Bounded Genericity

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January 2007

Notation: Strings and Languages

- Finite Alphabet $X = \{0, \ldots, r-1\}$, cardinality $|X| = r \geq 2$
- Finite strings (words): $w = x_1 \cdots x_l \in X^*$, $x_i \in X$, length $|w| = l$
- Languages: $W \subseteq X^*$
- Infinite strings (ω-words): $\xi = x_1 \cdots x_l \cdots \ \in X^\omega$
- Prefixes of infinite strings: $\xi[0..n] \in X^*$, $\big|\xi[0..n]\big| = n$
- Prefix relation: $w \sqsubseteq \eta$, for $\eta \in X^* \cup X^\omega$
- Language of prefixes: $\mathbf{pref}(B) := \{w : \exists \eta (\eta \in B \wedge w \sqsubseteq \eta)\}$

X^ω as CANTOR space

Metric: $\rho(\eta,\xi) := \inf\{r^{-|w|} : w \sqsubset \eta \wedge w \sqsubset \xi\}$

Balls in (X^ω,ρ): $w \cdot X^\omega = \{\eta : w \sqsubset \eta\}$

Diameter: $\mathrm{diam}\, w \cdot X^\omega = r^{-|w|}$

Open sets: $W \cdot X^\omega = \bigcup_{w\in W} w \cdot X^\omega$

Closure: $\mathcal{C}(F) = \{\xi : \mathbf{pref}(\xi) \subseteq \mathbf{pref}(F)\}$

Property

$F \subseteq X^\omega$ is closed if and only if $\mathbf{pref}(\xi) \subseteq \mathbf{pref}(F)$ implies $\xi \in F$.

Extension functions (Ambos-Spies & Busse '03)

Extension function: $f : X^* \to X^*$

k-**bounded extension function:** $f : X^* \to X^k$

$f : X^* \to X^*$ **meets** $\xi \in X^\omega$**:** $\exists w(w \sqsubset \xi \wedge w \cdot f(w) \sqsubset \xi)$

$$M_f := \bigcup_{w \in X^*} w \cdot f(w) \cdot X^\omega,$$

$$F_f := X^\omega \setminus M_f$$

$$M_{\mathcal{F}} := \bigcap_{f \in \mathcal{F}} M_f,$$

$$F_{\mathcal{F}} := X^\omega \setminus M_{\mathcal{F}} = \bigcup_{f \in \mathcal{F}} F_f$$

ξ is $(f_n)_{n\in\mathbb{N}}$-**generic:** $\xi \in \bigcap_{n\in\mathbb{N}} M_{f_n} = X^\omega \setminus \bigcup_{n\in\mathbb{N}} F_{f_n}$

k-bounded automatic genericity

$f : X^* \to X^k$ is defined by a finite automaton $\mathcal{A} = (X, Q, q_0, \delta)$ and a function $g : Q \to X^k$ via

$$f(w) := g(\delta(q_0, w)) \,.$$

Lemma

If f is a k-bounded automatic extension function then F_f is definable by a finite automaton.

Theorem (Ambos-Spies & Busse '03)

Let $\{f_n^k : n \in \mathbb{N}\}$ be the family of all k-bounded automatic extension functions. Then

$$\bigcup_{n\in\mathbb{N}} F_{f_n^k} \subset \bigcup_{n\in\mathbb{N}} F_{f_n^{k+1}} \,.$$

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Special case: Forbidden subwords

For some $v \in X^*$, $f \equiv v$ is a constant function, that is,

$$E(\neg v) := F_f = X^\omega \setminus X^* \cdot v \cdot X^\omega .$$

Definition (Overlap-free words)

A word $u \in X^*$ is **overlap-free** if and only if no non-empty proper prefix of u is a suffix of u.

A hierarchy result by VOLKMANN

Theorem (Volkmann '53)

1. **length k:** *Let $v \in X^k$ be overlap-free, and $u \in X^k$ neither overlap-free nor of the form a^k, $a \in X$. Then*

$$\dim_{\mathbf{H}} E(\neg\, v) < \dim_{\mathbf{H}} E(\neg\, u) < \dim_{\mathbf{H}} E(\neg\, 0^k)\,.$$

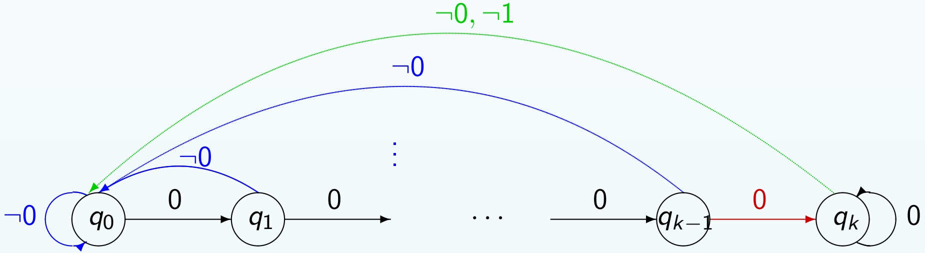
2. **length k and $k+1$:** *Let $u \in X^{k+1}$. If $|X| \geq 3$ or u is not overlap-free then*

$$\dim_{\mathbf{H}} \bigcup_{|v|=k} E(\neg\, v) < \dim_{\mathbf{H}} E(\neg\, u)\,.$$

The rôle of the alphabet

Observation. For $X = \{0,1\}$ we have

$$\begin{aligned} \dim_{\mathbf{H}} E(\neg\,\mathbf{0^k 1}) &= \dim_{\mathbf{H}} E(\neg\,\mathbf{0^k}) \\ E(\neg\,\mathbf{0^k 1}) &\subseteq \bigcup\nolimits_{n\in\mathbb{N}} F_{f_n^k} \; (f_n^k \text{ is } k\text{-bounded automatic}) \end{aligned}$$



Structure function

Definition (Structure Function of $W \subseteq X^*$ and $F \subseteq X^\omega$)

$$
\begin{aligned}
\mathsf{s}_W(n) &:= \left|\{w : w \in W \wedge |w| = n\}\right| = \left|W \cap X^n\right| \\
\mathsf{s}_F &:= \mathsf{s}_{\mathbf{pref}(F)}
\end{aligned}
$$

Special case: $E(\neg 0^{k-1}1)$ and $E(\neg 0^k)$

$$\begin{array}{lclcl} \underline{E}_k & := & E(\neg 0^{k-1}1) & = & X^\omega \setminus X^* \cdot \mathbf{0}^{\mathbf{k-1}}\mathbf{1} \cdot X^\omega \\ E_k & := & E(\neg 0^k) & = & X^\omega \setminus X^* \cdot \mathbf{0}^{\mathbf{k}} \cdot X^\omega \end{array}$$

Their structure functions satisfy

$$\begin{array}{lclcll} \underline{s}_k(n) & = & s_k(n) & = & r^n & , \text{if } 0 \le n < k \text{ and} \\ \underline{s}_k(k) & = & s_k(k) & = & r^k - 1 & , \end{array} \tag{1}$$

and the recurrences

$$\begin{array}{lcll} \underline{s}_k(n) & = & r \cdot \underline{s}_k(n-1) - \underline{s}_k(n-k) & , \text{if } n > k \\ s_k(n) & = & (r-1) \cdot \sum_{i=1}^k s_k(n-i) & , \text{if } n > k\,. \end{array} \tag{2}$$

Special case: $E(\neg\, 0^{k-1}1)$ and $E(\neg\, 0^k)$

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\underline{E}_k & := & E(\neg\, 0^{k-1}1) & = & X^\omega \setminus X^* \cdot \mathbf{0^{k-1}1} \cdot X^\omega \\
E_k & := & E(\neg\, 0^k) & = & X^\omega \setminus X^* \cdot \mathbf{0^k} \cdot X^\omega
\end{array}$$

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Asymptotic behaviour of $\underline{s}_k$ and s_k

Let $\underline{\lambda}_k$ and λ_k be the maximum roots of the polynomials

$$\begin{aligned} \underline{p}_k(z) &= z^k - r \cdot z^{k-1} + 1 \text{ and} \\ p_k(z) &= z^k - \sum_{i=0}^{k-1} (r-1) \cdot z^i \text{, respectively.} \end{aligned}$$

Remark. $\underline{\lambda}_k < \lambda_k$, and if $r > 2$ then $\lambda_k < \underline{\lambda}_{k+1}$.

$$\underline{s}_k(n) \approx \underline{\lambda}_k^n \quad \text{and} \quad s_k(n) \approx \lambda_k^n$$

Overlapping

Definition

Let $f : X^* \to X^k$ be an extension function.

1. We call f **overlap-free** $:\Longleftrightarrow$
$\forall w \forall v (v \sqsubset w \sqsubset v \cdot f(v) \to v \cdot f(v) \not\sqsubseteq w \cdot f(w))$.
2. We call f **fully overlapping** $:\Longleftrightarrow$
$\forall w \forall v (v \sqsubset w \sqsubset v \cdot f(v) \to v \cdot f(v) \sqsubseteq w \cdot f(w))$.

Lemma

Let $f : X^* \to X^k$ be an extension function.

1. *There is an overlap-free extension function $g : X^* \to X^k$ such that $F_g \subseteq F_f$.*
2. *There is a fully overlapping extension function $h : X^* \to X^k$ such that $F_h \supseteq F_f$.*

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Minimal and maximal elements

$$\mathcal{E}_k := \{F_f : \quad f : X^* \to X^k\}$$

Lemma

If F_f is a minimal (maximal) w.r.t. set inclusion element in $\mathcal{E}_k$ then there is an overlap-free extension function $g : X^* \to X^k$ (a fully overlapping extension function $h : X^* \to X^k$) such that $F_f = F_g$ ($F_f = F_h$).

Theorem

1. *If $f : X^* \to X^k$ then $\underline{s}_k \leq s_{F_f} \leq s_k$.*
2. *If $g : X^* \to X^k$ is overlap-free then $s_{F_g} = \underline{s}_k$ and F_g is minimal in $\mathcal{E}_k$.*
3. *If $h : X^* \to X^k$ is fully overlapping then $s_{F_h} = s_k$ and F_h is maximal in $\mathcal{E}_k$.*

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3. *If $h : X^* \to X^k$ is fully overlapping then $\mathsf{s}_{F_h} = \mathsf{s}_k$ and F_h is maximal in $\mathcal{E}_k$.*

Minkowski or box-counting dimensions

$$\begin{aligned} \underline{\dim}_{\mathbf{B}} F &:= \liminf_{n\to\infty} \tfrac{1}{n} \cdot \log_{|X|}(s_F(n)+1) \\ \overline{\dim}_{\mathbf{B}} F &:= \limsup_{n\to\infty} \tfrac{1}{n} \cdot \log_{|X|}(s_F(n)+1) \end{aligned}$$

Property

$$\begin{aligned} \underline{\dim}_{\mathbf{B}} F &= \underline{\dim}_{\mathbf{B}} \mathcal{C}(F) \\ \overline{\dim}_{\mathbf{B}} F &= \overline{\dim}_{\mathbf{B}} \mathcal{C}(F) \\ E \subseteq F &\rightarrow \underline{\dim}_{\mathbf{B}} E \leq \underline{\dim}_{\mathbf{B}} F \\ \overline{\dim}_{\mathbf{B}} E \cup F &= \max\{\overline{\dim}_{\mathbf{B}} E, \overline{\dim}_{\mathbf{B}} F\} \end{aligned}$$

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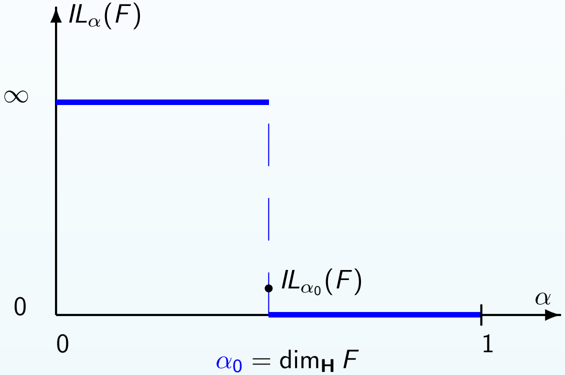
Hausdorff measure

$$IL_\alpha(F) := \lim_{l\to\infty} \inf\left\{ \sum_{w\in W} r^{-\alpha\cdot|w|} : F \subseteq \bigcup_{w\in W} w\cdot X^\omega \wedge \forall w(w \in W \to |w| \geq l)\right\}$$



Hausdorff dimension

$$\dim_{\mathbf{H}} F = \sup\{\alpha : I\!L_\alpha(F) = \infty\} = \inf\{\alpha : I\!L_\alpha(F) = 0\}$$

Proposition

1. $\dim_{\mathbf{H}} \bigcup_{i\in\mathbb{N}} F_i = \sup\{\dim_{\mathbf{H}} F_i : i \in \mathbb{N}\}$
2. $\dim_{\mathbf{H}} w \cdot F = \dim_{\mathbf{H}} F$
3. $\dim_{\mathbf{H}} F \leq \underline{\dim}_{\mathbf{B}} F \leq \overline{\dim}_{\mathbf{B}} F$

Relations between the dimensions

Definition ("Subtree" rooted at $w \in X^*$)

$F/w := \{\xi : w \cdot \xi \in F\}$

Theorem (*St.*'89)

Let $\log_r \gamma(n) = o(n)$ *for some* $\gamma : \mathbb{N} \to \mathbb{N}$. *If* $s_{F/w}(n) \leq \gamma(|w|) \cdot s_F(n)$ *for all* $w \in X^*$ *and all* $n \in \mathbb{N}$ *then*

$$\dim_{\mathbf{H}} F \; = \; \overline{\dim_{\mathbf{B}}} F \, .$$

Observation

If $f : X^* \to X^k$ is overlapfree or fully overlapping then $s_{F_f/w}(n) \leq s_{F_f}(n)$, for all $w \in X^*$ and all $n \in \mathbb{N}$.

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The dimensions of F_f

Corollary

If $f : X^* \to X^k$ is overlapfree or fully overlapping then

$$\dim_{\mathbf{H}} F_f = \overline{\dim}_{\mathbf{B}} F_f .$$

Lemma (Mauldin & Williams '88, Bandt '89, *St.* '89)

If $f : X^* \to X^k$ is automatic then

$$\dim_{\mathbf{H}} F_f = \overline{\dim}_{\mathbf{B}} F_f .$$

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If $f : X^* \to X^k$ *is overlapfree or fully overlapping then*

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General hierarchy result I

Theorem

If f is a k-bounded extension function and f' is a $(k+1)$-bounded extension function over X then $F_f \not\supseteq F_{f'}$ and

$$\log_{|X|} \underline{\lambda}_k \leq \dim_H F_f \leq \log_{|X|} \lambda_k \leq \log_{|X|} \underline{\lambda}_{k+1} \leq \dim_H F_{f'} \leq \log_{|X|} \lambda_{k+1} \,.$$

Equality of Hausdorff dimension $\dim_H F_f = \dim_H F_{f'}$ can hold only if $|X| = 2$, $\dim_H F_f = \log_2 \lambda_k$ and $\dim_H F_{f'} = \log_2 \underline{\lambda}_{k+1}$.

General hierarchy result II

Theorem

Let, for $k \in \mathbb{N}$, $\{f_n^k : n \in \mathbb{N}\}$ be a countable family of k-bounded extension functions containing all constant functions. Then

$$\bigcup_{n\in\mathbb{N}} F_{f_n^k} \not\supseteq \bigcup_{n\in\mathbb{N}} F_{f_n^{k+1}} .$$

Corollary

Let $\{f_n^k : n \in \mathbb{N}\}$ be the family of all k-bounded computable (recursive) extension functions. Then

$$\bigcup_{n\in\mathbb{N}} F_{f_n^k} \subset \bigcup_{n\in\mathbb{N}} F_{f_n^{k+1}} .$$

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