Compatible Funtions by Priestley Duality

Leonardo M. Cabrer and Marta Sagastume

Introduction

Introduction

- SOFRONIE-STOKKERMANS, V. *Duality and canonical extensions of bounded distributive lattices with operators, and applications to the semantics of non-classical logics I.* Studia Logica 64 (2000), 93-132.
- SOFRONIE-STOKKERMANS, V. *Duality and canonical extensions of bounded distributive lattices with operators, and applications to the semantics of non-classical logics II.* Studia Logica 64 (2000), 151-192.

Introduction

Definition

Given $\mathbf{L}_1, \ldots, \mathbf{L}_n, \mathbf{L}$ bounded lattices, a **join hemimorphism** is a function $f : L_1 \times \ldots \times L_n \rightarrow L$ such that:

Introduction

Definition

Given $\mathbf{L}_1, \ldots, \mathbf{L}_n, \mathbf{L}$ bounded lattices, a **join hemimorphism** is a function $f : L_1 \times \ldots \times L_n \to L$ such that:

1. $f(a_1, \ldots, a_{i-1}, 0, a_{i+1}, \ldots, a_n) = 0$.

Introduction

Definition

Given $\mathbf{L}_1, \ldots, \mathbf{L}_n, \mathbf{L}$ bounded lattices, a **join hemimorphism** is a function $f : L_1 \times \ldots \times L_n \rightarrow L$ such that:

1. $f(a_1, \ldots, a_{i-1}, 0, a_{i+1}, \ldots, a_n) = 0.$
2. $f(a_1, \ldots, a_{i-1}, b_1 \vee b_2, a_{i+1}, \ldots, a_n) =$
   $f(a_1, \ldots, a_{i-1}, b_1, a_{i+1}, \ldots, a_n) \vee f(a_1, \ldots, a_{i-1}, b_2, a_{i+1}, \ldots, a_n).$

Introduction

Definition

Given $\mathbf{L}_1, \ldots, \mathbf{L}_n, \mathbf{L}$ bounded lattices, a **join hemimorphism** is a function $f : L_1 \times \ldots \times L_n \to L$ such that:

1. $f(a_1, \ldots, a_{i-1}, 0, a_{i+1}, \ldots, a_n) = 0$.
2. $f(a_1, \ldots, a_{i-1}, b_1 \vee b_2, a_{i+1}, \ldots, a_n) =$ $f(a_1, \ldots, a_{i-1}, b_1, a_{i+1}, \ldots, a_n) \vee f(a_1, \ldots, a_{i-1}, b_2, a_{i+1}, \ldots, a_n)$.

Definition

Given $\mathbf{L}$ a bounded lattice and $\varepsilon_1, \ldots, \varepsilon_n, \varepsilon \in \{-1, +1\}$. A function $f : L^n \to L$ is and **operator of type** $\varepsilon_1, \ldots, \varepsilon_n \to \varepsilon$ if $f : L^{\varepsilon_1} \times \ldots \times L^{\varepsilon_n} \to L^{\varepsilon}$ is a join hemimorphism.

Let $f : L^n \longrightarrow L$ be an operator of type $\varepsilon_1, \ldots, \varepsilon_n \longrightarrow \varepsilon$.

Introduction

Let $f : L^n \longrightarrow L$ be an operator of type $\varepsilon_1, \ldots, \varepsilon_n \longrightarrow \varepsilon$.

$R_f \subseteq X(\mathbf{L})^{n+1}$ is defined by

Introduction

Let $f : L^n \to L$ be an operator of type $\varepsilon_1, \ldots, \varepsilon_n \to \varepsilon$.

$$R_f \subseteq X(\mathbf{L})^{n+1} \text{ is defined by}$$

$$(P_1, \ldots, P_n, P) \in R_f \text{ iff } f\left(P_1^{\varepsilon_1}, \ldots, P_n^{\varepsilon_n}\right) \subseteq P^{\varepsilon}.$$

Introduction

Introduction

- URQUHART, A. *Distributive Lattices with a Dual Homomorphic Operation*. Studia Logica 39 (1979) 201-209.
- CIGNOLI, R. *Quantifiers on distributive lattices*. Discrete Mathematics 96 (1991) 183-197.
- MARTINEZ, N.G. *A topological duality for some lattice oredered algebraic structures including l-groups*. Algebra universalis 31 (1996), 516-541.
- URQUHART, A. *Duality for Algebras of Relevant Logics*. Studia Logica 56 (1996) 263-276.
- PETROVICH, A. *Equational Calsses of Totally Ordered Modal Lattices*. Order 16 (1999) 1-17.
- CELANI, S.A. *Distributive lattices with a Negation Operator. Mthematical Logic Quaterly 42 (2) (1999) 208-218.*
- CELANI, S.A. *Distributive lattices with fusion and implication*. Southeast Asian Bulletin of Mathematics 28 (2004) 999-1010.

Congruences

Congruences

Definition

Let $P \in X(\mathbf{L})$. $(P_1, \ldots, P_n) \in MaxR_f^{-1}(P)$ if and only the following statements hold:

1. $(P_1, \ldots, P_n, P) \in R_f$.
2. $(P_1, \ldots, P_n)$ is maximal between the elements of $X(\mathbf{L})^{\varepsilon_1} \times \ldots \times X(\mathbf{L})^{\varepsilon_n}$ with the property 1.

Congruences

Definition

Let $P \in X(\mathbf{L})$. $(P_1, \ldots, P_n) \in MaxR_f^{-1}(P)$ if and only the following statements hold:

1. $(P_1, \ldots, P_n, P) \in R_f$.
2. $(P_1, \ldots, P_n)$ is maximal between the elements of $X(\mathbf{L})^{\varepsilon_1} \times \ldots \times X(\mathbf{L})^{\varepsilon_n}$ with the property 1.

Definition

A set closed $Y \subseteq X(\mathbf{L})$ is called R_f-**closed** if and only if $MaxR_f^{-1}(P) \subseteq Y^n$ for each $P \in Y$.

Congruences

Theorem

The lattice of R_f-closed subsets of $X(\mathbf{L})$ is dually isomorphic to the lattice of congruences of $\langle L, \wedge, \vee, 0, 1, f\rangle$.

Subdirectly irreducible and simple algebras

If $Y \subseteq X(\mathbf{L})$, we define Y_n by

$$Y_0 = Y$$
$$Y_{n+1} = Y_n \cup MaxR_f^{-1}(Y_n).$$

Subdirectly irreducible and simple algebras

If $Y \subseteq X(\mathbf{L})$, we define Y_n by

$$Y_0 = Y$$
$$Y_{n+1} = Y_n \cup MaxR_f^{-1}(Y_n).$$

Consider the set

$$Y_\omega = \bigcup_{n \in \mathbb{N}} Y_n.$$

Subdirectly irreducible and simple algebras

Subdirectly irreducible and simple algebras

Theorem

An algebra $\langle L, \wedge, \vee, 0, 1, f\rangle$ *is simple if and only if for each* $P \in X(\mathbf{L})$, $\{P\}_\omega$ *is dense in* $\mathcal{X}(\mathbf{L})$.

Subdirectly irreducible and simple algebras

Theorem

An algebra $\langle L, \wedge, \vee, 0, 1, f \rangle$ *is simple if and only if for each* $P \in X(\mathbf{L})$, $\{P\}_\omega$ *is dense in* $\mathcal{X}(\mathbf{L})$.

Theorem

An algebra $\langle L, \wedge, \vee, 0, 1, f \rangle$ *is subdirectly irreducible but non simple if and only if one of the following statements hold:*

Subdirectly irreducible and simple algebras

Theorem

An algebra $\langle L, \wedge, \vee, 0, 1, f \rangle$ *is simple if and only if for each* $P \in X(\mathbf{L})$, $\{P\}_{\omega}$ *is dense in* $\mathcal{X}(\mathbf{L})$.

Theorem

An algebra $\langle L, \wedge, \vee, 0, 1, f \rangle$ *is subdirectly irreducible but non simple if and only if one of the following statements hold:*

1. $\left\{P \in X(\mathbf{L}) : \left(MaxR_f^{-1}(P)\right)_{\omega} \text{ is dense in } \mathcal{X}(\mathbf{L})\right\}$ *is a non empty open proper subset of* $\mathcal{X}(\mathbf{L})$.

Subdirectly irreducible and simple algebras

Theorem

An algebra $\langle L, \wedge, \vee, 0, 1, f\rangle$ *is simple if and only if for each* $P \in X(\mathbf{L})$, $\{P\}_{\omega}$ *is dense in* $\mathcal{X}(\mathbf{L})$.

Theorem

An algebra $\langle L, \wedge, \vee, 0, 1, f\rangle$ *is subdirectly irreducible but non simple if and only if one of the following statements hold:*

1. $\left\{P \in X(\mathbf{L}) : \left(MaxR_f^{-1}(P)\right)_{\omega} \text{ is dense in } \mathcal{X}(\mathbf{L})\right\}$ *is a non empty open proper subset of* $\mathcal{X}(\mathbf{L})$.
2. *There exists* $P \in X(\mathbf{L})$, *such that* $P \notin Cl\left(MaxR_f^{-1}(P)_{\omega}\right) \neq \emptyset$ *and* $X(\mathbf{L}) = \{P\} \cup Cl\left(\left(MaxR_f^{-1}(P)\right)_{\omega}\right)$.

Applications: Compatible operators on distributive lattices

Applications: Compatible operators on distributive lattices

Theorem

Let $f : L^n \longrightarrow L$ be an operator of type $\varepsilon_1, \ldots, \varepsilon_n \longrightarrow \varepsilon$. The following are equivalent:

1. *f is a compatible with the congruences of the lattice* $\mathbf{L}$.

Applications: Compatible operators on distributive lattices

Theorem

Let $f : L^n \to L$ be an operator of type $\varepsilon_1, \ldots, \varepsilon_n \to \varepsilon$. The following are equivalent:

1. *f is a compatible with the congruences of the lattice* $\mathbf{L}$.
2. *For every $Y \subseteq X(\mathbf{L})$ closed and every $P \in Y$ it holds that $MaxR_f^{-1}(P) \subseteq Y^n$.*

Applications: Compatible operators on distributive lattices

Theorem

Let $f : L^n \to L$ be an operator of type $\varepsilon_1, \ldots, \varepsilon_n \to \varepsilon$. The following are equivalent:

1. *f is a compatible with the congruences of the lattice $\mathbf{L}$.*
2. *For every $Y \subseteq X(\mathbf{L})$ closed and every $P \in Y$ it holds that $MaxR_f^{-1}(P) \subseteq Y^n$.*
3. *$MaxR_f^{-1}(P) \subseteq \{P\}^n$ for each $P \in X(\mathbf{L})$.*

Applications: Compatible operators on distributive lattices

Theorem

Let $f : L^n \to L$ be an operator of type $\varepsilon_1, \ldots, \varepsilon_n \to \varepsilon$. The following are equivalent:

1. *f is a compatible with the congruences of the lattice $\mathbf{L}$.*
2. *For every $Y \subseteq X(\mathbf{L})$ closed and every $P \in Y$ it holds that $MaxR_f^{-1}(P) \subseteq Y^n$.*
3. *$MaxR_f^{-1}(P) \subseteq \{P\}^n$ for each $P \in X(\mathbf{L})$.*
4. *$(P_1, \ldots, P_n, P) \in R_f$ if and only if $(P_1, \ldots, P_n) \leq (P, \ldots, P)$ in $X(\mathbf{L})^{\varepsilon_1} \times \ldots \times X(\mathbf{L})^{\varepsilon_n}$ and $(P, \ldots, P, P) \in R_f$.*

Applications: Compatible operators on distributive lattices

Applications: Compatible operators on distributive lattices

Theorem

Let $f : L^n \to L$ be a join hemimorphism. Then the following are equivalent:

1. *f is compatible.*
2. $f(a_1, \ldots, a_n) = a_1 \wedge \ldots \wedge a_n \wedge f(1, \ldots, 1)$.

Applications: Compatible operators on distributive lattices

Theorem

Let $f : L^n \to L$ be a join hemimorphism. Then the following are equivalent:

1. *f is compatible.*
2. $f(a_1, \ldots, a_n) = a_1 \wedge \ldots \wedge a_n \wedge f(1, \ldots, 1)$.

Theorem

Let $f : L^n \to L$ be a meet hemimorphism (type $-1, \ldots, -1 \to -1$). Then the following are equivalent:

1. *f is compatible.*
2. $f(a_1, \ldots, a_n) = a_1 \vee \ldots \vee a_n \vee f(0, \ldots, 0)$.

Applications: Compatible operators on distributive lattices

Applications: Compatible operators on distributive lattices

Theorem

Let f be compatible operator of type $\varepsilon_1, \ldots, \varepsilon_n \rightarrow \varepsilon$ over a distributive lattice $\mathbf{L}$ such that $X_{1,f}(\mathbf{L}) = X(\mathbf{L})$, thus if there exists $1 \leq i \leq n$ such that $\varepsilon_i \neq \varepsilon$, then $\mathbf{L}$ is a boolean lattice.

Applications: Compatible operators on distributive lattices

Theorem

Let f be compatible operator of type $\varepsilon_1, \ldots, \varepsilon_n \longrightarrow \varepsilon$ over a distributive lattice $\mathbf{L}$ such that $X_{1,f}(\mathbf{L}) = X(\mathbf{L})$, thus if there exists $1 \leq i \leq n$ such that $\varepsilon_i \neq \varepsilon$, then $\mathbf{L}$ is a boolean lattice.

Corollary

Let $\langle L, \wedge, \vee, \neg, 0, 1 \rangle$ be a pseudocomplemented lattice. Thus the following are equivalent:

1. *$\neg$ is compatible with the congruences of its lattice reduct.*
2. *$\langle L, \wedge, \vee, \neg, 0, 1 \rangle$ is a boolean algebra*

Applications: Compatible operators on Heyting algebras

Applications: Compatible operators on Heyting algebras

Theorem

Let $f : H^n \to H$ of type $\varepsilon_1, \ldots, \varepsilon_n \to \varepsilon$. The following statements are equivalent:

1. *f is a compatible.*

Applications: Compatible operators on Heyting algebras

Theorem

Let $f : H^n \to H$ of type $\varepsilon_1, \ldots, \varepsilon_n \to \varepsilon$. The following statements are equivalent:

1. *f is a compatible.*
2. *For every $Y \subseteq X(\mathbf{H})$ closed and increasing and every $P \in Y$ it holds that $MaxR_f^{-1}(P) \subseteq Y^n$.*

Applications: Compatible operators on Heyting algebras

Theorem

Let $f : H^n \to H$ *of type* $\varepsilon_1, \ldots, \varepsilon_n \to \varepsilon$. *The following statements are equivalent:*

1. f *is a compatible.*
2. *For every* $Y \subseteq X(\mathbf{H})$ *closed and increasing and every* $P \in Y$ *it holds that* $MaxR_f^{-1}(P) \subseteq Y^n$.
3. $MaxR_f^{-1}(P) \subseteq [P)^n$ *for each* $P \in X(\mathbf{H})$.

Applications: Compatible operators on Heyting algebras

Theorem

Let $f : H^n \to H$ of type $\varepsilon_1, \ldots, \varepsilon_n \to \varepsilon$. The following statements are equivalent:

1. *f is a compatible.*
2. *For every $Y \subseteq X(\mathbf{H})$ closed and increasing and every $P \in Y$ it holds that $MaxR_f^{-1}(P) \subseteq Y^n$.*
3. *$MaxR_f^{-1}(P) \subseteq [P)^n$ for each $P \in X(\mathbf{H})$.*
4. *If $(P_1, \ldots, P_n, P) \in R_f$, then the following propositions hold:*

Applications: Compatible operators on Heyting algebras

Theorem

Let $f : H^n \to H$ of type $\varepsilon_1, \ldots, \varepsilon_n \to \varepsilon$. The following statements are equivalent:

1. *f is a compatible.*
2. *For every $Y \subseteq X(\mathbf{H})$ closed and increasing and every $P \in Y$ it holds that $MaxR_f^{-1}(P) \subseteq Y^n$.*
3. *$MaxR_f^{-1}(P) \subseteq [P)^n$ for each $P \in X(\mathbf{H})$.*
4. *If $(P_1, \ldots, P_n, P) \in R_f$, then the following propositions hold:*
    1. *If $\varepsilon_i = -1$, then $P \subseteq P_i$ and*

Applications: Compatible operators on Heyting algebras

Theorem

Let $f : H^n \to H$ of type $\varepsilon_1, \ldots, \varepsilon_n \to \varepsilon$. The following statements are equivalent:

1. *f is a compatible.*
2. *For every $Y \subseteq X(\mathbf{H})$ closed and increasing and every $P \in Y$ it holds that $MaxR_f^{-1}(P) \subseteq Y^n$.*
3. *$MaxR_f^{-1}(P) \subseteq [P)^n$ for each $P \in X(\mathbf{H})$.*
4. *If $(P_1, \ldots, P_n, P) \in R_f$, then the following propositions hold:*
    1. *If $\varepsilon_i = -1$, then $P \subseteq P_i$ and*
    2. *if $\varepsilon_i = 1$, then there exists P_i' such that $P_i \cup P \subseteq P_i'$ and $(P_1, \ldots, P_i', \ldots, P_n, P) \in R_f$.*

Applications: Compatible operators on Heyting algebras

Applications: Compatible operators on Heyting algebras

Theorem

Let $f : H^n \to H$ of type $\varepsilon_1, \ldots, \varepsilon_n \to \varepsilon$. If $\varepsilon = -1$, then the following statements are equivalent:

1. *f is a compatible function.*

Applications: Compatible operators on Heyting algebras

Theorem

Let $f : H^n \to H$ *of type* $\varepsilon_1, \ldots, \varepsilon_n \to \varepsilon$*. If* $\varepsilon = -1$*, then the following statements are equivalent:*

1. f *is a compatible function.*
2. *For every* $a_1, \ldots, a_n, b \in H$ *the following propositions hold:*

Applications: Compatible operators on Heyting algebras

Theorem

Let $f : H^n \longrightarrow H$ of type $\varepsilon_1, \ldots, \varepsilon_n \longrightarrow \varepsilon$. If $\varepsilon = -1$, then the following statements are equivalent:

1. *f is a compatible function.*
2. *For every $a_1, \ldots, a_n, b \in H$ the following propositions hold:*
    1. *If $\varepsilon_i = -1$, then $a_i \leq f(a_1, \ldots, a_n)$,*

Applications: Compatible operators on Heyting algebras

Theorem

Let $f : H^n \to H$ of type $\varepsilon_1, \ldots, \varepsilon_n \to \varepsilon$. If $\varepsilon = -1$, then the following statements are equivalent:

1. *f is a compatible function.*
2. *For every $a_1, \ldots, a_n, b \in H$ the following propositions hold:*
   1. *If $\varepsilon_i = -1$, then $a_i \leq f(a_1, \ldots, a_n)$,*
   2. *if $\varepsilon_i = 1$, then $f(a_1, \ldots, a_i \wedge b, \ldots, a_n) \wedge b \leq f(a_1, \ldots, a_n)$.*

Applications: Compatible operators on Heyting algebras

Theorem

Let $f : H^n \to H$ of type $\varepsilon_1, \ldots, \varepsilon_n \to \varepsilon$. If $\varepsilon = -1$, then the following statements are equivalent:

1. *f is a compatible function.*
2. *For every $a_1, \ldots, a_n, b \in H$ the following propositions hold:*
   1. *If $\varepsilon_i = -1$, then $a_i \leq f(a_1, \ldots, a_n)$,*
   2. *if $\varepsilon_i = 1$, then $f(a_1, \ldots, a_i \wedge b, \ldots, a_n) \wedge b \leq f(a_1, \ldots, a_n)$.*

If $\varepsilon = 1$, then the following statements are equivalent:

1. *f is a compatible function.*

Applications: Compatible operators on Heyting algebras

Theorem

Let $f : H^n \to H$ *of type* $\varepsilon_1, \ldots, \varepsilon_n \to \varepsilon$. *If* $\varepsilon = -1$, *then the following statements are equivalent:*

1. f *is a compatible function.*
2. *For every* $a_1, \ldots, a_n, b \in H$ *the following propositions hold:*
    1. *If* $\varepsilon_i = -1$, *then* $a_i \leq f(a_1, \ldots, a_n)$,
    2. *if* $\varepsilon_i = 1$, *then* $f(a_1, \ldots, a_i \wedge b, \ldots, a_n) \wedge b \leq f(a_1, \ldots, a_n)$.

If $\varepsilon = 1$, *then the following statements are equivalent:*

1. f *is a compatible function.*
2. *For every* $a_1, \ldots, a_n, b \in H$ *the following propositions hold:*

Applications: Compatible operators on Heyting algebras

Theorem

Let $f : H^n \to H$ of type $\varepsilon_1, \ldots, \varepsilon_n \to \varepsilon$. If $\varepsilon = -1$, then the following statements are equivalent:

1. *f is a compatible function.*
2. *For every $a_1, \ldots, a_n, b \in H$ the following propositions hold:*
    1. *If $\varepsilon_i = -1$, then $a_i \leq f(a_1, \ldots, a_n)$,*
    2. *if $\varepsilon_i = 1$, then $f(a_1, \ldots, a_i \wedge b, \ldots, a_n) \wedge b \leq f(a_1, \ldots, a_n)$.*

If $\varepsilon = 1$, then the following statements are equivalent:

1. *f is a compatible function.*
2. *For every $a_1, \ldots, a_n, b \in H$ the following propositions hold:*
    1. *If $\varepsilon_i = -1$, then $a_i \wedge f(a_1, \ldots, a_n) = 0$,*

Applications: Compatible operators on Heyting algebras

Theorem

Let $f : H^n \to H$ of type $\varepsilon_1, \ldots, \varepsilon_n \to \varepsilon$. If $\varepsilon = -1$, then the following statements are equivalent:

1. *f is a compatible function.*
2. *For every $a_1, \ldots, a_n, b \in H$ the following propositions hold:*
    1. *If $\varepsilon_i = -1$, then $a_i \leq f(a_1, \ldots, a_n)$,*
    2. *if $\varepsilon_i = 1$, then $f(a_1, \ldots, a_i \wedge b, \ldots, a_n) \wedge b \leq f(a_1, \ldots, a_n)$.*

If $\varepsilon = 1$, then the following statements are equivalent:

1. *f is a compatible function.*
2. *For every $a_1, \ldots, a_n, b \in H$ the following propositions hold:*
    1. *If $\varepsilon_i = -1$, then $a_i \wedge f(a_1, \ldots, a_n) = 0$,*
    2. *if $\varepsilon_i = 1$, then $f(a_1, \ldots, a_n) \wedge b \leq f(a_1, \ldots, a_i \wedge b, \ldots, a_n)$.*

Applications: Negations

Applications: Negations

A negation an operator f of type $+1 \to -1$ or type $-1 \to +1$.

Applications: Negations

A negation an operator f of type $+1 \to -1$ or type $-1 \to +1$.

Theorem

Let $f : H \to H$ function of type $+1 \to -1$. Then the following are equivalent:

1. *f is a compatible function.*
2. *For every $a, b \in H$, $f(a \wedge b) \wedge b \leq f(a)$.*

Applications: Negations

A negation an operator f of type $+1 \to -1$ or type $-1 \to +1$.

Theorem

Let $f : H \to H$ function of type $+1 \to -1$. Then the following are equivalent:

1. *f is a compatible function.*
2. *For every $a, b \in H$, $f(a \wedge b) \wedge b \leq f(a)$.*

Theorem

Let $f : H \to H$ function of type -1 $\to +1$. Then the following are equivalent:

1. *f is a compatible function.*
2. *For every $a \in H$, $a \wedge f(a) = 0$.*

Applications: Meet hemimorphisms

Applications: Meet hemimorphisms

Theorem

Let $f : H^n \rightarrow H$ a meet hemimorphism. Then the following are equivalent:

1. *f is a compatible function.*
2. *For every $a_1, \ldots, a_n, b \in H$, $a_1 \vee \ldots \vee a_n \leq f(a_1, a_2, \ldots, a_n)$.*

Open Problems

Open Problems

- Compatibility between operators.
- Compatible operators on residuated lattices.
- Subalgebras.

e-mails: lcabrer@exa.unicen.edu.ar
marta@mate.unlp.edu.ar