Randomness, Lowness Notions, Measure and Domination

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Buenos Aires, Argentina
10 January, 2007

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- Part III: Domination and Measure
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- Part IV: Domination and Lowness Notions
  Pull the two parts of the talk together. Discuss positive measure domination, due to Kjos-Hanssen.

Part I: Randomness and Lowness Notions

Martin-Löf Randomness

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- A is 1-X-random if it is 1-random relative to X.

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Important observation: a Σ^0_n class is not necessarily open, hence not necessarily a $\Sigma^0_1[\emptyset^{(n-1)}]$ class.

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Important observation: a Σ^0_n class is not necessarily open, hence not necessarily a $\Sigma^0_1[\emptyset^{(n-1)}]$ class.

However, Kurtz proved that these definitions are equivalent.

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- 2-random $\implies$ weak 2-random $\implies$ 1-random.
- The reverse implications fail (Kurtz; Kautz).

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This is well defined up to a constant.

Incompressibility

Prefix-free complexity gives us a nice characterization of 1-randomness.

Theorem (Schnorr)

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Open Question

Is there an initial segment complexity characterization of weak 2-randomness?

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- [Zambella (1990)] A is low for 1-randomness if every 1-random is 1-A-random (i.e., $\mathrm{RAND}^A = \mathrm{RAND}$).
- [Nies (2005)] $A \leq_{LR} B$ if every 1-B-random is 1-A-random.

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- [Hirschfeldt, Nies, Stephan] There is a 1-A-random $X \geq_T A$.

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I.e., low for weak 2-tests $\implies$ low for weak 2-randomness.

There is no simple reason for the other direction to hold.

Relating these Lowness Notions

Theorem (Downey, Nies, Weber, Yu)

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We well see that:

low for 1-randomness $\implies$ low for weak 2-tests.

Part II: Lowness Notions and Classes

Relativizing Classes and Preserving Measure

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- Every Σ^0_3 set is $\Sigma^0_2[\emptyset']$.

Relativizing Classes and Preserving Measure

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Theorem (— | Kurtz)

- Every Π^0_2 set is $\Pi^0_1[\emptyset']$.
- Every Σ^0_3 set is $\Sigma^0_2[\emptyset']$.

- For every $\varepsilon > 0$, every Π^0_2 class contains a $\Pi^0_1[\emptyset']$ subclass of measure within ε.
- Every Σ^0_3 class contains a $\Sigma^0_2[\emptyset']$ class of the same measure.

The Main Result

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Theorem (— | Kjos-Hanssen, M, Solomon)

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TFAE:

- $A \leq_{LR} B$ and $A \leq_T B'$,
- Every $\Sigma^0_2[A]$ class contains a $\Sigma^0_2[B]$ subclass of the same measure.

Low and Low for Random

Take $B = \emptyset$ in the main result.

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Here we use the fact that $A \leq_{LR} \emptyset$ implies $A \leq_T \emptyset'$ (Nies).

Lowness Revisited

Corollary (Kjos-Hanssen, M, Solomon)

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Together with the work of Downey, Nies, Weber, Yu:

Corollary

low for 1-randomness $\iff$ low for weak 2-randomness $\iff$ low for weak 2-tests.

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low for 1-randomness $\iff$ low for weak 2-randomness $\iff$ low for weak 2-tests.

Note. This corollary was actually first proved using the Golden Run machinery of (Nies, 2005), independently by Nies & M.

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Corollary (Martin)

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- B is high for random ($\emptyset' \leq_{LR} B$),
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Our next goal is to discuss u.a.e. domination.

Part III: Domination and Measure

Domination

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Definition (Dobrinen, Simpson (2004))

- A is almost everywhere (a.e.) dominating if for almost all $X \in 2^\omega$ and all functions $g \leq_T X$, there is a $f \leq_T A$ that dominates g.

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- A is uniformly a.e. dominating if there is a function $f \leq_T A$ such that for almost all $X \in 2^\omega$, f dominates all $g \leq_T X$.

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Questions (Dobrinen, Simpson)

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- u.a.e. dominating $\implies$ a.e. dominating.
- [Martin] f dominates all computable functions iff f is high ($f' \geq_T \emptyset''$).
- u.a.e. dominating $\implies$ high.
- [Kurtz] $\emptyset'$ is u.a.e. dominating.

Questions (Dobrinen, Simpson)

- Does a.e. dominating $\implies$ u.a.e. dominating? Does it imply high?
- Does high $\implies$ a.e. dominating?
- Does u.a.e. dominating $\implies$ complete ($A \geq_T \emptyset'$)?

Answers

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So these definitions capture a(n arguably natural) class strictly between high and complete.

Measure and Domination

Theorem (Dobrinen, Simpson)

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So as started earlier, B is high for random iff B is u.a.e. dominating.

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- B is a.e. dominating,
- For every Π^0_2 class P and $\varepsilon > 0$, there is a $\Pi^0_1[B]$ subclass of P with measure within ε.

Regularity of Measure

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Our results about u.a.e. domination help us analyze the proof-theoretic strength of G_δ-REG.

For example, the existence of incomplete u.a.e. dominating degrees can be strengthened to prove:

Theorem (Kjos-Hanssen; Cholak, Greenberg, M)

$RCA_0 + G_\delta$-REG does not imply ACA_0 (or even $WWKL_0$).

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For the other direction, note that there is a Π^0_2 class with measure shared by no Σ^0_2 class. So, $\langle\omega,\mathrm{REC}\rangle \not\models G_\delta$-REG. □

Part IV: Domination and Lowness Notions

Domination and Lowness

The connection between domination and lowness notions was understood through a series of results.

Theorem (Binns, Kjos-Hanssen, Lerman, Solomon)

If B is a.e. dominating, then $\emptyset' \leq_{LR} B$ (every 1-B-random is 2-random; B is high for random).

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Building on this work:

Definition (Kjos-Hanssen)

B is positive measure (p.m.) dominating if for every Turing functional $\Phi\colon 2^{\omega} \to \omega^{\omega}$, if $\Phi[X]$ is total for positive measure many X, then there is an $f \leq_T B$ that dominates $\Phi[X]$ for positive measure many X.

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Proof. Let B be a.e. dominating and Φ have domain with positive measure. For a.e. $X \in \text{dom}(\Phi)$, some $f \leq_T B$ dominates $\Phi[X]$. But B computes only countably many functions, so one must work on a set of positive measure. □

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Domination Revisited

From the main result, $\emptyset' \leq_{LR} B$ iff B is u.a.e. dominating, so:

Corollary (Kjos-Hanssen, M, Solomon)

u.a.e. dominating $\iff$ a.e. dominating $\iff$ p.m. dominating.

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The proof goes through over WWKL_0 (but not RCA_0).

Corollary (WWKL_0)

Weak G_δ-REG is equivalent to G_δ-REG.

Another Corollary

Recall:

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What about that Hypothesis?

Main Result

If $A \leq_T B'$ and $A \leq_{LR} B$, then every $\Sigma^0_2[A]$ class has a $\Sigma^0_2[B]$ subclass of the same measure.

Is $A \leq_T B'$ a necessary assumption?

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But does $A \leq_{LR} B$ imply $A \leq_T B'$?

No, there is a B with continuum many $A \leq_{LR} B$ (M, Yu).

Part V: About the Proof of the Main Result

Proof Method

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The proof method has also been used to show:

Theorem (Kjos-Hanssen, M, Solomon)

$A \leq_{LR} B$ iff $A \leq_{\mathrm{LK}} B$.

Where:

Definition (Nies)

$A \leq_{\mathrm{LK}} B$ if $(\forall \sigma)\ K^B(\sigma) \leq K^A(\sigma) + O(1)$.

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This is a relativization of low for K in the same sense that $A \leq_{LR} B$ is a relativization of low for 1-randomness.

Ingredients in the Proof

We need two results. First:

Lemma 1 (Kjos-Hanssen)

$A \leq_{LR} B$ iff every $\Pi^0_1[A]$ class of positive measure has a $\Pi^0_1[B]$ subclass of positive measure.

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From analysis:

Lemma 2

Let $\{a_i\}_{i\in\omega}$ be a sequence of real numbers with $a_i \in [0, 1)$, for all i. Then $\prod_{i\in\omega}(1 - a_i) > 0$ iff $\sum_{i\in\omega} a_i$ converges.

Proof Sketch

Notation

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and if $I \subseteq 2^{<\omega} \times 2^{<\omega}$, then $\mu\left(\bigcap_{p\in I}[V_p]^c\right) = \prod_{\langle\sigma,\tau\rangle\in I}(1 - 2^{-|\tau|})$.

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and if $I \subseteq 2^{<\omega} \times 2^{<\omega}$, then $\mu\left(\bigcap_{p \in I} [V_p]^c\right) = \prod_{\langle \sigma, \tau \rangle \in I} (1 - 2^{-|\tau|})$.

Intuitively, each V_p is independent from all of the others.

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Let $I = \{\langle \sigma, \tau \rangle \colon \tau \in S^A$ with use $\sigma\}$. Consider the $\Pi^0_1[A]$ class $P = \bigcap_{p \in I} [V_p]^c$. Using Lemma 2, $\mu(P) > 0$. Therefore by Lemma 1, there is a $\Pi^0_1[B]$ class $Q \subseteq P$ such that $\mu(Q) > 0$.

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Let $I = \{\langle\sigma, \tau\rangle \colon \tau \in S^A \text{ with use } \sigma\}$. Consider the $\Pi^0_1[A]$ class $P = \bigcap_{p \in I}[V_p]^c$. Using Lemma 2, $\mu(P) > 0$. Therefore by Lemma 1, there is a $\Pi^0_1[B]$ class $Q \subseteq P$ such that $\mu(Q) > 0$.

Define $J = \{\langle\sigma, \tau\rangle \colon [V_{\langle\sigma,\tau\rangle}] \cap Q = \emptyset\}$.

Proof Sketch

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Intuitively, J approximates I with only finitely much error.

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Let $\{A_s\}_{s\in\omega}$ be a B-computable sequence approximating A.

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So $Y = \bigcup_{s\in\omega}[U_s]^c$ is the desired Σ_2^B class. □

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