Computably enumerable, strongly jump-traceable reals

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Traces

Definition (Terwijn, Zambella)

A trace for a (partial) function $f\colon \omega \to \omega$ is a sequence of finite sets $\langle F_x \rangle$ such that for all $x \in \operatorname{dom} f$,

$$f(x) \in F_x.$$

A trace is computable if the sequence of (canonical indexes for the) finite sets is computable. A trace is c.e. if the sequence of finite sets is uniformly c.e.

Orders

Definition

An order is a computable, non-decreasing, and unbounded function $h\colon \omega \to \omega$.

A trace $\langle F_x \rangle$ for a function f respects an order h if for all x,

$$|F_x| \leqslant h(x).$$

COMPUTABLE TRACEABILITY

DEFINITION

A Turing degree **a** is computably traceable if there is some order *h* such that every (total) $f \leqslant_T \mathbf{a}$ has a computable trace which respect *h*.

THEOREM (TERWIJN, ZAMBELLA, KJOS-HANSSEN)

A degree **a** *is computably traceable iff it is low for Schnorr randomness.*

There are $2^{\aleph_0}$ many computably traceable degrees. They are all hymperimmune-free (or **0**-dominated) and so none are Δ^0_2.

C.E. TRACEABILITY

DEFINITION

A degree is c.e. traceable if there is some order h such that every (total) $f \leqslant_T \mathbf{a}$ has a c.e. trace which respects h.

THEOREM (ISHMUKHAMETOV)

A c.e. degree is c.e. traceable iff it is array computable. As a result, a c.e. degree has a strong minimal cover iff it is array computable.

THEOREM (STEPHAN)

A degree is computably traceable iff it is both c.e. traceable and hyperimmune-free.

Strong traceability

Let $\Gamma \in \{\text{c.e.}, \text{computably}\}$.

Definition

A degree **a** is strongly Γ-traceable if for *every* order h, every $f \leqslant_T$ **a** has a Γ-trace which respects h.

Theorem (Terwijn, Zembella)

A degree is Γ-traceable iff it is strongly Γ-traceable.

JUMP-TRACEABILITY

DEFINITION (NIES)

A degree **a** is jump-traceable if there is an order h such that every function which is *partial* computable in **a** has a c.e. trace which respects h.

THEOREM (NIES)

1. *There are* $2^{\aleph_0}$ *many jump-traceable degrees.*
2. *Every* K*-trivial degree is jump-traceable.*
3. *On the c.e. degrees, superlowness coincides with jump-traceability. They differ on the* ω*-c.e. degrees.*

Strong jump-traceability

Definition (Figueira, Nies, Stephan)

A degree **a** is strongly jump-traceable if for all orders h, every function which is partial computable in **a** has a c.e. trace which respects h.

Figueira, Nies and Stephan showed that not every jump-traceable degree is strongly jump-traceable.

Theorem (Figueira, Nies, Stephan)

A set A has strongly jump-traceable degree iff it is "almost low for C" in the sense that for every order h, for almost all x,

$$C(x) - C^A(x) \leqslant h(C^A(x)).$$

Existence

Figueira, Nies and Stephan proved that there is a non-computable, c.e. strongly jump-traceable set.

Fix a slow-growing order h, and let us enumerate a set A which will be jump-traceable respecting h. The requirements to meet are:

$$P_e\colon\ A \neq \bar{W}_e.$$
$$N_e\colon\ \text{Trace } J^A(e) \text{ with fewer than } h(e) \text{ many errors.}$$

Here J^A is the universal A-partial computable function.

In the beginning, the requirements are ordered thus:

$$\underbrace{N_0\, N_1\, N_2\, \cdots\, N_e\, \cdots}_{h(e)=1}\; P_0\; \underbrace{\cdots\, N_e\, \cdots}_{h(e)=2}\; P_1\; \underbrace{\cdots\, N_e\, \cdots}_{h(e)=3}\; P_2\; \cdots$$

This works because each positive requirement acts at most once.

A FAILED CONSTRUCTION

Suppose we wanted more. Let us try to build a perfect Π^0_1 class of h-jump-traceable sets.

The negative requirement N_e make sure that the width of the tree we build, at the level at which all $J^X(e)$ computations already appear, is at most $h(e)$. The positive requirements add splits.
However, if a traced value for $J^X(e)$ is cut off, we cannot take it out of the trace. Thus N_e has to become stronger and requires a narrower tree at its level.

The turn-around

Theorem (Downey, G)
Every strongly jump-traceable set is Δ^0_2.

Thus there are only countably many.

The c.e. case

Theorem (Cholak, Downey, G)

In the c.e. degrees, the strongly jump-traceable degrees form a proper sub-ideal of the K-trivial degrees.

A CONJECTURE

Every strongly jump-traceable degree is computable from a c.e. one. Or at least one which is h-jump-traceable (given a slow order h.) As a result, every strongly jump-traceable set is K-trivial.

Other classes?

Some other classes, possibly smaller than the K-trivials:

- ML non-cuppable degrees: no incomplete Martin-Löf random joins them above $\mathbf{0}'$.
- ML coverable degrees: they are computable from an incomplete Martin-Löf random real.
- Degrees which are computable from all almost complete random reals.

Theorem (Cholak, Downey, G)

Every c.e., strongly jump-traceable degree is ML non-cuppable.

NO LEAST ORDER

THEOREM (NG)

For every order h there is an order h' and a set A which is h-jump-traceable but not h'-jump-traceable.

HIGHNESS

Using Jockusch-Shore pseudo jump inversions, we see that the following classes of c.e. degrees **a** are distinct:

- $\mathbf{0}'$-tracing (or "ultra-high"): $\mathbf{0}'$ is strongly jump-traceable relative to **a**;
- Almost complete (or "$\mathbf{0}'$-trivialising"): $\mathbf{0}'$ is K-trivial over **a**;
- Superhigh degrees.

HIGHNESS AND CAPPING

THEOREM (NG)
There is a cappable $\mathbf{0}'$-tracing c.e. degree.

THEOREM (NG; SHORE)
There is a minimal pair of superhigh c.e. degrees.

We conjecture that there is no minimal pair of c.e. *cero*$'$-tracing degrees. This is related to cone-avoidance questions about uniform a.e. domination and pseudo-jump inversions.