Low upper bounds of ideals

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The main result

- There is a low T-upper bound for the class of K-trivials
- Ideals in Δ_2^0 degrees which have a low T-upper bound

Standard notation

2^ω denotes the set of infinite binary sequences (the Cantor set)
$2^{<\omega}$ denotes the set of finite binary strings
Σ^0_1 classes, Π^0_1 classes
relativized Π^0_1 classes ($\Pi^{0,A}_1$ classes)

Algorithmic randomness

K denotes prefix-free Kolmogorov complexity
$\{\mathcal{U}_n : n \in \omega\}$ denotes universal ML test
concepts of 1-randomness and their relativizations

Algorithmic weakness

There are several notions of computational weakness related to 1-randomness

Definition

1. $\mathcal{L}$ denotes the class of sets which are low for 1-randomness, i.e. sets A such that every 1-random set is also 1-random relative to A.
2. $\mathcal{K}$ denotes the class of K-trivial sets, i.e. the class of sets A such that for all n, $K(A \upharpoonright n) \leq K(0^n) + O(1)$.
3. $\mathcal{M}$ denotes the class of sets that are low for K, i.e. sets A such that for all σ, $K(\sigma) \leq K^A(\sigma) + O(1)$.
4. A set A is a basis for 1-randomness if $A \leq_T Z$ for some Z such that Z is 1-random relative to A. The collection of such sets is denoted by $\mathcal{B}$.

Theorem (Nies, Hirschfeldt, Stephan)

$\mathcal{K} = \mathcal{L} = \mathcal{M} = \mathcal{B}$

More precisely:

- Nies: $\mathcal{L} = \mathcal{M}$
- Hirschfeldt, Nies: $\mathcal{K} = \mathcal{M}$
- Hirschfeldt, Nies, Stephan: $\mathcal{K} = \mathcal{B}$

Four different characterizations of the same class!
However, these characterizations yield different information content

Basic facts about $\mathcal{K}$

- $\mathcal{K} \subseteq \Delta^0_2$
- $\mathcal{K} \subseteq L_1$ (i.e. K-trivials are low)

More precisely:

- Chaitin: $\mathcal{K} \subseteq \Delta^0_2$
- A.K.: $\mathcal{L} \subseteq GL_1$ (thus, $\mathcal{L} = \mathcal{K} \subseteq L_1$)

Nowadays there are easier ways to prove lowness of K-trivials

Theorem (Nies; Downey,Hirschfeldt,Nies,Stephan)

- *r.e. K-trivial sets induce a Σ^0_3 ideal in the r.e. T-degrees*
- *K-trivial sets induce an ideal in the ω-r.e. T-degrees generated by its r.e. members (in fact, a Σ^0_3 ideal in the ω-r.e. T-degrees)*

Theorem (Downey,Hirschfeldt,Nies,Stephan; Nies)

- *There is an effective sequence* $\{B_e, d_e\}_e$ *of all the r.e.* K*-trivial sets and of constants such that each* B_e *is* K*-trivial via* d_e
- *There is no effective sequence* $\{B_e, c_e\}_e$ *of all the r.e. low for* K *sets with appropriate constants*
- *There is no effective way to obtain from a pair* (B, d)*, where* B *is an r.e. set that is* K*-trivial via* d*, a constant* c *such that* B *is low for* K *via* c
- *There is no effective listing of all the r.e.* K*-trivial sets together with their low indices*

Theorem (Nies)

For each low r.e. set B, there is an r.e. K-trivial set A such that $A \not\leq_T B$.

Thus, no low r.e. set can be a T-upper bound for the class $\mathcal{K}$.

Comment

The proof uses Robinson low guessing technique which is compatible for r.e. sets with a technique do what is cheap.

Cheap is defined

- either by a cost function in case of K-trivials,
- or by having a small measure in case of low for random sets.

However, in the more general case of Δ^0_2 instead of r.e. sets, the Robinson low guessing technique does not seem to be compatible with a technique do what is cheap. In fact, it is not.

Since all K-trivials are low and every K-trivial set is recursive in some r.e. K-trivial set, we have, as a corollary, that the ideal (induced by) $\mathcal{K}$ is nonprincipal (in the Δ_2^0 T-degrees)

A more general result.

Theorem (Nies)

For any effective listing $\{B_e, z_e\}_e$ of low r.e. sets and of their low indices there is an r.e. K-trivial set A such that $A \not\leq_T B_e$ for all e.

This result is, in fact, used to prove that there is no effective way to obtain low indices of (r.e.) K-trivial sets

Theorem (Nies)

- *There is a* low_2 *r.e. set which is a* T*-upper bound for the class of* K*-trivials.*
- *Any proper* Σ^0_3 *ideal in the r.e.* T*-degrees has a* low_2 *r.e.* T*-upper bound*

Question

Is there a low Δ^0_2 T-upper bound for the class $\mathcal{K}$?

The problem is mentioned:

- Nies, Computability and randomness, draft: question 5.46
- Miller, Nies, Open Questions: Question 4.3.
- AIM/ARCC Open Problem List: Problem 3.6.

Several attempts: Nies, Downey, A.K., Barmpalias, ...

Theorem (Yates)

For any r.e. set A TFAE:

1. $A'' \equiv_T \emptyset''$
2. $\{x : W_x \leq_T A\}$ *is a* Σ^0_3 *set*
3. *the class* $\{W_x : W_x \leq_T A\}$ *is uniformly r.e.*

Together with a result of Nies, this provides information about the existence of low_2 r.e. T-upper bounds for Σ^0_3 ideals in the r.e. T-degrees

Open

A characterization of Σ^0_3 ideals in the r.e. T-degrees for which there is a low T-upper bound, not necessarily r.e.(!)
(similarly for ideals in Δ^0_2 T-degrees)

Theorem

Let $\mathcal{C}$ be a Σ^0_3 ideal in the r.e. T-degrees. Then TFAE:

1. *there is a function F recursive in $\emptyset'$ which dominates all partial functions recursive in any member of the ideal $\mathcal{C}$,*
2. *there is a low T-upper bound for $\mathcal{C}$*

A slightly more general result.

Theorem

Let $\mathcal{C}$ be an ideal in Δ^0_2 T-degrees such that there is a uniformly recursively in $\emptyset'$ a sequence of sets $\{A_n\}_n$ which generate the ideal $\mathcal{C}$. Then TFAE:

1. *there is a function F recursive in $\emptyset'$ which dominates all partial functions recursive in any member of the ideal $\mathcal{C}$,*
2. *there is a low T-upper bound for $\mathcal{C}$*

Corollary

There is a low T-upper bound for the class $\mathcal{K}$ (the class of K-trivials).

Proof

Nies proved that the ideal (induced by) $\mathcal{K}$ is generated by its r.e. members and r.e. K-trivial sets induce a Σ_3^0 ideal in the r.e. T-degrees.
A.K. proved that there is a function F recursive in $\emptyset'$ which dominates all partial functions recursive in any member of $\mathcal{K}$.
Thus, the result follows from the previous Theorem.

Remark

We may equivalently require that a low T-upper bound (in the above) is *PA* since every low set has a low *PA* set T-above it. Thus, T-upper bounds which are *PA* are the most general case in this characterization

Proof

$1 \to 2$ (the opposite direction is trivial)

Given $\mathcal{C}$, $\{A_n\}_n$, and F recursive in $0'$ eventually dominating every function recursive in an A_n, we want a T-upper bound A of $\mathcal{C}$.

Obstacle

Lowness of A

versus

$A \geq_T A_n$ without having low indices for the A_n.

- A possible solution: use of Π_1^{0,A_n}-classes.
- Question: how to code into Π_1^{0,A_n}-classes.
- Answer: use rich Π_1^{0,A_n}-classes.

We replace the missing lowness indices of A_n by guesses relative to the function F. Properties of F guarantee that in a finite injury style an infinite path through Π_1^{0,A_n} classes is found (more details on that later)

Definition

Let $\mathcal{PA}(B)$ denote the class of $\{0,1\}$-valued B-DNR functions, i.e. the class of functions $f \in 2^\omega$ such that $f(x) \neq \Phi_x(B)(x)$ for all x . If B is $\emptyset$ we simply speak of $\mathcal{PA}$.

Definition (Simpson)

$\mathbf{b} << \mathbf{a}$ means that every infinite tree $T \subseteq 2^{<\omega}$ of degree $\leq \mathbf{b}$ has an infinite path of degree $\leq \mathbf{a}$.

Theorem (D. Scott and others)

The following conditions are equivalent:

1. $\mathbf{a}$ *is a degree of a* $\{0,1\}$*-DNR function*
2. $\mathbf{a} >> \mathbf{0}$
3. $\mathbf{a}$ *is a degree of a complete extension of PA*
4. $\mathbf{a}$ *is a degree of a set separating some effectively inseparable pair of r.e. sets.*

Remark

1. $\mathcal{PA}$ is a kind of a "universal" Π^0_1 class
2. $\{0,1\}$-valued DNR functions are also called PA sets and degrees $>> \mathbf{0}$ are called PA degrees.
3. Similarly for $\mathcal{PA}(B)$.

Fact (Simpson)

1. The partial ordering $<<$ is dense
2. $\mathbf{a} << \mathbf{b}$ implies $\mathbf{a} < \mathbf{b}$

Definition

Let M be an infinite set and $\{m_0, m_1, m_2, \ldots\}$ be an increasing list of all members of M.

- If $f \in 2^\omega$ then by $Restr(f, M)$ we denote $g \in 2^\omega$ defined for all i by $g(i) = f(m_i)$
- Similarly, if $\mathcal{A} \subseteq 2^\omega$ then by $Restr(\mathcal{A}, M)$ we denote a class of functions $\{g : g = Restr(f, M) \ \wedge \ f \in \mathcal{A}\}$

Lemma (A.K.)

- *For every Π^0_1 class $\mathcal{A} \subseteq \mathcal{PA}$ there is an infinite recursive set M such that if $\mathcal{A}$ is nonempty then $Restr(\mathcal{A}, M) = 2^\omega$, i.e. for every $g \in 2^\omega$ there is a function $f \in \mathcal{A}$ such that $Restr(f, M) = g$.*
- *For every $\Pi^{0,B}_1$ class $\mathcal{A} \subseteq \mathcal{PA}(B)$ there is an infinite recursive set M such that if $\mathcal{A}$ is nonempty then $Restr(\mathcal{A}, M) = 2^\omega$, where (an index of) M can be found uniformly from an index of $\mathcal{A}$, i.e. it does not depend on B.*

Remark

This is basically Gödel incompleteness phenomenon.

The Lemma is crucial for coding into $\Pi_1^{0,B}$ classes which are subclasses of $\mathcal{PA}(B)$

We may

- code either an individual set C (by $\mathit{Restr}(\mathcal{A}, M) = \{C\}$)
- or nest another class $\mathcal{B} \subseteq 2^\omega$ (by $\mathit{Restr}(\mathcal{A}, M) = \mathcal{B}$)

Nesting in this way a $\Pi_1^{0,C}$ class into a $\Pi_1^{0,B}$ class we obtain $\Pi_1^{0,B\oplus C}$ class.

Key ingredients of the proof ($1 \to 2$).
WLOG: $A_0 = \emptyset$, $A_e \leq_T A_{e+1}$ for all e
We construct sequences of

- Π_1^{0,A_e} classes $\mathcal{A}_e$, $\mathcal{A}_0 = \mathcal{PA}$, $\mathcal{A}_{e+1} \subseteq \mathcal{A}_e$ for all e
- infinite recursive sets M_e, $M_0 = \omega$, $M_{e+1} \subseteq M_e$ for all e,

such that $Restr(\mathcal{A}_e, M_e) = \mathcal{PA}(A_e)$ for all e.

Condition $e \in A'$.
Take Φ_e (a T.r.f.), provided we have a low index of A_e, use LBT technique to find a Π_1^{0,A_e} class $\mathcal{A}_e^* \subseteq \mathcal{A}_e$ which decides this condition.
Next find an infinite recursive set $M_{e+1} \subseteq M_e$ such that $Restr(\mathcal{A}_e^*, M_{e+1}) = 2^\omega$.
Then take a $\Pi_1^{0,A_{e+1}}$ class $\mathcal{A}_{e+1} \subseteq \mathcal{A}_e^*$ so that $Restr(\mathcal{A}_{e+1}, M_{e+1}) = \mathcal{PA}(A_{e+1})$.

Remark

We do not code directly sets A_e into our constructed set A.
Instead, we nest $\mathcal{PA}(A_{e+1})$ into $\mathcal{A}_e$ (or $\mathcal{A}_e^*$) , thus ensuring that a $\{0,1\}$-valued A_{e+1}-DNR function f_{e+1} is coded in A, and obviously $A_{e+1} \leq_T f_{e+1}$.
This way of nesting $\mathcal{PA}(A_e)$ classes at each level leaves enough space for later coding requirements.

Let A be the only set in all classes $\mathcal{A}_e$.

Obviously, $A_e \leq_T A$,

since for $f_e = Restr(A, M_e)$ we have $f_e \in \mathcal{PA}(A_e)$,
A_e is recursive in any $\{0, 1\}$-valued A_e-DNR function,
and, thus, $A_e \leq_T f_e$ and $A_e \leq_T A$.

Here: low indices of A_e are uniformly recursive in $\emptyset'$ (a trivial case).

However, in a more general case with function F instead of low inidices of A_e, we replace questions about ω-extendability of a string on a tree recursive in A_e by its finite-extendability, where the depth to which extendability is required is computed by F.
We have to be able, for each A_e, to construct a A_e-recursive (partial) function which F has to dominate and which will ensure that guesses computed by F are, eventually, correct guesses, i.e. yield ω-extendability. We adapt a finite injury style to do that. R.sp., whenever necessary, we leave the current version of a Π_1^{0,A_e} class and nest a completely new copy of $\mathcal{PA}(A_e)$ in the currently available $\Pi_1^{0,A_{e-1}}$ class (in an appropriate way).
Observe, $\mathcal{A}_0$ is a Π_1^0 class $\mathcal{PA}$ so that by means of $\emptyset'$ we always have correct answers about ω-extandability here and, thus, there is no injury at the 0-level.

As a corollary of a result of Shore there is an exact pair for the class $\mathcal{K}$ in Δ^0_2 T-degrees.

Question

Is there an exact pair for the class $\mathcal{K}$ in the r.e. T-degrees? I.e. are there r.e. T-degrees $\mathbf{a}$, $\mathbf{b}$ such that $[\mathbf{0}, \mathbf{a}] \cap [\mathbf{0}, \mathbf{b}]$ is equal to T-degrees of $\mathcal{K}$?

Appendix

Applications of coding into $\mathcal{PA}$

- Posner-Robinson: For every nonrecursive set Z
  $\exists A(Z \oplus A \equiv_T A') \quad$ (with $A \leq_T Z \oplus \emptyset'$)
- Shore-Slaman: For every $Z \notin \Delta^0_n, \quad n \geq 1$
  $\exists A(Z \oplus A \equiv_T A^{(n)}) \quad$ (with $A \leq_T Z \oplus \emptyset^{(n)}$)

In the above, A may be chosen to be a PA set (i.e. $A \in \mathcal{PA}$).

Appendix

The main idea (for $n = 1$).

Isolated paths through recursive trees are recursive.

Given a Π^0_1 class $\mathcal{A} \subseteq \mathcal{PA}$,
and an infinite recursive set M with $Restr(\mathcal{A}, M) = 2^\omega$,

find $\sigma \prec Z$ and Π^0_1 classes $\mathcal{B}_i \subseteq \mathcal{A}$, $i = 0, 1$
with $Restr(\mathcal{B}_i, M) = \sigma * i * 2^\omega$, *(r.sp. σ is coded in $\mathcal{A}$)*
such that

- either $\mathcal{B}_i$ forces a Σ^0_1 property with $|\sigma|$ longer than all Σ^0_1 witnesses in question, for both $i = 0, 1$
- or for both $i = 0, 1$, $\mathcal{B}_i$ forces its negation, i.e. a Π^0_1 property.

Then take $\mathcal{B}_i$ for $i \neq Z(|\sigma|)$, i.e. code a difference from Z.

Appendix

For $n = 2$,
a coding into $\omega^{<\omega}$ rather than a linear coding is used,
where Z can find upper bounds of relevant finitely branching subtree of $\omega^{<\omega}$
and, by an analogous technique,
Z can recognize first sufficiently many Σ^0_1 or Π^0_1 events,
so that eventually Z can also recognize whether a Σ^0_2 or a Π^0_2 event happens.

Appendix

Definition

$A \leq_{LR} B$ if every set 1-random in B is also 1-random in A.

Fact

TFAE:

1. $A \leq_{LR} B$
2. for some level e of a universal ML test relative to A, i.e. $\mathcal{U}_e^A$, there is a V^B which is $\Sigma_1^{0,B}$ such that $\mu(V^B) < 1$ and $\mathcal{U}_e^A \subseteq V^B$

Proof.

- Terwijn and A.K. $2 \rightarrow 1$ (for $B = \emptyset$)
- Nies and Stephan $1 \rightarrow 2$

Appendix

Definition

B is almost complete if $\emptyset'$ is K-trivial relative to B, i.e.
$\forall n \; (\; K^B(\emptyset' \upharpoonright n) \leq K^B(0^n) + O(1) \;)$

Lemma (Nies)

B is almost complete $\iff \emptyset' \oplus B \leq_{LR} B$

Thus, for Δ^0_2 sets: B is almost complete $\iff \emptyset' \leq_{LR} B$

Appendix

Psedo-jump inversion.

Theorem (Jockusch,Shore,1983)

For every r.e. operator W, there is an r.e. set B such that $B \oplus W^B \equiv_T \emptyset'$.

Corollary (Nies)

There is an almost complete r.e. set $B <_T \emptyset'$.

Apply Theorem to the r.e. operator obtained by relativizing
- r.e. K-trivial set construction, or
- r.e. low for random set construction.

Appendix

Theorem

- *There is an almost complete PA set* $A <_T \emptyset'$
- *For every nonrecursive* $Z \leq_T \emptyset'$*, there is an almost complete PA set* A *such that* $A \oplus Z \equiv_T \emptyset'$*.*

The same technique as in a version of Posner-Robinson theorem for *PA* sets (above) applied to the r.e. operator obtained by relativizing a low for random r.e. set construction

Appendix

Coding into Π^0_1 classes of positive measure

Theorem (A.K,1989)

- *There is an incomplete high 1-random set* $A <_T \emptyset'$*.*
- *For every set B r.e.a. in* $\emptyset'$ *and every nonrecursive* Δ^0_2 *set C there is a* Δ^0_2 *1-random set A such that* $A' \equiv_T B$ *and* $A \not\geq_T C$*.*

Theorem (Nies)

- *There is an almost complete 1-random set* $A <_T \emptyset'$
- *For every r.e. operator W there is a 1-random set* $A \leq_T \emptyset'$ *such that* $A \oplus W^A \equiv_T \emptyset'$*.*

Remark

In fact, the jump inversion technique in Theorem (A.K.,1989) yields immediately also a pseudo-jump inversion method.

Appendix

A direct construction of an r.e. almost complete set $A <_T \emptyset'$.

This was done first by Nies and Shore who used a cost function method (by building an appropriate oracle KC-set).
Not surprisingly, one can use for that also do what is cheap method based on measure as in low for random r.e. set construction (of Terwijn and A.K.).

Remark

There is a direct construction of an almost complete r.e. set $A <_T \emptyset'$ using a method:

$\emptyset' \leq_{LR} A \iff$
for some level e of a universal ML test relative to $\emptyset'$, i.e. $\mathcal{U}_e^{\emptyset'}$, there is a V^A which is $\Sigma_1^{0,A}$ such that $\mu(V^A) < 1$ and $\mathcal{U}_e^{\emptyset'} \subseteq V^A$.

Appendix

Comment

The idea is to construct an r.e. set A which is, in a sense, very close to $\emptyset'$.

Typically, to make $A <_T \emptyset'$ we sometimes want to put some x into $\emptyset'$ but not into A.

To make A almost complete, we are allowed to do that only when such x is cheap relatively to A. Nonrecursivness of A makes this more difficult.

Instead of a recursive version *to be cheap* we have here a version *to be cheap* relativized to A. This leads to approximations and to more subtle construction.

Appendix

Definition

Let $\mathcal{K}_0$ denote the set of r.e. K-trivials which are T-below all almost complete 1-random sets.

Remark

Hirschfeldt and Miller proved that $\mathcal{K}_0$ is a subclass of the r.e. members of $\mathcal{K}$, containing also nonrecursive r.e. sets.
Sets in $\mathcal{K}_0$ are ML-noncuppable, i.e. for such sets A, $A \oplus Z <_T \emptyset'$ for all Δ^0_2 1-random sets $Z <_T \emptyset'$.
(Nies was the first proving the existence of a ML-noncuppable r.e. K-trivial set).

Appendix

Open questions

- Does for every K-trivial set A exist a 1-random set Z such that $A \leq_T Z \ \wedge \ \emptyset' \not\leq Z$?
- Is $\mathcal{K}_0$ equal to r.e. members of $\mathcal{K}$?
- Are there minimal pairs of r.e. almost complete sets ?
- Can a K-trivial set be ML-cuppable?

Appendix

Comment

Many obstacles in solving the above questions concerning 1-randomness are connected with a problem of coding an information into 1-random sets.
While we can code an infinitary information into *PA* sets (or into Π^0_1 subclasses of $\mathcal{PA}$),
coding an information into 1-random sets
(or into Π^0_1 classes of positive measure) is less powerful and it is still not completely understood.

Among others it should demonstrate

Splendors and miseries of Π^0_1 classes

Thank you