Simply Typed Lambda-Calculus Modulo Type Isomorphisms

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Propositional equivalence vs definitional equality

$\forall x.\forall y.(x \;=\; y) \Rightarrow (\mathsf{even}(x) = \mathsf{even}(y))$

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$\forall x.\forall y.(x = y) \Rightarrow (\text{even}(x) = \text{even}(y))$

$(2 + 2 = 4) \Rightarrow (\text{even}(2 + 2) = \text{even}(4))$

$(2 + 2 = 5) \Rightarrow (\text{even}(2 + 2) = \text{even}(5))$

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Proposition

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$\forall X.\forall Y.(X \equiv Y) \Rightarrow (X \Leftrightarrow Y)$

Propositional

$(A \equiv B) \Rightarrow (A \Leftrightarrow B)$

Decidable! `[DiCosmo et al]`

(for some settings)

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We want to go further:

$$(A \equiv B) \Rightarrow (\pi : A \Leftrightarrow \pi : B)$$

The goal is to identify isomorphic types

The basic setting

- Simply types with conjunction and implication

$$A, B, C ::= \tau \mid A \Rightarrow B \mid A \wedge B$$

- An equivalence relation between types (based on the isomorphisms[1])
  1. $A \wedge B \equiv B \wedge A$ (comm)
  2. $A \wedge (B \wedge C) \equiv (A \wedge B) \wedge C$ (asso)
  3. $(A \wedge B) \Rightarrow C \equiv A \Rightarrow B \Rightarrow C$ (curry)
  4. $A \Rightarrow (B \wedge C) \equiv (A \Rightarrow B) \wedge (A \Rightarrow C)$ (distrib)

[1] Bruce, Di Cosmo, Longo, MSCS 2(2), 231–247, 1992

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We want

$$[A\equiv B]\frac{\mathbf{r} : A}{\mathbf{r} : B}$$

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Associative and commutative conjunction

$$\frac{\Gamma \vdash \mathbf{r} : A \quad \Gamma \vdash \mathbf{s} : B}{\Gamma \vdash \langle \mathbf{r}, \mathbf{s} \rangle : A \wedge B} \; (\wedge_i)$$

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So
$$\langle \mathbf{r}, \mathbf{s} \rangle \leftrightarrows \langle \mathbf{s}, \mathbf{r} \rangle$$
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What about the elimination?

$$\frac{\Gamma \vdash \langle \mathbf{r}, \mathbf{s} \rangle : A \wedge B}{\Gamma \vdash \pi_1 \langle \mathbf{r}, \mathbf{s} \rangle : A} \ (\wedge_e)$$

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Workaround: Church-style – Project w.r.t. a type

If $\mathbf{r} : A$ then $\pi_A \langle \mathbf{r}, \mathbf{s} \rangle \to \mathbf{r}$

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We are interested in the proof theory and both **r** and **s** are valid proofs of A

Curryfication

$$(A \wedge B) \Rightarrow C \quad \equiv \quad A \Rightarrow B \Rightarrow C$$

induces

$$\mathbf{r}\langle \mathbf{s}, \mathbf{t} \rangle \quad \leftrightarrows \quad \mathbf{rst}$$

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$$\mathbf{r}\langle \mathbf{s}, \mathbf{t} \rangle \quad \leftrightarrows \quad \mathbf{rst}$$

$$(\lambda x^A.\mathbf{r})\mathbf{s} \quad \rightarrow \quad \mathbf{r}[\mathbf{s}/x]$$

Curryfication

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If $\mathbf{s} : A$, $\quad (\lambda x^A.\mathbf{r})\mathbf{s} \quad \to \quad \mathbf{r}[\mathbf{s}/x]$

Distributivity of implication over conjunction

$$A \Rightarrow (B \wedge C) \quad \equiv \quad (A \Rightarrow B) \wedge (A \Rightarrow C)$$

induces

$\lambda x^A.\langle \mathbf{r}, \mathbf{s}\rangle \leftrightarrows \langle \lambda x^A.\mathbf{r}, \lambda x^A.\mathbf{s}\rangle$

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Example

$$\dfrac{\dfrac{\vdash \lambda x^{A\wedge B}.x : (A\wedge B) \Rightarrow (A \wedge B)}{\vdash \lambda x^{A\wedge B}.x : ((A\wedge B)\Rightarrow A) \wedge ((A\wedge B) \Rightarrow B)}\ (\equiv)}{\vdash \pi_{(A\wedge B)\Rightarrow A}(\lambda x^{A\wedge B}.x) : (A\wedge B) \Rightarrow A}\ (\wedge_e)$$

$$\pi_{(A\wedge B)\Rightarrow A}(\lambda x^{A\wedge B}.x) \leftrightarrows \lambda x^{A\wedge B}.\pi_A(x)$$

Distributivity of implication over conjunction

Multiple choices

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$$\begin{array}{rcl} \langle \mathbf{r}, \mathbf{s} \rangle \mathbf{t} & \leftrightarrows & \langle \mathbf{rt}, \mathbf{st} \rangle \\ \pi_{A \Rightarrow B}(\mathbf{r})\mathbf{s} & \leftrightarrows & \pi_B(\mathbf{rs})^* \end{array} \qquad \begin{array}{rcl} \Rightarrow_e, \wedge_i & \leftrightarrows & \wedge_i, \Rightarrow_e \\ \Rightarrow_e, \wedge_e & \leftrightarrows & \wedge_e, \Rightarrow_e \end{array}$$

* if $\mathbf{r} : A \Rightarrow (B \wedge C)$

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* if $\mathbf{r} : A \Rightarrow (B \wedge C)$

$\pi_{A\Rightarrow B}(\mathbf{r})\mathbf{s} \leftrightarrows \pi_B(\mathbf{rs})$ suffices ... plus η and δ (surjective pairing)

α-equivalence

Rules

- If $A \equiv B$, $\mathbf{r} \leftrightarrows \mathbf{r}[A/B]$
- If $\mathbf{r} =_\alpha \mathbf{r}'$, $\mathbf{r} \leftrightarrows \mathbf{r}'$

Example

Let $A \equiv B$ $\qquad \overbrace{\langle \lambda x^A.\mathbf{r}, \lambda y^B.\mathbf{s} \rangle}^{A \Rightarrow (C_1 \wedge C_2)}$

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Example

$$\text{Let } A \equiv B \qquad \overbrace{\langle \lambda x^A.\mathbf{r}, \lambda y^B.\mathbf{s} \rangle}^{A \Rightarrow (C_1 \wedge C_2)} \leftrightarrows^* \langle \lambda x^A.\mathbf{r}, \lambda x^A.\mathbf{s}[x/y][A/B] \rangle$$

$$\leftrightarrows \lambda x^A.\langle \mathbf{r}, \mathbf{s}[x/y][A/B] \rangle$$

The full operational semantics

Symmetric relation:

$$\langle \mathbf{r},\mathbf{s}\rangle \rightleftarrows \langle \mathbf{s},\mathbf{r}\rangle \qquad \text{(comm)}$$

$$\langle\langle \mathbf{r},\mathbf{s}\rangle,\mathbf{t}\rangle \rightleftarrows \langle \mathbf{r},\langle\mathbf{s},\mathbf{t}\rangle\rangle \qquad \text{(asso)}$$

$$\mathbf{rst} \rightleftarrows \mathbf{r}\langle \mathbf{s},\mathbf{t}\rangle \qquad \text{(curry)}$$

$$\text{If } \mathbf{r} : A \Rightarrow C,\ \pi_{A\Rightarrow B}(\mathbf{r})\mathbf{s} \rightleftarrows \pi_B(\mathbf{rs}) \qquad \text{(dist)}$$

$$\text{If } A \equiv B,\ \mathbf{r} \rightleftarrows \mathbf{r}[A/B] \qquad (\alpha\text{–Types})$$

$$\text{If } \mathbf{r} =_\alpha \mathbf{s},\ \mathbf{r} \rightleftarrows \mathbf{s} \qquad (\alpha\text{–Terms})$$

If $\mathbf{r} \rightleftarrows \mathbf{s}$, then $K[\mathbf{r}] \rightleftarrows K[\mathbf{s}]$ for any context $K[\cdot]$

Reductions:

$$\text{If } \mathbf{s} : A,\ (\lambda x^A.\mathbf{r})\mathbf{s} \rightharpoonup \mathbf{r}[\mathbf{s}/x] \qquad (\beta)$$

$$\text{If } \mathbf{r} : A,\ \pi_A\langle \mathbf{r},\mathbf{s}\rangle \rightharpoonup \mathbf{r} \qquad (\pi_n)$$

$$\text{If } \mathbf{r} : A,\ \pi_A(\mathbf{r}) \rightharpoonup \mathbf{r} \qquad (\pi_1)$$

Extensionality:

$$\text{If } \left\{ \begin{array}{l} x \notin FV(\mathbf{r}), \\ \mathbf{r} : A \Rightarrow B, \\ \mathbf{r} \not\rightleftarrows^* \lambda y^A.\mathbf{r}' \end{array} \right\},\ \mathbf{r} \rightharpoonup \lambda x^A.\mathbf{r}x \qquad (\eta)$$

$$\text{If } \left\{ \begin{array}{l} \mathbf{r} : A \wedge B, \\ \mathbf{r} \not\rightleftarrows^* \langle \mathbf{r}_1,\mathbf{r}_2\rangle \\ \text{with } \mathbf{r}_1 : A, \mathbf{r}_2 : B \end{array} \right\},\ \mathbf{r} \rightharpoonup \langle \pi_A(\mathbf{r}),\pi_B(\mathbf{r})\rangle \qquad (\delta)$$

$$\frac{\mathbf{r} \rightharpoonup \mathbf{s}}{\mathbf{r} \to \mathbf{s}} \quad \frac{\mathbf{r} \to \mathbf{s}}{\lambda x^A.\mathbf{r} \to \lambda x^A.\mathbf{s}} \quad \frac{\mathbf{r} \to \mathbf{s}}{\mathbf{tr} \to \mathbf{ts}} \quad \frac{\mathbf{r} \overset{\neg\eta}{\to} \mathbf{s}}{\mathbf{rt} \to \mathbf{st}} \quad \frac{\mathbf{r} \to \mathbf{s}}{\langle \mathbf{t},\mathbf{r}\rangle \to \langle \mathbf{t},\mathbf{s}\rangle} \quad \frac{\mathbf{r} \overset{\neg\delta}{\to} \mathbf{s}}{\pi_A(\mathbf{r}) \to \pi_A(\mathbf{s})}$$

Normalisation

$\mathbf{r}$ is in normal form, if it can only continue reducing by relation $\leftrightarrows$

Normal form

$$\mathrm{Red}(\mathbf{r}) = \{\mathbf{s} \mid \mathbf{r} \leftrightarrows^* \mathbf{r}' \to \mathbf{s}' \leftrightarrows^* \mathbf{s}\}$$

$\mathbf{r}$ in normal form if $\mathrm{Red}(\mathbf{r}) = \emptyset$

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r in normal form if $\mathrm{Red}(\mathbf{r}) = \emptyset$

Theorem (Strong normalisation)

If $\Gamma \vdash \boldsymbol{r} : A$ *then* $\boldsymbol{r}$ *strongly normalising*

Proof. Reducibility method

Finding an interpretation

The standard interpretation does not work

$$\boxed{[\![A \wedge B]\!] = \{\mathbf{r} \mid \pi_A(\mathbf{r}) \in [\![A]\!] \text{ and } \pi_B(\mathbf{r}) \in [\![B]\!]\}}$$

Counter-example: $\mathbf{r} = x^A, y^B, \Omega \in [\![A \wedge B]\!]$

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$\pi_B(\mathbf{r}, \mathbf{s}) \to \mathbf{s}$

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$$\pi_A(\mathbf{r}, \mathbf{s}) \to \mathbf{r}$$
$$\pi_B(\mathbf{r}, \mathbf{s}) \to \mathbf{s}$$

but...

let $\begin{cases} A = A_1 \wedge A_2 \\ B = A_2 \wedge B_2 \end{cases}$

$$\pi_{A_1 \wedge A_2}(\underbrace{\mathbf{r}_1, \mathbf{r}_2}_{A_1 \wedge A_2}, \underbrace{\mathbf{s}_1, \mathbf{s}_2}_{A_2 \wedge B_2}) \to \mathbf{r}_1, \mathbf{s}_1$$

We need something more subtle

Finding an interpretation

Conjunction free

Conjunction-free type	$S, R ::= \tau \mid S \Rightarrow R$

Finding an interpretation

Conjunction free

Conjunction-free type	$S, R ::= \tau \mid S \Rightarrow R$

Lemma

$$\forall A, \qquad A \equiv \bigwedge_{i=1}^{m} S_{i1} \Rightarrow \cdots \Rightarrow S_{in_i} \Rightarrow \tau$$

Example

$$(A \wedge B) \Rightarrow C \equiv A \Rightarrow B \Rightarrow C$$
$$A \Rightarrow (B \wedge C) \equiv (A \Rightarrow B) \wedge (A \Rightarrow C)$$

Finding an interpretation

Interpreting canonical types

$$\left[\!\left[\bigwedge_{i=1}^{n}\left(\overline{(S_{ij})}_{j=1}^{m_i} \Rightarrow \tau\right)\right]\!\right] = \left\{\mathbf{r} \mid \forall i,\ \left[\begin{array}{l}\mathbf{s}_{ij} \in [\![S_{ij}]\!] \\ j = 1, \dots, m_i\end{array} \quad \text{implies } \pi_{\overline{(S_{ij})}_{j=1}^{m_i} \Rightarrow \tau}(\mathbf{r})\vec{\mathbf{s}}_i \in \mathcal{SN}\right]\right\}$$

with $n \geq 1$, and $m \geq 0$

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Example

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Lemma

If $\boldsymbol{r} : A$ *then* $\boldsymbol{r} \in [\![A]\!]$

Corollary

If $\boldsymbol{r} : A$ *then* $\boldsymbol{r} \in \mathcal{SN}$

Computing with this calculus

Pairs

$$\pi_{Nat}(3,4) \to 3 \quad \text{and} \quad \pi_{Nat}(3,4) \to 4$$

But it is possible to encode pairs behaving in an standard way

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$A \Rightarrow A \Rightarrow A$
is a singleton!

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$A \Rightarrow A \Rightarrow A$
is a singleton!

But $(A \Rightarrow B) \Rightarrow B \Rightarrow B$ is not

So, it is possible to encode booleans too

Summarising

What have we done?

We introduced a new calculus where
isomorphic propositions have the same proofs

Why?

If $A \equiv B$, a **proof** of A
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Future work

- Add more connectives
- Introduce polymorphism (not trivial)
- Move to dependant types (provable isomorphisms?)