

Low upper bounds of ideals

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The main result

- ▶ There is a low T -upper bound for the class of K -trivials
- ▶ Ideals in Δ_2^0 degrees which have a low T -upper bound

Standard notation

2^ω denotes the set of infinite binary sequences (the Cantor set)

$2^{<\omega}$ denotes the set of finite binary strings

Σ_1^0 classes, Π_1^0 classes

relativized Π_1^0 classes ($\Pi_1^{0,A}$ classes)

Algorithmic randomness

K denotes prefix-free Kolmogorov complexity

$\{\mathcal{U}_n : n \in \omega\}$ denotes universal ML test

concepts of 1-randomness and their relativizations

Algorithmic weakness

There are several notions of computational weakness related to 1-randomness

Definition

1. \mathcal{L} denotes the class of sets which are low for 1-randomness, i.e. sets A such that every 1-random set is also 1-random relative to A .
2. \mathcal{K} denotes the class of K -trivial sets, i.e. the class of sets A such that for all n , $K(A \upharpoonright n) \leq K(0^n) + O(1)$.
3. \mathcal{M} denotes the class of sets that are low for K , i.e. sets A such that for all σ , $K(\sigma) \leq K^A(\sigma) + O(1)$.
4. A set A is a basis for 1-randomness if $A \leq_T Z$ for some Z such that Z is 1-random relative to A . The collection of such sets is denoted by \mathcal{B} .

Theorem (Nies, Hirschfeldt, Stephan)

$$\mathcal{K} = \mathcal{L} = \mathcal{M} = \mathcal{B}$$

More precisely:

- ▶ Nies: $\mathcal{L} = \mathcal{M}$
- ▶ Hirschfeldt, Nies: $\mathcal{K} = \mathcal{M}$
- ▶ Hirschfeldt, Nies, Stephan: $\mathcal{K} = \mathcal{B}$

Four different characterizations of the same class!

However, these characterizations yield different information content

Basic facts about \mathcal{K}

- ▶ $\mathcal{K} \subseteq \Delta_2^0$
- ▶ $\mathcal{K} \subseteq L_1$ (i.e. K -trivials are low)

More precisely:

- ▶ Chaitin: $\mathcal{K} \subseteq \Delta_2^0$
- ▶ A.K.: $\mathcal{L} \subseteq GL_1$ (thus, $\mathcal{L} = \mathcal{K} \subseteq L_1$)

Nowadays there are easier ways to prove lowness of K -trivials

Theorem (Nies; Downey,Hirschfeldt,Nies,Stephan)

- ▶ *r.e. K -trivial sets induce a Σ_3^0 ideal in the r.e. T -degrees*
- ▶ *K -trivial sets induce an ideal in the ω -r.e. T -degrees generated by its r.e. members (in fact, a Σ_3^0 ideal in the ω -r.e. T -degrees)*

Theorem (Downey,Hirschfeldt,Nies,Stephan; Nies)

- ▶ *There is an effective sequence $\{B_e, d_e\}_e$ of all the r.e. K -trivial sets and of constants such that each B_e is K -trivial via d_e*
- ▶ *There is no effective sequence $\{B_e, c_e\}_e$ of all the r.e. low for K sets with appropriate constants*
- ▶ *There is no effective way to obtain from a pair (B, d) , where B is an r.e. set that is K -trivial via d , a constant c such that B is low for K via c*
- ▶ *There is no effective listing of all the r.e. K -trivial sets together with their low indices*

Theorem (Nies)

For each low r.e. set B , there is an r.e. K -trivial set A such that $A \not\leq_T B$.

Thus, no low r.e. set can be a T -upper bound for the class \mathcal{K} .

Comment

The proof uses Robinson low guessing technique which is compatible for r.e. sets with a technique **do what is cheap**.

Cheap is defined

- ▶ either by a cost function in case of K -trivials,
- ▶ or by having a small measure in case of low for random sets.

However, in the more general case of Δ_2^0 instead of r.e. sets, the Robinson low guessing technique does not seem to be compatible with a technique **do what is cheap**. In fact, it is not.

Since all K -trivials are low and every K -trivial set is recursive in some r.e. K -trivial set, we have, as a corollary, that the ideal (induced by) \mathcal{K} is nonprincipal (in the Δ_2^0 T -degrees)

A more general result.

Theorem (Nies)

For any effective listing $\{B_e, z_e\}_e$ of low r.e. sets and of their low indices there is an r.e. K -trivial set A such that $A \not\leq_T B_e$ for all e .

This result is, in fact, used to prove that there is no effective way to obtain low indices of (r.e.) K -trivial sets

Theorem (Nies)

- ▶ *There is a low_2 r.e. set which is a T -upper bound for the class of K -trivials.*
- ▶ *Any proper Σ_3^0 ideal in the r.e. T -degrees has a low_2 r.e. T -upper bound*

Question

Is there a low Δ_2^0 T -upper bound for the class \mathcal{K} ?

The problem is mentioned:

- ▶ Nies, Computability and randomness, draft: question 5.46
- ▶ Miller, Nies, Open Questions: Question 4.3.
- ▶ AIM/ARCC Open Problem List: Problem 3.6.

Several attempts: Nies, Downey, A.K., Barmpalias, ...

Theorem (Yates)

For any r.e. set A TFAE:

1. $A'' \equiv_T \emptyset''$
2. $\{x : W_x \leq_T A\}$ is a Σ_3^0 set
3. the class $\{W_x : W_x \leq_T A\}$ is uniformly r.e.

Together with a result of Nies, this provides information about the existence of low₂ r.e. T -upper bounds for Σ_3^0 ideals in the r.e. T -degrees

Open

A characterization of Σ_3^0 ideals in the r.e. T -degrees for which there is a low T -upper bound, not necessarily r.e.(!)
(similarly for ideals in Δ_2^0 T -degrees)

Theorem

Let \mathcal{C} be a Σ_3^0 ideal in the r.e. T -degrees. Then TFAE:

1. there is a function F recursive in \emptyset' which dominates all partial functions recursive in any member of the ideal \mathcal{C} ,
2. there is a low T -upper bound for \mathcal{C}

A slightly more general result.

Theorem

Let \mathcal{C} be an ideal in Δ_2^0 T -degrees such that there is a uniformly recursively in \emptyset' a sequence of sets $\{A_n\}_n$ which generate the ideal \mathcal{C} . Then TFAE:

1. there is a function F recursive in \emptyset' which dominates all partial functions recursive in any member of the ideal \mathcal{C} ,
2. there is a low T -upper bound for \mathcal{C}

Corollary

There is a low T -upper bound for the class \mathcal{K} (the class of K -trivials).

Proof

Nies proved that the ideal (induced by) \mathcal{K} is generated by its r.e. members and r.e. K -trivial sets induce a Σ_3^0 ideal in the r.e. T -degrees.

A.K. proved that there is a function F recursive in \emptyset' which dominates all partial functions recursive in any member of \mathcal{K} . Thus, the result follows from the previous Theorem.

Remark

We may equivalently require that a low T -upper bound (in the above) is PA since every low set has a low PA set T -above it. Thus, T -upper bounds which are PA are the most general case in this characterization

Proof

1 \rightarrow 2 (the opposite direction is trivial)

Given \mathcal{C} , $\{A_n\}_n$, and F recursive in $0'$ eventually dominating every function recursive in an A_n , we want a T -upper bound A of \mathcal{C} .

Obstacle

Lowness of A

versus

$A \geq_T A_n$ without having low indices for the A_n .

- ▶ A possible solution: use of Π_1^{0, A_n} -classes.
- ▶ Question: how to code into Π_1^{0, A_n} -classes.
- ▶ Answer: use rich Π_1^{0, A_n} -classes.

We replace the missing lowness indices of A_n by guesses relative to the function F . Properties of F guarantee that in a finite injury style an infinite path through Π_1^{0, A_n} classes is found (more details on that later)

Definition

Let $\mathcal{PA}(B)$ denote the class of $\{0, 1\}$ -valued B -DNR functions, i.e. the class of functions $f \in 2^\omega$ such that $f(x) \neq \Phi_x(B)(x)$ for all x . If B is \emptyset we simply speak of \mathcal{PA} .

Definition (Simpson)

$\mathbf{b} \ll \mathbf{a}$ means that every infinite tree $T \subseteq 2^{<\omega}$ of degree $\leq \mathbf{b}$ has an infinite path of degree $\leq \mathbf{a}$.

Theorem (D. Scott and others)

The following conditions are equivalent:

1. \mathbf{a} is a degree of a $\{0, 1\}$ -DNR function
2. $\mathbf{a} \gg \mathbf{0}$
3. \mathbf{a} is a degree of a complete extension of \mathcal{PA}
4. \mathbf{a} is a degree of a set separating some effectively inseparable pair of r.e. sets.

Remark

1. \mathcal{PA} is a kind of a “universal” Π_1^0 class
2. $\{0,1\}$ -valued DNR functions are also called PA sets and degrees $\gg \mathbf{0}$ are called PA degrees.
3. Similarly for $\mathcal{PA}(B)$.

Fact (Simpson)

1. The partial ordering \ll is dense
2. $\mathbf{a} \ll \mathbf{b}$ implies $\mathbf{a} < \mathbf{b}$

Definition

Let M be an infinite set and $\{m_0, m_1, m_2, \dots\}$ be an increasing list of all members of M .

- ▶ If $f \in 2^\omega$ then by $\text{Restr}(f, M)$ we denote $g \in 2^\omega$ defined for all i by $g(i) = f(m_i)$
- ▶ Similarly, if $\mathcal{A} \subseteq 2^\omega$ then by $\text{Restr}(\mathcal{A}, M)$ we denote a class of functions $\{g : g = \text{Restr}(f, M) \wedge f \in \mathcal{A}\}$

Lemma (A.K.)

- ▶ For every Π_1^0 class $\mathcal{A} \subseteq \mathcal{P}\mathcal{A}$ there is an infinite recursive set M such that if \mathcal{A} is nonempty then $\text{Restr}(\mathcal{A}, M) = 2^\omega$, i.e. for every $g \in 2^\omega$ there is a function $f \in \mathcal{A}$ such that $\text{Restr}(f, M) = g$.
- ▶ For every $\Pi_1^{0,B}$ class $\mathcal{A} \subseteq \mathcal{P}\mathcal{A}(B)$ there is an infinite recursive set M such that if \mathcal{A} is nonempty then $\text{Restr}(\mathcal{A}, M) = 2^\omega$, where (an index of) M can be found uniformly from an index of \mathcal{A} , i.e. it does not depend on B .

Remark

This is basically Gödel incompleteness phenomenon.

The Lemma is crucial for coding into $\Pi_1^{0,B}$ classes which are subclasses of $\mathcal{PA}(B)$

We may

- ▶ code either an individual set C (by $\text{Restr}(\mathcal{A}, M) = \{C\}$)
- ▶ or nest another class $\mathcal{B} \subseteq 2^\omega$ (by $\text{Restr}(\mathcal{A}, M) = \mathcal{B}$)

Nesting in this way a $\Pi_1^{0,C}$ class into a $\Pi_1^{0,B}$ class we obtain $\Pi_1^{0,B \oplus C}$ class.

Key ingredients of the proof (1 \rightarrow 2).

WLOG: $A_0 = \emptyset$, $A_e \leq_T A_{e+1}$ for all e

We construct sequences of

- ▶ Π_1^{0, A_e} classes \mathcal{A}_e , $A_0 = \mathcal{P}\mathcal{A}$, $\mathcal{A}_{e+1} \subseteq \mathcal{A}_e$ for all e
- ▶ infinite recursive sets M_e , $M_0 = \omega$, $M_{e+1} \subseteq M_e$ for all e ,

such that $\text{Restr}(\mathcal{A}_e, M_e) = \mathcal{P}\mathcal{A}(A_e)$ for all e .

Condition $e \in A'$.

Take Φ_e (a T.r.f.), provided we have a low index of A_e , use LBT technique to find a Π_1^{0, A_e} class $\mathcal{A}_e^* \subseteq \mathcal{A}_e$ which decides this condition.

Next find an infinite recursive set $M_{e+1} \subseteq M_e$ such that $\text{Restr}(\mathcal{A}_e^*, M_{e+1}) = 2^\omega$.

Then take a $\Pi_1^{0, A_{e+1}}$ class $\mathcal{A}_{e+1} \subseteq \mathcal{A}_e^*$ so that $\text{Restr}(\mathcal{A}_{e+1}, M_{e+1}) = \mathcal{P}\mathcal{A}(A_{e+1})$.

Remark

We do not code directly sets A_e into our constructed set A . Instead, we nest $\mathcal{PA}(A_{e+1})$ into \mathcal{A}_e (or \mathcal{A}_e^*), thus ensuring that a $\{0, 1\}$ -valued A_{e+1} -DNR function f_{e+1} is coded in A , and obviously $A_{e+1} \leq_T f_{e+1}$. This way of nesting $\mathcal{PA}(A_e)$ classes at each level leaves enough space for later coding requirements.

Let A be the only set in all classes \mathcal{A}_e .

Obviously, $A_e \leq_T A$,

since for $f_e = \text{Restr}(A, M_e)$ we have $f_e \in \mathcal{PA}(A_e)$,
 A_e is recursive in any $\{0, 1\}$ -valued A_e -DNR function,
and, thus, $A_e \leq_T f_e$ and $A_e \leq_T A$.

Here: low indices of A_e are uniformly recursive in \emptyset' (a trivial case).

However, in a more general case with function F instead of low indices of A_e , we replace questions about ω -extendability of a string on a tree recursive in A_e by its finite-extendability, where the depth to which extendability is required is computed by F .

We have to be able, for each A_e , to construct a A_e -recursive (partial) function which F has to dominate and which will ensure that guesses computed by F are, eventually, correct guesses, i.e. yield ω -extendability. We adapt a finite injury style to do that.

R.sp., whenever necessary, we leave the current version of a Π_1^{0, A_e} class and nest a completely new copy of $\mathcal{PA}(A_e)$ in the currently available $\Pi_1^{0, A_{e-1}}$ class (in an appropriate way).

Observe, \mathcal{A}_0 is a Π_1^0 class \mathcal{PA} so that by means of \emptyset' we always have correct answers about ω -extendability here and, thus, there is no injury at the 0-level.

As a corollary of a result of Shore there is an exact pair for the class \mathcal{K} in Δ_2^0 T -degrees.

Question

Is there an exact pair for the class \mathcal{K} in the r.e. T -degrees? I.e. are there r.e. T -degrees \mathbf{a} , \mathbf{b} such that $[\mathbf{0}, \mathbf{a}] \cap [\mathbf{0}, \mathbf{b}]$ is equal to T -degrees of \mathcal{K} ?

Appendix

Applications of coding into \mathcal{PA}

- ▶ Posner-Robinson: For every nonrecursive set Z
 $\exists A(Z \oplus A \equiv_T A')$ (with $A \leq_T Z \oplus \emptyset'$)
- ▶ Shore-Slaman: For every $Z \notin \Delta_n^0$, $n \geq 1$
 $\exists A(Z \oplus A \equiv_T A^{(n)})$ (with $A \leq_T Z \oplus \emptyset^{(n)}$)

In the above, A may be chosen to be a PA set (i.e. $A \in \mathcal{PA}$).

Appendix

The main idea (for $n = 1$).

Isolated paths through recursive trees are recursive.

Given a Π_1^0 class $\mathcal{A} \subseteq \mathcal{P}\mathcal{A}$,

and an infinite recursive set M with $\text{Restr}(\mathcal{A}, M) = 2^\omega$,

find $\sigma \prec Z$ and Π_1^0 classes $\mathcal{B}_i \subseteq \mathcal{A}$, $i = 0, 1$

with $\text{Restr}(\mathcal{B}_i, M) = \sigma * i * 2^\omega$, (r.sp. σ is coded in \mathcal{A})

such that

- ▶ either \mathcal{B}_i forces a Σ_1^0 property with $|\sigma|$ longer than all Σ_1^0 witnesses in question, for both $i = 0, 1$
- ▶ or for both $i = 0, 1$, \mathcal{B}_i forces its negation, i.e. a Π_1^0 property.

Then take \mathcal{B}_i for $i \neq Z(|\sigma|)$, i.e. code a difference from Z .

Appendix

For $n = 2$,
a coding into $\omega^{<\omega}$ rather than a linear coding is used,
where Z can find upper bounds of relevant finitely branching
subtree of $\omega^{<\omega}$
and, by an analogous technique,
 Z can recognize first sufficiently many Σ_1^0 or Π_1^0 events,
so that eventually Z can also recognize whether a Σ_2^0 or a Π_2^0
event happens.

Appendix

Definition

$A \leq_{LR} B$ if every set 1-random in B is also 1-random in A .

Fact

TFAE:

1. $A \leq_{LR} B$
2. for some level e of a universal ML test relative to A , i.e. \mathcal{U}_e^A , there is a V^B which is $\Sigma_1^{0,B}$ such that $\mu(V^B) < 1$ and $\mathcal{U}_e^A \subseteq V^B$

Proof.

- ▶ Terwijn and A.K. $2 \rightarrow 1$ (for $B = \emptyset$)
- ▶ Nies and Stephan $1 \rightarrow 2$

Appendix

Definition

B is almost complete if \emptyset' is K -trivial relative to B , i.e.

$$\forall n (K^B(\emptyset' \upharpoonright n) \leq K^B(0^n) + O(1))$$

Lemma (Nies)

B is almost complete $\iff \emptyset' \oplus B \leq_{LR} B$

Thus, for Δ_2^0 sets: B is almost complete $\iff \emptyset' \leq_{LR} B$

Appendix

Pseudo-jump inversion.

Theorem (Jockusch, Shore, 1983)

For every r.e. operator W , there is an r.e. set B such that $B \oplus W^B \equiv_T \emptyset'$.

Corollary (Nies)

There is an almost complete r.e. set $B <_T \emptyset'$.

Apply Theorem to the r.e. operator obtained by relativizing

- r.e. K -trivial set construction, or
- r.e. low for random set construction.

Appendix

Theorem

- ▶ *There is an almost complete PA set $A <_T \emptyset'$*
- ▶ *For every nonrecursive $Z \leq_T \emptyset'$, there is an almost complete PA set A such that $A \oplus Z \equiv_T \emptyset'$.*

The same technique as in a version of Posner-Robinson theorem for PA sets (above) applied to the r.e. operator obtained by relativizing a low for random r.e. set construction

Appendix

Coding into Π_1^0 classes of positive measure

Theorem (A.K.,1989)

- ▶ *There is an incomplete high 1-random set $A <_T \emptyset'$.*
- ▶ *For every set B r.e.a. in \emptyset' and every nonrecursive Δ_2^0 set C there is a Δ_2^0 1-random set A such that $A' \equiv_T B$ and $A \not\leq_T C$.*

Theorem (Nies)

- ▶ *There is an almost complete 1-random set $A <_T \emptyset'$*
- ▶ *For every r.e. operator W there is a 1-random set $A \leq_T \emptyset'$ such that $A \oplus W^A \equiv_T \emptyset'$.*

Remark

In fact, the jump inversion technique in Theorem (A.K.,1989) yields immediately also a pseudo-jump inversion method.

Appendix

A direct construction of an r.e. almost complete set $A <_T \emptyset'$.

This was done first by Nies and Shore who used a cost function method (by building an appropriate oracle KC-set).

Not surprisingly, one can use for that also **do what is cheap** method based on measure as in low for random r.e. set construction (of Terwijn and A.K.).

Remark

There is a direct construction of an almost complete r.e. set $A <_T \emptyset'$ using a method:

$$\emptyset' \leq_{LR} A \iff$$

for some level e of a universal ML test relative to \emptyset' , i.e. $\mathcal{U}_e^{\emptyset'}$, there is a V^A which is $\Sigma_1^{0,A}$ such that $\mu(V^A) < 1$ and $\mathcal{U}_e^{\emptyset'} \subseteq V^A$.

Appendix

Comment

The idea is to construct an r.e. set A which is, in a sense, very close to \emptyset' .

Typically, to make $A <_T \emptyset'$ we sometimes want to put some x into \emptyset' but not into A .

To make A almost complete, we are allowed to do that only when such x is **cheap** relatively to A . Nonrecursiveness of A makes this more difficult.

Instead of a recursive version *to be cheap* we have here a version *to be cheap* relativized to A . This leads to approximations and to more subtle construction.

Appendix

Definition

Let \mathcal{K}_0 denote the set of r.e. K -trivials which are T -below all almost complete 1-random sets.

Remark

Hirschfeldt and Miller proved that \mathcal{K}_0 is a subclass of the r.e. members of \mathcal{K} , containing also nonrecursive r.e. sets.

Sets in \mathcal{K}_0 are ML-noncuppable, i.e. for such sets A , $A \oplus Z <_T \emptyset'$ for all Δ_2^0 1-random sets $Z <_T \emptyset'$.

(Nies was the first proving the existence of a ML-noncuppable r.e. K -trivial set).

Appendix

Open questions

- ▶ Does for every K -trivial set A exist a 1-random set Z such that $A \leq_T Z \wedge \emptyset' \not\leq Z$?
- ▶ Is \mathcal{K}_0 equal to r.e. members of \mathcal{K} ?
- ▶ Are there minimal pairs of r.e. almost complete sets ?
- ▶ Can a K -trivial set be ML-cuppable?

Appendix

Comment

Many obstacles in solving the above questions concerning 1-randomness are connected with a problem of coding an information into 1-random sets.

While we can code an infinitary information into PA sets (or into Π_1^0 subclasses of \mathcal{PA}), coding an information into 1-random sets (or into Π_1^0 classes of positive measure) is less powerful and it is still not completely understood.

Among others it should demonstrate

Splendors and miseries of Π_1^0 classes



Thank you