), (6) are non-empty, we see that $\{\Omega_{\widehat{V}}[X] : X \subseteq 2^{<\omega}\}$ is not the union of less than $n+1$ intervals.

However, even though universal functions take each value infinitely many times, an optimal function, such as U, may take some values only once. Therefore, $\widehat{V}$ may be non-surjective, hence non-optimal. We have to insure that $U(p_0), \ldots, U(p_{i_{n-1}})$ are indeed values of V. In that purpose, observe that there are infinitely strings x such that $U^{-1}(x)$ has at least two elements. Else, for x large enough, $K_U(x)$ would be $|p|$ where p is the unique element such that $U(p) = x$, which would make K_U computable, contradicting optimality of U.

Now, compute i_n many distinct indexes j, all $\ge i_n$, such that the outputs $U(p_j)$ are distinct and each $U^{-1}(U(j))$ has at least two elements. Let $j_0, \ldots, j_{i_{n-1}}$ be such indexes and set

$$V(p_i) = \begin{cases} \widehat{V}(p_i) & \text{if } i < i_n; \\ U(p_\ell) & \text{if } \ell < i_n \text{ and } i = j_\ell; \\ U(p_i) & \text{if } i \ge i_n \text{ and } i \text{ is not among } j_0, \ldots, j_{i_n - 1}. \end{cases}$$

V is still a finite modification of U but has the same range as U, hence is surjective. Being surjective and equal to the optimal U almost everywhere, V is also optimal. Finally, observe that inclusions (5), (6) are still true for V since V and $\widehat{V}$ coincide on p_i for $i < i_n$. □

4.3 $\Omega_U[X]$ is n-random for some Δ^0_{n+1} sets

As a corollary of Theorem 4.2, we get the following result which is in contrast with Theorems 2.7, 2.8 and 2.9.

Corollary 4.4. *Let U be any optimal machine:*

1. For any $A \subseteq \mathbb{N}$ there is a $\Delta_2^{0,A}$ set X such that $\Omega_U[X]$ is random in A.

2. For every $n \geq 2$ there is a Δ_{n+1}^0 set X such that $\Omega_U[X]$ is n-random. For $n = 1$, there is a computable such X.

Proof. 1. Let b_1 be as in Point 1 of Theorem 4.2, let $r = \Omega_{U^A}[2^{<\omega}]$ associated to some optimal machine U^A with oracle A and $k \in \mathbb{N}$ be such $r2^{-k} < b_1$. Then r and $r2^{-k}$ are $\Delta_2^{0,A}$ and random in A. Theorem 4.2 insures that there exists some set X which is computable in $r2^{-k} \oplus \emptyset' \leq_T A'$ such that $r2^{-k} = \Omega_U[X]$.

2. If $n = 1$, set $X = 2^{<\omega}$ and apply Chaitin's celebrated result. If $n \geq 2$, apply Point 1. $\square$

5 Varying U and X in $\Omega_U[X]$

From point 2 of Theorem 4.2, it follows that, for any given optimal machine U, every c.e. random real small enough is $\Omega_U[X]$ for some $X \subseteq 2^{<\omega}$ which is Δ_2^0. We now show that X can be any Σ_1^0 set if we pick an appropriate optimal machine U.

To prove this, we need some well-known facts. In [3] Calude et al. showed that for any c.e. real a there exists an r.e. prefix-free set $R \subseteq 2^{<\omega}$ such that $a = \mu(R2^\omega)$.

Let us recall the definition of Solovay's domination between c.e. reals: Let a and b be c.e. reals. We say that a *dominates* b, and write $b \leq_S a$ iff there is a constant c and a partial computable function $f : \mathbb{Q} \to \mathbb{Q}$ such that for each rational $q < a$, $f(q)$ is defined and $f(q) < b$ and $b - f(q) \leq c(a - q)$.

In [7], Downey et al. proved that if a and b are c.e. reals such that $b \leq_S a$, then there is a c.e. real d and constant $c \in \mathbb{N}$ such that $ca = b + d$. Note that c can be taken as large as we like.

Using these results, we can prove the following:

Theorem 5.1. *Let $X \subseteq 2^{<\omega}$ be Σ_1^0, $X \neq \emptyset$, and let $a \in (0, 1)$ be c.e. random real. There is an optimal machine V such that $a = \Omega_V[X]$.*

Proof. Let U be the usual optimal machine such that

$$U(0^{e-1}1p) = M_e(p).$$

By Chaitin's Theorem (Cf. Point 1 of Theorem 1.4), $\Omega_U[X]$ is a c.e. random real and following [10] we know that $a \equiv_S \Omega_U[X]$ (this just means $a \leq_S \Omega_U[X]$ and $\Omega_U[X] \leq_S a$). Hence, from [7] there is a c large enough such that $2^{-c}\Omega_U < \min\{a, 1-a\}$ and $a - 2^{-c}\Omega_U[X]$ is a c.e. real in $(0, 1)$. From [3] there is an r.e. prefix-free set R such that $a - 2^{-c}\Omega_U[X] = \mu(R2^\omega)$.

We define the Kraft-Chaitin list for V with the axioms $\{(|r|, y) : r \in R\}$ and $\{(|p| + c, U(p)) : U(p) \downarrow\}$, where $y \in X$. Since for any p, if $U(p) \downarrow$ then $U(p) = V(q)$, for some q with $|q| = |p| + c$, we conclude that V is optimal. By construction, we have $\Omega_V[X] = \mu(R2^\omega) + 2^{-c}\Omega_U[X] = a$. $\square$

6 On the notion of optimality

Conjecture 1.2 is valid for finite sets for machines that are *optimal by adjunction* (Cf. Part 2 of Theorem 1.4). This refinement of the notion of optimality corresponds to the most natural use of optimal machines.

Let $(M_e)_{e\in\mathbb{N}}$ be a recursive enumeration of all prefix Turing machines.

Definition 6.1 (Optimality and optimality by adjunction)**.** Let

$$U : 2^{<\omega} \to 2^{<\omega}$$

be a prefix Turing machine.

1. U is *optimal* if and only if

$$\forall e\ \exists c_e\ \forall p\ \exists \sigma_{e,p}\ (U(\sigma_{e,p}) = M_e(p)\ \wedge |\sigma_{e,p}| \leq |p| + c_e).$$

   U is *effectively optimal* if there is a total recursive function $c : \mathbb{N} \times 2^{<\omega} \to 2^{<\omega}$ such that we can take $\sigma_{e,p} = c(e,p)$.

2. U is *optimal by adjunction* if and only if

$$\forall e\ \exists \sigma_e\ \forall p\ U(\sigma_e p) = M_e(p).$$

   Hence, in this case, $c_e = |\sigma_e|$ and $\sigma_{e,p} = \sigma_e p$ (concatenation of words σ_e and p).

Clearly, U is optimal if and only if it satisfies the Invariance Theorem (of program-size complexity) which states that for all e there is a constant c_e such that $K_U(y) \leq K_{M_e}(y) + c_e$ for all y.

Optimality by adjunction can be obtained from effective optimality plus some extra conditions on the coding function c.

Proposition 6.2. *Let V be effectively optimal such that the associated $c\colon \mathbb{N} \times 2^{<\omega} \to 2^{<\omega}$ is injective and has recursive range. Then there exists a machine U optimal by adjunction such that*

$$(\forall \sigma \in 2^{<\omega})\, \Omega_U[\{\sigma\}] = \Omega_V[\{\sigma\}]. \tag{7}$$

Proof. Since V is optimal, $\Omega_V[2^{<\omega}]$ is random, hence less than 1 and so there exists k such that $\Omega_V[2^{<\omega}] < 1 - 2^{-k}$. Fix such a k. The idea of the proof is as follows: first, define U on a prefix-free subset of $0^{k+1}2^{<\omega}$ in a way that insures that U is optimal by adjunction. Then define U on a prefix-free subset of $0^{\leq k}12^{<\omega}$ to get condition (7).

For (e,p) such that $c(e,p) \in \mathrm{dom}(V)$, and $n \in \mathbb{N}$ such that $|c(e,p)| \leq |p| + n$, we set

$$U(0^{k+1+n}\ 1^{e+1}\ 0\ p) = V(c(e,p)). \tag{8}$$

Since $V(c(e,p)) = M_e(p)$ we see that $U(0^{k+1+n}1^{e+1}0p) = M_e(p)$ for all $n \geq |c(e,p)| - |p|$. The optimality of V insures that there exists c_e such that $|c(e,p)| \leq |p| + c_e$ for all p. Then

$U(0^{k+1+c_e}1^{e+1}0p) = M_e(p)$ for all p. This proves that U is optimal by adjunction with $\sigma_e = 0^{k+1+c_e}1^{e+1}0$. Observe that for given e and p,

$$\begin{aligned}\sum_{n\geq m} 2^{-|0^{k+1+n}1^{e+1}0p|} &= 2^{-(k+e+3)}\sum_{n\geq m} 2^{-(n+|p|)} \\ &= 2^{-(k+2+e+\max(|p|,|c(e,p)|))}.\end{aligned}$$

where $m = \max(0, |c(e,p)| - |p|)$. Let $Q_{e,p}$ be the finite subset of $\mathbb{N}$ such that

$$\sum_{j\in Q_{e,p}} 2^{-j} = 2^{-|c(e,p)|} - 2^{-(k+2+e+\max(|p|,|c(e,p)|))}.$$

To define U on $\bigcup_{n\leq k} 0^n 1 2^{<\omega}$, we introduce the following Kraft-Chaitin set

$$\begin{aligned} W &= \{(j, V(c(e,p)))\colon (e,p)\in \mathrm{dom}(V\circ c), j\in Q_{e,p}\} \\ &\quad \cup \{(|q|, V(q))\colon q\in\mathrm{dom}(V)\setminus \mathrm{range}(c)\}.\end{aligned}$$

Since the range of c is recursive, there is a recursive enumeration $(l_n, \sigma_n)_{n\in\mathbb{N}}$ of W. Let us show that W is indeed a Kraft-Chaitin set.

$$\begin{aligned}\sum_{(l,\sigma)\in W} 2^{-l} &= \sum_{(e,p)\in\mathrm{dom}(V\circ c)}\ \sum_{j\in Q_{e,p}} 2^{-j} + \sum_{q\in\mathrm{dom}(V)\setminus\mathrm{range}(c)} 2^{-|q|} \\ &\leq \sum_{(e,p)\in\mathrm{dom}(V\circ c)} 2^{-|c(e,p)|} + \sum_{q\in\mathrm{dom}(V)\setminus\mathrm{range}(c)} 2^{-|q|} \\ &\leq \sum_{q\in\mathrm{dom}(V)\cap\mathrm{range}(c)} 2^{-|q|} + \sum_{q\in\mathrm{dom}(V)\setminus\mathrm{range}(c)} 2^{-|q|} \\ &< 1-2^{-k}.\end{aligned}$$

A straightforward extension of the Kraft-Chaitin theorem shows that there is a r.e. set $\{p_n\colon n\in\mathbb{N}\}$ which is a prefix-free subset of $0^{\leq k}12^{<\omega}$ and $|p_n| = l_n$ for all n. We complete the definition of U on $0^{\leq k}12^{<\omega}$ by setting for all n

$$U(p_n) = \sigma_n. \tag{9}$$

Observe that U, as defined by (8) and (9), has prefix-free domain. Also, for $\sigma\in 2^{<\omega}$, we have

$$\Omega_V[\{\sigma\}] = \sum \{2^{-|q|}\colon q\in\mathrm{dom}(V)\cap\mathrm{range}(c)\wedge V(q)=\sigma\} \tag{10}$$

$$+\sum \{2^{-|q|}\colon q\in\mathrm{dom}(V)\setminus\mathrm{range}(c)\wedge V(q)=\sigma\}. \tag{11}$$

Take q as in (10). Since c is injective, there is a unique pair (e,p) such that $q = c(e,p)$. Thus, the sum (10) is exactly

$$\sum_{V(c(e,p))=\sigma} 2^{-|c(e,p)|}.$$

The U-descriptions of type (8) of σ add $2^{-(k+2+e+\max(|p|,|c(e,p)|))}$ to $\Omega_U[\{\sigma\}]$, for any (e,p) such that $V(c(e,p)) = \sigma$; the U-descriptions of type (9) add $2^{-|c(e,p)|} - 2^{-(k+2+e+\max(|p|,|c(e,p)|))}$ to $\Omega_U[\{\sigma\}]$ for any (e,p) such that $V(c(e,p)) = \sigma$. So, in total, for every (e,p) such that $V(c(e,p)) = \sigma$ contributes $2^{-|c(e,p)|}$ to $\Omega_U[\{\sigma\}]$.

For $q \in \mathrm{dom}(V)\backslash\mathrm{range}(c)$ we have $(|q|, V(q)) \in W$ so there are additional U-descriptions that contribute to $\Omega_U[\{\sigma\}]$ exactly the amount in (11). Thus, $\Omega_U[\{\sigma\}] = \Omega_V[\{\sigma\}]$. $\square$

Remark 6.3. In the proposition above, the condition that c has recursive image is used to see that W is r.e. This condition can be replaced by the assumption that $\mathrm{dom}(V) \setminus \mathrm{range}(c)$ is r.e.

7 Conjecture for infinite computations

For infinite computations we are going to consider *monotone* Turing machines with infinite inputs. Such a machine has a one-way and write-only output tape (no erasing nor overwriting is possible). Thus, the sequence of symbols written on the output tape increases monotonically with respect to the prefix ordering as the number of computation steps grows. The input/output behavior of a monotone Turing machine is a map $M^\infty : 2^\omega \to 2^{\leq\omega}$, where $2^{\leq\omega} = 2^{<\omega} \cup 2^\omega$, such that

$$M^\infty(Z) = \lim_{t\to\infty} M_t(Z \upharpoonright t)$$

where $M_t : 2^{<\omega} \to 2^{<\omega}$ is total recursive monotone increasing with respect to the prefix ordering on words. For more details on infinite computations, Cf. [2].

For any optimal machine U, and for $\mathcal{X} \subseteq 2^{\leq\omega}$ we define

$$\Omega_U^\infty[\mathcal{X}] = \mu((U^\infty)^{-1}(\mathcal{X})).$$

i.e. $\Omega_U^\infty[\mathcal{X}]$ is the probability that U^∞ gives an output in $\mathcal{X}$.

An analog of Conjecture 1.2 can be stated for infinite computations.

Conjecture 7.1. For any $\mathcal{X} \subsetneq 2^{\leq\omega}$, the probability $\Omega_U^\infty[\mathcal{X}]$ that an arbitrary infinite input to an optimal monotone machine performing infinite computations gives an output in $\mathcal{X}$ is random. And the harder the set $\mathcal{X}$, the more random $\Omega_U^\infty[\mathcal{X}]$.

Relatively to monotone Turing machines which are optimal by adjunction (Cf. Def.6.1), this conjecture has been proved in [2, 1] for many $\mathcal{X} \subseteq 2^{\leq\omega}$, considering the effective levels of the Borel hierarchy on $2^{\leq\omega}$ with a spectral topology (for which the basic open sets are of the form $s2^{\leq\omega}$, for $s \in 2^{<\omega}$).

Theorem 7.2 ([2, 1])**.** *Let $\mathcal{X} \subseteq 2^{\leq\omega}$ be Σ_n^0(spectral) and hard for the class $\Sigma_n^0(2^\omega)$ with respect to effective Wadge reductions, for any $n \geq 1$. Then, $\Omega_U^\infty[\mathcal{X}]$ is n-random.*

However, the conjecture is not always true. The proof uses an adequate version of Lemma 2.6.

Lemma 7.3. *Let U be a monotone prefix Turing machine which is optimal by adjunction. Then $\exists d\, \forall n\, \exists \sigma\ 2^{-n-d} \leq \Omega_U^\infty[\{\sigma\}] \leq 2^{-n+d}$. In fact, for some constant d', there are at least $2^n/(d'\, n^2)$ strings $\sigma \in 2^{<\omega}$ satisfying the inequalities.*

Proof. Fix some total recursive injective function $\theta : 2^{<\omega} \to 2^{<\omega}$ with recursive prefix-free range. Thanks to Lemma 2.6, it suffices to prove that there exists k such that for any $\sigma \in 2^{<\omega}$,

$$2^{-k}\Omega_U[\{\sigma\}] \leq \Omega_U^\infty[\{\theta(\sigma)\}] \leq 2^k \Omega_U[\{\sigma\}],$$

Consider the relation $R \subset 2^{<\omega} \times 2^{<\omega}$ such that $(p, u) \in R$ if and only if the computation of U^∞ on any infinite extension of p has current output u. Let $M : 2^{<\omega} \to 2^{<\omega}$ be the machine such that $M(p)$ halts and outputs σ if and only if $(p, \theta(\sigma)) \in R$ but $(q, \theta(\sigma)) \notin R$ for any proper prefix of p. Clearly, M is partial recursive and has prefix-free domain.

Using optimality by adjunction, let $\tau \in 2^{<\omega}$ be such that $M(p) = U(\tau p)$ for all p. Thus, for any $Z \in 2^\omega$ if $U^\infty(Z) = \theta(\sigma)$, then there exists n such that $U(\tau(Z \restriction n))$ halts and $U(\tau(Z \restriction n)) = \sigma$. Hence,

$$\begin{aligned}
\Omega_U^\infty[\{\theta(\sigma)\}] &\leq \mu(\{Z \in 2^\omega : \exists n\ U(\tau(Z \restriction n)) = \sigma\}) \\
&= \sum_{U(\tau p)=\sigma} 2^{-|p|} \\
&\leq 2^{|\tau|}\, \Omega_U[\{\sigma\}].
\end{aligned}$$

For the other inequality, let $N : 2^{<\omega} \to 2^{<\omega}$ be the machine such that $N(p) = \theta(U(p))$ and let ρ be such that $U(\rho p) = N(p) = \theta(U(p))$. Then

$$\begin{aligned}
\Omega_U[\{\theta(\sigma)\}] &\geq 2^{-|\rho|} \sum_{U(\rho p)=\theta(\sigma)} 2^{-|p|} \\
&= 2^{-|\rho|} \sum_{U(p)=\sigma} 2^{-|p|} \\
&= 2^{-|\rho|}\, \Omega_U[\{\sigma\}].
\end{aligned}$$

To conclude, observe that

$$\Omega_U^\infty[\{\theta(\sigma)\}] \geq \Omega_U[\{\theta(\sigma)\}]$$

and take $k = \max(|\tau|, |\rho|)$. □

From Lemma 7.3, the proofs of Theorems 2.7 and 2.8 adapt easily to Ω_U^∞, giving counterexamples which are included in the subset $2^{<\omega}$ of $2^{\leq\omega}$. However, oracle $\emptyset''$ is needed to check inequalities $\Omega_U^\infty[\{\sigma\}] > \tau$ and check if a given bit of $\Omega_U^\infty[X]$ is zero for finite subsets X of $2^{<\omega}$. Which gives a shift to Δ_3^0. We state the analog of Theorem 2.8.

Theorem 7.4. *For every optimal U and any $A \subseteq \mathbb{N}$, there is a $\Delta_3^{0,A}$ set $X \subseteq 2^{<\omega}$ which is $\Sigma_1^{0,A}$-hard and such that $\Omega_U^\infty[X]$ is not random. In particular, if $n \geq 1$ there is a Δ_{n+2}^0 set $X \subseteq 2^{<\omega}$ which is Σ_n^0-hard and such that $\Omega_U^\infty[X]$ is not random.*

The counterparts of Theorem 4.2 and Proposition 4.3 are as follows.

Theorem 7.5. *Let U be a monotone Turing machine optimal by adjunction.*

1. *The set $\{\Omega_U^\infty[X] : X \subseteq 2^{<\omega}\}$ is the union of finitely many pairwise disjoint closed intervals.*

   For every real s in the above set there exists $X \subseteq 2^{<\omega}$ recursive in $s \oplus \emptyset''$ such that $s = \Omega_U^\infty[X]$.

2. *$\{\Omega_U^\infty[\mathcal{X}] : \mathcal{X} \subseteq 2^{\leq\omega} \ \wedge \ (U^\infty)^{-1}(\mathcal{X})$ is measurable$\}$ $= \{\Omega_U^\infty[\mathcal{X}] : \mathcal{X} \subseteq 2^{\leq\omega} \ \wedge \ \mathcal{X}$ is a Borel set in $2^{\leq\omega}\}$ and this set is equal to the union of finitely many pairwise disjoint closed intervals.*

Proof. 1. Points 1i, 1ii of the proof of Theorem 4.2 adapt easily. To adapt point 1iii, we show that Ω_U^∞ yields a continuous map $P(2^{<\omega}) \to [0,1]$ where $P(2^{<\omega})$ is endowed with the compact Cantor topology.

Observe that, for all $s \in 2^{<\omega}$, the set $(U^\infty)^{-1}(\{s\})$ is a Borel subset of 2^ω. In fact it is the difference of two open sets since $U^\infty(\alpha) = s$ if and only if at all times t, the current output is a prefix of s and at some time it is s. Now, $\Omega_U^\infty[X] = \sum_{x \in X} \mu((U^\infty)^{-1}(\{x\})$ proves that $X \mapsto \Omega_U^\infty[X]$ yields a continuous map $P(2^{<\omega}) \to [0,1]$.

2. The proof uses an argument in the spirit of the Radon-Nikodym theorem. First, observe that

$$(\dagger) \qquad (U^\infty)^{-1}(2^\omega) \text{ and } (U^\infty)^{-1}(\{\xi\}) \text{ are Borel subsets of } 2^\omega,\ \forall \xi \in 2^{\leq\omega}$$

The case $\xi \in 2^{<\omega}$ has been checked above. If $\xi \in 2^\omega$ then $U^\infty(\alpha) = \xi$ (resp. $U^\infty(\alpha) \in 2^\omega$) if and only if for all n there is some time at which the current output is $\xi \upharpoonright n$ (resp. has length n). Which shows that $(U^\infty)^{-1}(\{\xi\})$ (resp. $(U^\infty)^{-1}(2^\omega)$) are G_δ Borel sets. Now, let

$$\begin{array}{rcl} \mathcal{A} & = & 2^{<\omega} \ \cup \ \{\alpha \in 2^\omega : \mu((U^\infty)^{-1}(\{\alpha\})) > 0\}; \\ \mathcal{B} & = & \{\alpha \in 2^\omega : \mu((U^\infty)^{-1}(\{\alpha\})) = 0\}. \end{array}$$

Clearly, $\mathcal{A}$ is countable. We prove that for some finite sequence $0 = a_1 < b_1 < \ldots < a_n < b_n$,

$$\begin{array}{rl} (*) & \{\Omega_U^\infty[\mathcal{X}] : \mathcal{X} \subseteq \mathcal{A}\} = [a_1, b_1] \cup \ldots \cup [a_n, b_n] \\ (**) & \{\Omega_U^\infty[\mathcal{X}] : \mathcal{X} \subseteq \mathcal{B} \ \wedge \ (U^\infty)^{-1}(\mathcal{X}) \text{ is measurable}\} \\ & = \ \{\Omega_U^\infty[\mathcal{X}] : \mathcal{X} \subseteq \mathcal{B} \ \wedge \ \mathcal{X} \text{ is a } G_\delta \text{ Borel set in } 2^{\leq\omega}\} \\ & = \ \text{a closed interval.} \end{array}$$

Point 2 then follows: if $\mathcal{X} \subseteq 2^{\leq\omega}$ is such that $(U^\infty)^{-1}(\mathcal{X})$ is measurable, then, for some $\mathcal{Y} \subseteq \mathcal{B}$ which is G_δ in $2^{\leq\omega}$,

$$\begin{array}{rcl} \Omega_U^\infty[\mathcal{X}] & = & \Omega_U^\infty[\mathcal{X} \cap \mathcal{A}] + \Omega_U^\infty[\mathcal{X} \cap \mathcal{B}] \\ & = & \Omega_U^\infty[\mathcal{X} \cap \mathcal{A}] + \Omega_U^\infty[\mathcal{Y}] \\ & = & \Omega_U^\infty[(\mathcal{X} \cap \mathcal{A}) \cup \mathcal{Y}]). \end{array}$$

But $\mathcal{X} \cap \mathcal{A}$ is countable and $\mathcal{Y}$ is G_δ, so that their union is Borel in $2^{\leq\omega}$.

The proof of (*) is an easy adaptation of that of Theorem 4.2 since Ω_U^∞ yields a continuous map $P(\mathcal{A}) \to [0,1]$ with the compact Cantor topology on $P(\mathcal{A})$.

Let us prove (**). Consider the lexicographic ordering $\prec$ on 2^ω, which is a total ordering, and let $f : 2^\omega \to [0,1]$ be the map such that

$$f(\alpha) = \mu((U^\infty)^{-1}(\mathcal{B} \cap \{\beta : \beta \preceq \alpha\})).$$

Let us see that this map is well defined. Since $\mathcal{A} \cap 2^\omega$ and $\mathcal{B}$ partition 2^ω, we have $(U^\infty)^{-1}(\mathcal{B}) = (U^\infty)^{-1}(2^\omega) \setminus (U^\infty)^{-1}(\mathcal{A} \cap 2^\omega)$. Since $\mathcal{A}$ is countable, (†) shows that $(U^\infty)^{-1}(\mathcal{B})$ is Borel. Also, $U^\infty(\gamma) \in 2^\omega \wedge U^\infty(\gamma) \preceq \alpha$ if and only if for all n there is some time at which the current output has length n and is $\preceq \alpha$. Thus, $(U^\infty)^{-1}(\{\beta : \beta \preceq \alpha\})$ is G_δ. This shows that $(U^\infty)^{-1}(\mathcal{B} \cap \{\beta : \beta \preceq \alpha\})$ is Borel, so that $f(\alpha)$ is the measure of a Borel set, hence is well defined. Facts 1 and 2 below prove that the range of f is a closed interval $[0, c]$. To conclude the proof of $(**)$, observe that

a. If $\mathcal{X} \subseteq \mathcal{B}$ and $(U^\infty)^{-1}(\mathcal{X})$ is measurable then $\Omega_U^\infty[\mathcal{X}] \leq \Omega_U^\infty[\mathcal{B}] = f(1^\omega) = c$. Thus, there is some α such that $\Omega_U^\infty[\mathcal{X}] = f(\alpha) = \Omega_U^\infty[\mathcal{B} \cap \{\beta : \beta \preceq \alpha\}]$.

b. The set $\mathcal{B} \cap \{\beta : \beta \preceq \alpha\}$ is G_δ in 2^ω, hence also G_δ in $2^{\leq\omega}$. In fact, $\mathcal{B}$ is G_δ in 2^ω as the complement of the countable set $\mathcal{A} \cap 2^\omega$, and $\{\beta : \beta \preceq \alpha\}$ is closed in 2^ω.

Fact 1. *1. f is monotone increasing with respect to $\prec$.*

2. $f(\alpha)$ is also equal to $\mu((U^\infty)^{-1}(\mathcal{B} \cap \{\beta : \beta \prec \alpha\}))$.

3. $f(1^\omega) = \mu((U^\infty)^{-1}(\mathcal{B}))$, $f(0^\omega) = 0$, and $f(u01^\omega) = f(u10^\omega)$ for all $u \in 2^{<\omega}$.

4. f is continuous.

Fact 2. *Suppose $g : 2^\omega \to [0,1]$ is a continuous map such that*

$$g(u01^\omega) = g(u10^\omega) \text{ for all } u \in 2^{<\omega}. \tag{12}$$

Then the range of g is a closed interval.

Proof of Fact 1. Point 1 is obvious.

2. Observe that $f(\alpha) - \mu((U^\infty)^{-1}(\mathcal{B} \cap \{\beta : \beta \prec \alpha\})) = \mu((U^\infty)^{-1}(\mathcal{B} \cap \{\alpha\}))$. Now, 2 is obvious if $\alpha \notin \mathcal{B}$. Else, use the definition of $\mathcal{B}$.

3. The assertion about $f(1^\omega)$ is obvious. For $f(0^\omega)$, use 2. Finally, using 2 again, and the fact that $u01^\omega$ is the predecessor of $u10^\omega$, we get

$$\begin{aligned} f(u10^\omega) &= \mu((U^\infty)^{-1}(\mathcal{B} \cap \{\beta : \beta \prec u10^\omega\})) \\ &= \mu((U^\infty)^{-1}(\mathcal{B} \cap \{\beta : \beta \preceq u01^\omega\})) \\ &= f(u01^\omega). \end{aligned}$$

4. It is sufficient to show that if $(\alpha_n)_{n\in\mathbb{N}}$ is a monotone increasing or decreasing sequence in 2^ω with limit α and every α_n is different from α then $f(\alpha) = \lim_n f(\alpha_n)$. If $(\alpha_n)_{n\in\mathbb{N}}$ is increasing with limit α, we have $\beta \prec \alpha$ if and only if $\beta \preceq \alpha_n$ for some n. Thus,

$$\begin{aligned}
f(\alpha) &= \mu((U^\infty)^{-1}(\mathcal{B} \cap \{\beta : \beta \prec \alpha\})) \\
&= \mu((U^\infty)^{-1}(\mathcal{B} \cap \bigcup_{n\in\mathbb{N}} \{\beta : \beta \preceq \alpha_n\})) \\
&= \sup_{n\in\mathbb{N}} \ \mu((U^\infty)^{-1}(\mathcal{B} \cap \{\beta : \beta \preceq \alpha_n\})) \\
&= \sup_{n\in\mathbb{N}} f(\alpha_n).
\end{aligned}$$

In the case where $(\alpha_n)_{n\in\mathbb{N}}$ is decreasing, we argue similarly, using the fact that $\beta \preceq \alpha$ if and only if $\beta \preceq \alpha_n$ for all n. $\square$

Proof of Fact 2. Let $\theta : [0,1] \to (2^\omega \setminus 2^{<\omega}0^\omega) \cup \{0^\omega\}$ be the bijective map such that $\theta(0) = 0^\omega$ and, for $0 < r \leq 1$, $\theta(r)$ is the sequence of dyadic digits of r which lies in $2^\omega \setminus 2^{<\omega}0^\omega$.

Using (12), we see that range(g) = range$(g \circ \theta)$. Since the range of a continuous map $[0,1] \to [0,1]$ is always a closed interval, it suffices to prove that $g \circ \theta$ is continuous. I.e. to prove that if $(r_n)_{n\in\mathbb{N}}$ is a monotone increasing (resp. decreasing) sequence of reals in $[0,1]$ with limit $r > 0$ (resp. $r < 1$) and such that every r_n is different from r, then $g(\theta(r)) = \lim_n g(\theta(r_n))$.

Case r is not dyadic rational. Then $\theta(r)$ is the limit of $\theta(r_n)$ and we can apply the continuity of g.

Case r is dyadic rational and $(r_n)_{n\in\mathbb{N}}$ is increasing. Then the limit of $\theta(r_n)$ is the dyadic expansion of r of the form $u01^\omega$ where $u \in 2^{<\omega}$, which is exactly $\theta(r)$. Again, we apply the continuity of g.

Case r is dyadic rational and $(r_n)_{n\in\mathbb{N}}$ is decreasing. Then the limit of $\theta(r_n)$ is the dyadic expansion of r of the form $u10^\omega$ where $u \in 2^{<\omega}$. Applying the continuity of g, we see that the $\lim_n g(\theta(r_n)) = g(u10^\omega)$. To conclude, observe that $u01^\omega = \theta(r)$ and that (12) insures $g(u10^\omega) = g(u01^\omega)$. $\square$

This completes the proof of the whole result. $\square$

As in the case of finite computations, one can force that $\{\Omega_U^\infty[\mathcal{X}] : \mathcal{X} \subseteq 2^{\leq\omega}\}$ has arbitrarily many disjoint sections.

Proposition 7.6. *Let U be optimal. For each $n \geq 1$, there exists a finite modification V of U which is still optimal and such that neither of the sets $\{\Omega_U^\infty[X] : X \subseteq 2^{<\omega}\}$ and $\{\Omega_U^\infty[\mathcal{X}] : \mathcal{X} \subseteq 2^{\leq\omega}\}$ is the union of less than n intervals.*

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