
Simply Typed Lambda-Calculus Modulo Type Isomorphisms

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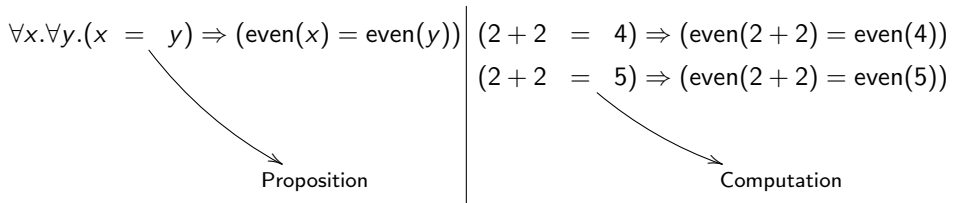
Propositional equivalence vs definitional equality

$$\forall x. \forall y. (x = y) \Rightarrow (\text{even}(x) = \text{even}(y))$$

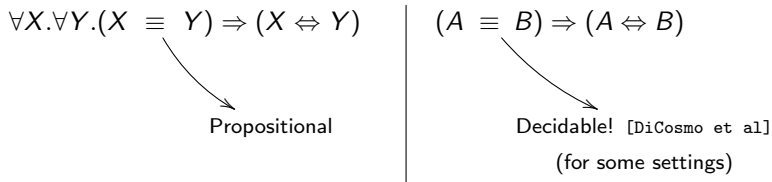
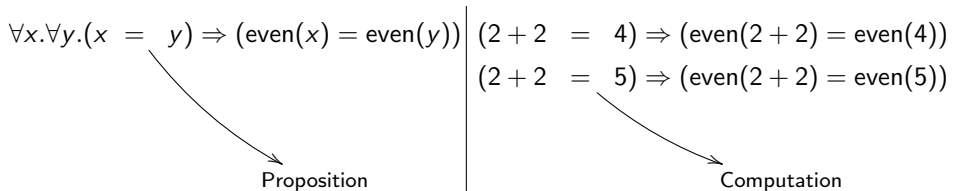
Propositional equivalence vs definitional equality

$$\forall x. \forall y. (x = y) \Rightarrow (\text{even}(x) = \text{even}(y)) \quad \left| \begin{array}{l} (2 + 2 = 4) \Rightarrow (\text{even}(2 + 2) = \text{even}(4)) \\ (2 + 2 = 5) \Rightarrow (\text{even}(2 + 2) = \text{even}(5)) \end{array} \right.$$

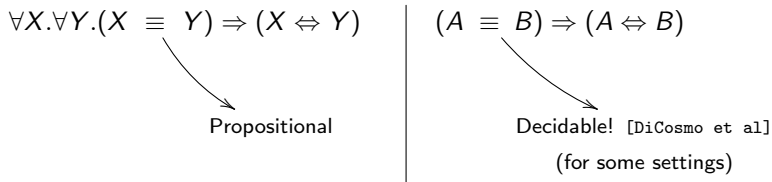
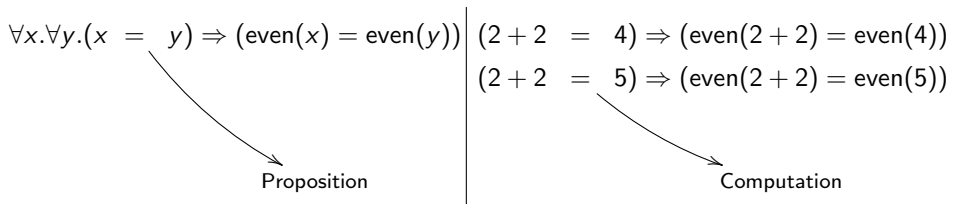
Propositional equivalence vs definitional equality



Propositional equivalence vs definitional equality



Propositional equivalence vs definitional equality



We want to go further:

$$(A \equiv B) \Rightarrow (\pi : A \Leftrightarrow \pi : B)$$

The goal is to identify isomorphic types

The basic setting

- ▶ Simply types with conjunction and implication

$$A, B, C ::= \tau \mid A \Rightarrow B \mid A \wedge B$$

- ▶ An equivalence relation between types (based on the isomorphisms¹)
 1. $A \wedge B \equiv B \wedge A$ (comm)
 2. $A \wedge (B \wedge C) \equiv (A \wedge B) \wedge C$ (asso)
 3. $(A \wedge B) \Rightarrow C \equiv A \Rightarrow B \Rightarrow C$ (curry)
 4. $A \Rightarrow (B \wedge C) \equiv (A \Rightarrow B) \wedge (A \Rightarrow C)$ (distrib)

¹Bruce, Di Cosmo, Longo, MSCS 2(2), 231–247, 1992

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We want

$$[A \equiv B] \frac{r : A}{r : B}$$

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Associative and commutative conjunction

$$\frac{\Gamma \vdash \mathbf{r} : A \quad \Gamma \vdash \mathbf{s} : B}{\Gamma \vdash \langle \mathbf{r}, \mathbf{s} \rangle : A \wedge B} (\wedge_i)$$

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$$A \wedge B \equiv B \wedge A$$

$$A \wedge (B \wedge C) \equiv (A \wedge B) \wedge C$$

$$\text{So } \langle \mathbf{r}, \mathbf{s} \rangle \leftrightarrow \langle \mathbf{s}, \mathbf{r} \rangle$$

$$\langle \mathbf{r}, \langle \mathbf{s}, \mathbf{t} \rangle \rangle \leftrightarrow \langle \langle \mathbf{r}, \mathbf{s} \rangle, \mathbf{t} \rangle$$

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What about the elimination?

$$\frac{\Gamma \vdash \langle \mathbf{r}, \mathbf{s} \rangle : A \wedge B}{\Gamma \vdash \pi_1 \langle \mathbf{r}, \mathbf{s} \rangle : A} (\wedge_e)$$

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Moreover $\langle \mathbf{r}, \mathbf{s} \rangle = \langle \mathbf{s}, \mathbf{r} \rangle$ so $\pi_1 \langle \mathbf{r}, \mathbf{s} \rangle = \pi_1 \langle \mathbf{s}, \mathbf{r} \rangle !!$

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Workaround: Church-style – Project w.r.t. a type

If $\mathbf{r} : A$ then $\pi_A \langle \mathbf{r}, \mathbf{s} \rangle \rightarrow \mathbf{r}$

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This induces non-determinism

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If $\mathbf{s} : A$ then $\pi_A \langle \mathbf{r}, \mathbf{s} \rangle \rightarrow \mathbf{s}$

We are interested in the proof theory and
both \mathbf{r} and \mathbf{s} are valid proofs of A

Curryfication

$$(A \wedge B) \Rightarrow C \quad \equiv \quad A \Rightarrow B \Rightarrow C$$

induces

$$\mathbf{r}\langle \mathbf{s}, \mathbf{t} \rangle \quad \Leftrightarrow \quad \mathbf{rst}$$

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$$\mathbf{r}\langle \mathbf{s}, \mathbf{t} \rangle \quad \Leftrightarrow \quad \mathbf{rst}$$

$$(\lambda x^A. \mathbf{r})\mathbf{s} \quad \rightarrow \quad \mathbf{r}[\mathbf{s}/x]$$

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Distributivity of implication over conjunction

$$A \Rightarrow (B \wedge C) \equiv (A \Rightarrow B) \wedge (A \Rightarrow C)$$

induces

$$\lambda x^A. \langle \mathbf{r}, \mathbf{s} \rangle \Leftrightarrow \langle \lambda x^A. \mathbf{r}, \lambda x^A. \mathbf{s} \rangle$$

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$$\lambda x^A. \langle \mathbf{r}, \mathbf{s} \rangle \Leftrightarrow \langle \lambda x^A. \mathbf{r}, \lambda x^A. \mathbf{s} \rangle \quad \text{and} \quad \lambda x^A. \pi_B(\mathbf{r}) \Leftrightarrow \pi_{A \Rightarrow B}(\lambda x^A. \mathbf{r})$$

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Example

$$\frac{\frac{\vdash \lambda x^{A \wedge B}. x : (A \wedge B) \Rightarrow (A \wedge B)}{\vdash \lambda x^{A \wedge B}. x : ((A \wedge B) \Rightarrow A) \wedge ((A \wedge B) \Rightarrow B)} \quad (\equiv)}{\vdash \pi_{(A \wedge B) \Rightarrow A}(\lambda x^{A \wedge B}. x) : (A \wedge B) \Rightarrow A} \quad (\wedge_e)$$

$$\pi_{(A \wedge B) \Rightarrow A}(\lambda x^{A \wedge B}. x) \Leftrightarrow \lambda x^{A \wedge B}. \pi_A(x)$$

Distributivity of implication over conjunction

Multiple choices

$$A \Rightarrow (B \wedge C) \equiv (A \Rightarrow B) \wedge (A \Rightarrow C)$$

$$\begin{array}{ll} \lambda x^A. \langle \mathbf{r}, \mathbf{s} \rangle \Leftrightarrow \langle \lambda x^A. \mathbf{r}, \lambda x^A. \mathbf{s} \rangle & \Rightarrow_i, \wedge_i \Leftrightarrow \wedge_i, \Rightarrow_i \\ \lambda x^A. \pi_B(\mathbf{r}) \Leftrightarrow \pi_{A \Rightarrow B}(\lambda x^A. \mathbf{r}) & \Rightarrow_i, \wedge_e \Leftrightarrow \wedge_e, \Rightarrow_i \end{array}$$

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$$\begin{array}{ll} \langle \mathbf{r}, \mathbf{s} \rangle \mathbf{t} \Leftrightarrow \langle \mathbf{r}\mathbf{t}, \mathbf{s}\mathbf{t} \rangle & \Rightarrow_e, \wedge_i \Leftrightarrow \wedge_i, \Rightarrow_e \\ \pi_{A \Rightarrow B}(\mathbf{r})\mathbf{s} \Leftrightarrow \pi_B(\mathbf{r}\mathbf{s})^* & \Rightarrow_e, \wedge_e \Leftrightarrow \wedge_e, \Rightarrow_e \end{array}$$

* if $\mathbf{r} : A \Rightarrow (B \wedge C)$

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* if $\mathbf{r} : A \Rightarrow (B \wedge C)$

$\pi_{A \Rightarrow B}(\mathbf{r})\mathbf{s} \Leftrightarrow \pi_B(\mathbf{r}\mathbf{s})$ suffices ... plus η and δ (surjective pairing)

α -equivalence

Rules

- ▶ If $A \equiv B$, $\mathbf{r} \leftrightarrow \mathbf{r}[A/B]$
- ▶ If $\mathbf{r} =_{\alpha} \mathbf{r}'$, $\mathbf{r} \leftrightarrow \mathbf{r}'$

Example

Let $A \equiv B$ $\underbrace{\langle \lambda x^A . \mathbf{r}, \lambda y^B . \mathbf{s} \rangle}_{A \Rightarrow (C_1 \wedge C_2)}$

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Example

$$\begin{aligned} \text{Let } A \equiv B \quad & \overbrace{\langle \lambda x^A. \mathbf{r}, \lambda y^B. \mathbf{s} \rangle}^{A \Rightarrow (C_1 \wedge C_2)} \Leftrightarrow^* \langle \lambda x^A. \mathbf{r}, \lambda x^A. \mathbf{s}[x/y][A/B] \rangle \\ & \Leftrightarrow \lambda x^A. \langle \mathbf{r}, \mathbf{s}[x/y][A/B] \rangle \end{aligned}$$

The full operational semantics

Symmetric relation:

$$\langle \mathbf{r}, \mathbf{s} \rangle \rightleftharpoons \langle \mathbf{s}, \mathbf{r} \rangle \quad (\text{comm})$$

$$\langle \langle \mathbf{r}, \mathbf{s} \rangle, \mathbf{t} \rangle \rightleftharpoons \langle \mathbf{r}, \langle \mathbf{s}, \mathbf{t} \rangle \rangle \quad (\text{asso})$$

$$\mathbf{rst} \rightleftharpoons \mathbf{r}\langle \mathbf{s}, \mathbf{t} \rangle \quad (\text{curry})$$

$$\text{If } \mathbf{r} : A \Rightarrow C, \pi_{A \Rightarrow B}(\mathbf{r}\mathbf{s}) \rightleftharpoons \pi_B(\mathbf{r}\mathbf{s}) \quad (\text{dist})$$

$$\text{If } A \equiv B, \mathbf{r} \rightleftharpoons \mathbf{r}[A/B] \quad (\alpha\text{-Types})$$

$$\text{If } \mathbf{r} =_{\alpha} \mathbf{s}, \mathbf{r} \rightleftharpoons \mathbf{s} \quad (\alpha\text{-Terms})$$

If $\mathbf{r} \rightleftharpoons \mathbf{s}$, then $K[\mathbf{r}] \rightleftharpoons K[\mathbf{s}]$ for any context $K[\cdot]$

Reductions:

$$\text{If } \mathbf{s} : A, (\lambda x^A. \mathbf{r})\mathbf{s} \rightarrow \mathbf{r}[\mathbf{s}/x] \quad (\beta)$$

$$\text{If } \mathbf{r} : A, \pi_A(\mathbf{r}, \mathbf{s}) \rightarrow \mathbf{r} \quad (\pi_n)$$

$$\text{If } \mathbf{r} : A, \pi_A(\mathbf{r}) \rightarrow \mathbf{r} \quad (\pi_1)$$

Extensionality:

$$\text{If } \left\{ \begin{array}{l} x \notin FV(\mathbf{r}), \\ \mathbf{r} : A \Rightarrow B, \\ \mathbf{r} \not\stackrel{*}{\rightarrow} \lambda y^A. \mathbf{r}' \end{array} \right\}, \mathbf{r} \rightarrow \lambda x^A. \mathbf{r}x \quad (\eta)$$

$$\text{If } \left\{ \begin{array}{l} \mathbf{r} : A \wedge B, \\ \mathbf{r} \not\stackrel{*}{\rightarrow} \langle \mathbf{r}_1, \mathbf{r}_2 \rangle \\ \text{with } \mathbf{r}_1 : A, \mathbf{r}_2 : B \end{array} \right\}, \mathbf{r} \rightarrow \langle \pi_A(\mathbf{r}), \pi_B(\mathbf{r}) \rangle \quad (\delta)$$

$$\frac{\mathbf{r} \rightarrow \mathbf{s}}{\mathbf{r} \rightarrow \mathbf{s}} \quad \frac{\mathbf{r} \rightarrow \mathbf{s}}{\lambda x^A. \mathbf{r} \rightarrow \lambda x^A. \mathbf{s}} \quad \frac{\mathbf{r} \rightarrow \mathbf{s}}{\mathbf{tr} \rightarrow \mathbf{ts}} \quad \frac{\mathbf{r} \xrightarrow{\eta} \mathbf{s}}{\mathbf{rt} \rightarrow \mathbf{st}} \quad \frac{\mathbf{r} \rightarrow \mathbf{s}}{\langle \mathbf{t}, \mathbf{r} \rangle \rightarrow \langle \mathbf{t}, \mathbf{s} \rangle} \quad \frac{\mathbf{r} \xrightarrow{\delta} \mathbf{s}}{\pi_A(\mathbf{r}) \rightarrow \pi_A(\mathbf{s})}$$

Normalisation

\mathbf{r} is in normal form, if it can only continue reducing by relation \leftrightarrow

Normal form

$$\text{Red}(\mathbf{r}) = \{\mathbf{s} \mid \mathbf{r} \leftrightarrow^* \mathbf{r}' \rightarrow \mathbf{s}' \leftrightarrow^* \mathbf{s}\}$$

\mathbf{r} in normal form if $\text{Red}(\mathbf{r}) = \emptyset$

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Theorem (Strong normalisation)

If $\Gamma \vdash \mathbf{r} : A$ then \mathbf{r} strongly normalising

Proof. Reducibility method

Finding an interpretation

The standard interpretation does not work

$$\llbracket A \wedge B \rrbracket = \{ \mathbf{r} \mid \pi_A(\mathbf{r}) \in \llbracket A \rrbracket \text{ and } \pi_B(\mathbf{r}) \in \llbracket B \rrbracket \}$$

Counter-example: $\mathbf{r} = x^A, y^B, \Omega \in \llbracket A \wedge B \rrbracket$

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How to prove that $\mathbf{r} \in \llbracket A \rrbracket$ and $\mathbf{s} \in \llbracket B \rrbracket$ implies $\mathbf{r}, \mathbf{s} \in \llbracket A \wedge B \rrbracket$?

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$$\pi_A(\mathbf{r}, \mathbf{s}) \rightarrow \mathbf{r}$$

$$\pi_B(\mathbf{r}, \mathbf{s}) \rightarrow \mathbf{s}$$

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$$\begin{aligned} \pi_A(\mathbf{r}, \mathbf{s}) &\rightarrow \mathbf{r} \\ \pi_B(\mathbf{r}, \mathbf{s}) &\rightarrow \mathbf{s} \end{aligned}$$

but...

$$\begin{aligned} \text{let } \begin{cases} A = A_1 \wedge A_2 \\ B = A_2 \wedge B_2 \end{cases} \\ \pi_{A_1 \wedge A_2}(\underbrace{\mathbf{r}_1, \mathbf{r}_2}_{A_1 \wedge A_2}, \underbrace{\mathbf{s}_1, \mathbf{s}_2}_{A_2 \wedge B_2}) &\rightarrow \mathbf{r}_1, \mathbf{s}_1 \end{aligned}$$

We need something more subtle

Finding an interpretation

Conjunction free

Conjunction-free type

$S, R ::= \tau \mid S \Rightarrow R$

Finding an interpretation

Conjunction free

Conjunction-free type

$S, R ::= \tau \mid S \Rightarrow R$

Lemma

$$\forall A, \quad A \equiv \bigwedge_{i=1}^m S_{i1} \Rightarrow \cdots \Rightarrow S_{in_i} \Rightarrow \tau$$

Example

$$(A \wedge B) \Rightarrow C \equiv A \Rightarrow B \Rightarrow C$$

$$A \Rightarrow (B \wedge C) \equiv (A \Rightarrow B) \wedge (A \Rightarrow C)$$

Finding an interpretation

Interpreting canonical types

$$\left[\left[\bigwedge_{i=1}^n \left(\overline{(S_{ij})_{j=1}^{m_i}} \Rightarrow \tau \right) \right] \right] = \left\{ \mathbf{r} \mid \forall i, \left[\begin{array}{l} \mathbf{s}_{ij} \in \llbracket S_{ij} \rrbracket \\ j = 1, \dots, m_i \end{array} \right] \text{ implies } \pi_{\overline{(S_{ij})_{j=1}^{m_i}} \Rightarrow \tau}(\mathbf{r}) \vec{\mathbf{s}}_i \in \mathcal{SN} \right\}$$

with $n \geq 1$, and $m \geq 0$

Finding an interpretation

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Example

$$\llbracket (S \Rightarrow \tau) \wedge \tau \rrbracket = \left\{ \mathbf{r} \mid \begin{array}{l} \mathbf{s} \in \llbracket S \rrbracket \text{ implies } \pi_{S \Rightarrow \tau}(\mathbf{r}) \mathbf{s} \in \mathcal{SN} \\ \text{and } \pi_{\tau}(\mathbf{r}) \in \mathcal{SN} \end{array} \right\}$$

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Lemma

If $\mathbf{r} : A$ then $\mathbf{r} \in \llbracket A \rrbracket$

Corollary

If $\mathbf{r} : A$ then $\mathbf{r} \in \mathcal{SN}$

Computing with this calculus

Pairs

$$\pi_{Nat}(3, 4) \rightarrow 3 \quad \text{and} \quad \pi_{Nat}(3, 4) \rightarrow 4$$

But it is possible to **encode** pairs behaving in an standard way

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Booleans

$$\begin{aligned}(\lambda x^A. \lambda y^A. x) \mathbf{rs} &\Leftrightarrow (\lambda x^A. \lambda y^A. x)(\mathbf{r}, \mathbf{s}) \\ &\Leftrightarrow (\lambda x^A. \lambda y^A. x)(\mathbf{s}, \mathbf{r}) \\ &\Leftrightarrow (\lambda x^A. \lambda y^A. x) \mathbf{sr} \rightarrow^* \mathbf{s}\end{aligned}$$

Hence
true \sim **false**

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Hence
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But $(A \Rightarrow B) \Rightarrow B \Rightarrow B$ is not

So, it is possible to encode booleans too

Summarising

What have we done?

We introduced a new calculus where isomorphic propositions have the same proofs

Why?

If $A \equiv B$, a **proof** of A should **not be distinguishable** from a proof of B

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Future work

- ▶ Add more connectives
- ▶ Introduce polymorphism (not trivial)
- ▶ Move to dependant types (provable isomorphisms?)