Genericity Theory from the Randomness Viewpoint

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Genericity and Randomness

Definition

A real $x \in 2^\omega$ is

(i) *weakly n-generic* if $x \notin A$ for all Π^0_n meager set A;

(ii) *weakly n-random* if $x \notin A$ for all Π^0_n null set A.

Definition

A real $x \in 2^\omega$ is

(i) *n-generic* if either $x \in A$ or there exists a finite string $\sigma \prec x$ so that $[\sigma] \cap A = \emptyset$ for all Σ^0_n open set A;

(ii) *n-random* if $x \notin A$ for set A which is the intersection of a Σ^0_n-test.

Effective Category and Measure Theory

Theorem

Fix a universal Σ^0_n set $A \subseteq \omega \times 2^\omega$, then:

1. *(Sacks) $\{(i, n) | \mu(A_i) > 2^{-n}\}$ is Σ^0_n.*
2. *(Kechris) $\{i | A_i$ is not meager $\}$is Σ^0_n.*

Theorem

For every Σ^0_n set $A \subseteq 2^\omega$,

1. *(Kurtz) there is a recursive sequence of Σ^0_{n-1} closed sets $\{F_m\}_m$ so that $\bigcup_n F_m \subseteq A$ and $\mu(\bigcup_m F_m) = \mu(A)$.*
2. *(Forklore) A is comeager, then there is an recursive sequence of Σ^0_n dense open sets $\{U_m\}_m$ so that $\bigcap_m U_m \subseteq A$.*

Uniformizing the Notions

Corollary (Kurtz)

A real $x \in 2^\omega$ is

1. *weakly n-generic iff $x \in U$ for every Σ^0_n dense open set U.*
2. *weakly $n+1$-random iff $x \notin \bigcap_m U_m$ for all recursive sequence of Σ^0_n open sets $\{U_m\}_m$ with $\lim_m \mu(U_m) = 0$.*
3. *n-random iff $x \notin \bigcap_m U_m$ for all recursive sequence of Σ^0_n open sets $\{U_m\}_m$ with $\mu(U_m) \leq 2^{-m}$.*

So weak $n+1$ randomness $\implies$ n-randomness $\implies$ weak n-randomness and weak $n+1$-genericity $\implies$ n-genericity $\implies$ weak n-genericity

Definition

- A Turing degree is hyperimmune if it contains a function not dominated by any recursive functions. Otherwise, it is hyperimmune-free.
- A degree is recursively traceable if every function computed by it can be traced by a recursive function with identity bound.
- A Turing degree is *DNR* if it contains a function f so that $\forall n(f(n) \neq \Phi_n(n))$.
- A Turing degree is *PA* if it contains a real computing a completion of Peano's Axioms.

Characterizing Low Levels I

Theorem

1. *(Forklore) Every weakly 1-generic real is weakly 1-random.*
2.
   - *(Kurtz+Jockusch) x has a weakly 1-generic degree iff it has a hyperimmune degree, and every 1-generic real is REA.*
   - *(DNWY+Hirschfeldt, Miller)x has a weakly 2-random degree iff it has a 1-random degree and is incomparable with all of nonrecursive Δ^0_2-degrees.*
   - *(Kurtz) Every 2-random real is REA.*
3.
   - *(Forklore) Every 1-generic degree is GL_1.*
   - *(Kautz) Every 2-random degree is GL_1.*
   - *(Kucera) If $x \geq_T \emptyset'$, then it has a 1-random degree.*

Characterizing Low Levels II

Theorem

- *(Forklore) No 1-generic real has DNR-degree.*
- *(Forklore) Every 1-random degree is a DNR-degree.*
- *(Stephan) If x is 1-random, then x has a PA degree iff $x \geq_T \emptyset'$.*
- *(Yu) A real x is hyperimmune-free, then x is weakly 1-random iff x is weakly 2-random.*

Question

Characterizing weakly 1-random degrees.

Kolmogorov Complexity vs Randomness

Theorem

1. *(Schnorr) x is 1-random iff there is a constant c so that* $\forall n(K(x \upharpoonright n) \geq n - c)$.
2. *(Miller and Yu) x is 1-random iff for every computable function g with* $\sum_n 2^{-g(n)} < \infty$*, there is a constant c so that* $\forall n(C(x \upharpoonright n) \geq n - g(n) - c)$.
3. *(Miller and Yu)* $x \oplus y$ *is 1-random iff there is a constant so that* $\forall n(K(x \upharpoonright n) + C(y \upharpoonright n) \geq 2n - c)$.
4. *(NST+Miller) x is 2-random iff there is a constant c so that* $\exists^\infty n(C(x \upharpoonright n) \geq n - c)$.

Kolmogorov Complexity vs Genericity

Theorem

1. *(Nies) There exists a K-trivial 1-generic real.*
2. *(Forklore) x is weakly 2-generic then x is K-“very low” and K-random infinitely often.*

Question

Finding out a complexity characterization of weak 2-randomness and genericity.

Lowness for Randomness and Genericity I

Definition

Given a notion G and its relativization G^x, a real x is low for G if $\mathrm{G} = \mathrm{G}^x$.

Theorem

1. *(Stephan and Yu) A real is low for weakly 1-random then it must have a degree strictly between recursively traceability and hyperimmune-freeness.*
2. *(Stephan and Yu) A real is low for weakly 1-generic iff it hyperimmune-free and non-DNR.*

Lowness for Randomness and Genericity II

Theorem

1.
   - *(Hirschfeldt and Nies) A real x is low for 1-random iff it is there exists a constant c so that $\forall n (K(x \upharpoonright n) \leq K(n) + c)$.*
   - *(Greenberg, Miller + Yu) A real is low for 1-generic iff x is recursive.*
2. *(DNWY+Miller + Nies) A real x is low for weakly 2-random iff it is there exists a constant c so that $\forall n (K(x \upharpoonright n) \leq K(n) + c)$.*

Forcing vs Genericity and Randomness

Definition

Let $(P_n, \leq)$ be a forcing notation where $P_n = \{A \subseteq 2^\omega | A \in \Pi^0_n \wedge \mu(A) > 0\}$ and $\leq = \subseteq$. $A \Vdash \varphi$ if $\varphi(x)$ is true for all $x \in A$. x is Solovay n-generic if for every Π^0_n-formula, there is a condition $x \in A$, A decides φ.

Theorem

1. *(Jockusch) x is n-generic iff x forces all of Σ^0_n sentences in the Cohen forcing sense.*
2. *(Kurtz) x is weakly n-random iff x is Solovay n-generic.*

Van Lambalgen's Theorem

Theorem

1. *(van Lambalgen)* $x \oplus y$ *is n-random iff* x *is n-random and* y *is n-*x*-random.*
2. *(Forklore)* $x \oplus y$ *is n-generic iff* x *is n-generic and* y *is n-*x*-generic.*

Relativized Randomness

Theorem

1. *(Miller and Yu) For any real z and 1-random reals $x \leq_T y$, if y is 1-z-random then x is 1-z-random.*
2. *(CDGHM) M-Y Theorem holds for n-genericity if $n \geq 2$, but fails for 1-genericity.*

Higher Up

Definition

Given a class of sets of reals T,

1. A real x is T-random if x is not in any null set in T.
2. A real x is T-generic if x is in every dense set in T where T is also a class of open sets.

Theorem

1. *(Sacks+Hjorth, Nies+Chong, Nies, Yu)* Π^1_1*-randomness* $\subset$ Π^1_1*-Martin-Löf randomness* $\subset$ Δ^1_1*-randomness* $=$ Δ^1_1*-Martin-Löf randomness.*
2. Π^1_1*-genericity* $=$ Δ^1_1*-genericity.*

Tracebility

Definition

(i) Let $h : \omega \to \omega$ be a nondecreasing unbounded function that is hyperarithmetical. A Π^1_1- trace/Δ^1_1-trace with bound h is a uniformly Π^1_1/uniformly Δ^1_1 sequence $(T_e)_{e\in\omega}$ such that $|T_e| \leq h(e)$ for each e.

(ii) $A \subseteq \omega$ is Π^1_1-traceable/Δ^1_1-traceable if there is $h \in \Delta^1_1$ such that, for each $f \leq_h A$, there is a Π^1_1-trace/Δ^1_1 -trace with bound h such that, for each e, $f(e) \in T_e$.

Proposition (Chong, Nies and Yu)

If x is Π^1_1-traceable, then x is Δ^1_1-traceable.

Lowness properties

Theorem

1. *(Chong, Nies and Yu) Lowness for Δ^1_1 randomness =Δ^1_1-traceability.*
2. *(Hjorth and Nies) Lowness for Π^1_1-Martin-Löf randomness =Hyperarithmetic.*
3. *(Harrington, Nies and Slaman) Lowness for Π^1_1-randomness =Lowness for Δ^1_1 randomness + non-random-cuppable.*
4. *(Yu) Lowness for Δ^1_1-genericity $\supseteq$ Δ^1_1-traceability.*

Beyond Recursion Theory

Theorem

Assume PD if $n \geq 1$.

- *(Kechris) There exists a largest* Π^1_{2n+1} *and* Σ^1_{2n} *null set.*
- *(Kechris) There exists a largest* Π^1_{2n+1} *and* Σ^1_{2n} *meager set.*
- *(Sacks+Tanaka+Kechris)Each non-null* Π^1_{2n+1} *set contains a* Δ^1_{2n} *real.*
- *(Hinman+Kechris)Each non-meager* Π^1_{2n+1} *set contains a* Δ^1_{2n} *real.*

Some questions

Question

1. *How far can genericity and randomness theory go under PD?*
2. *Finding out an inner model to develop higher genericity and randomness theory.*

Thank you