UNIVERSITÉ DE GRENOBLE

Soutenance de thèse

Du typage vectoriel

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CAPP Team

LABORATOIRE D'INFORMATIQUE DE GRENOBLE

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23 - 09 - 2011

Lambda calculus [Church'36]

Formal system to study the definition of function

$$
\begin{aligned}
f(x) &\sim t_x \\
x \mapsto f(x) &\sim \lambda x.t_x \\
(x \mapsto f(x))r &\sim (\lambda x.t_x)\ r \\
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Type system [Church'40]

"a tractable syntactic framework for classifying phrases according to the kinds of values they compute"

—[Pierce'02]

$$\frac{\lambda x.\mathbf{t}_x : T \to R \qquad \mathbf{r} : T}{(\lambda x.\mathbf{t}_x)\ \mathbf{r} : R}$$

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System F [Girard'71]

TS with a universal quantification over types

$$\begin{aligned} \lambda x.x &: Int \to Int \\ \lambda x.x &: Bool \to Bool \\ &\dots \end{aligned}$$

$$\rightsquigarrow \quad \lambda x.x : \forall X.X \to X$$

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Curry-Howard correspondence

Correspondence between type systems and logic

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Church vs. Curry style whether the types are part of the terms or not

To capture probabilistic/quantum/quantitative constructions:

algebraic extensions

$\mathbf{t}, \mathbf{r} ::= x \mid \lambda x.\mathbf{t} \mid (\mathbf{t})\,\mathbf{r} \mid \mathbf{t} + \mathbf{r} \mid \alpha.\mathbf{t} \mid 0$ $\qquad \alpha \in (\mathcal{S}, +, \times)$, a ring.

Two origins:

- Differential λ-calculus [Ehrhard'03]: linearity *à la* Linear Logic

  Removing the differential operator: Algebraic λ-calculus (λ_{alg}) [Vaux'09]
- Quantum computing: superposition of programs

  Linearity as in algebra: Linear-algebraic λ-calculus (λ_{lin}) [Arrighi,Dowek'08]

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Beta reduction:

$$(\lambda x.\mathbf{t})\ \mathbf{r} \to \mathbf{t}[\mathbf{r}/x]$$

"Algebraic" reductions:

$$\begin{aligned} \alpha.\mathbf{t} + \beta.\mathbf{t} &\to (\alpha + \beta).\mathbf{t}, \\ \alpha.\beta.\mathbf{t} &\to (\alpha \times \beta).\mathbf{t}, \\ (\mathbf{t})\ (\mathbf{r}_1 + \mathbf{r}_2) &\to (\mathbf{t})\ \mathbf{r}_1 + (\mathbf{t})\ \mathbf{r}_2, \\ (\mathbf{t}_1 + \mathbf{t}_2)\ \mathbf{r} &\to (\mathbf{t}_1)\ \mathbf{r} + (\mathbf{t}_2)\ \mathbf{r}, \\ &\dots \end{aligned}$$

(oriented version of the axioms of vectorial spaces)[Arrighi,Dowek'07]

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$\mathcal{B} = \{\mathbf{t}_i : \mathbf{t}_i \text{ var. or abs. }\}$

Set of values $::= \mathrm{Span}(\mathcal{B})$

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	λ_{alg}	λ_{lin}
Origin	Linear Logic	Quantum computing
Evaluation strategy	Call-by-name	Call-by-base
Algebraic part	Equalities	Rewrite system

Contribution: CPS simulation [Díaz-Caro,Perdrix,Tasson,Valiron'10]

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Two base vectors: $\textbf{true} = \lambda x.\lambda y.x$
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Aim:

To provide a type system capturing the "vectorial" structure of terms

... to check for properties of probabilistic processes

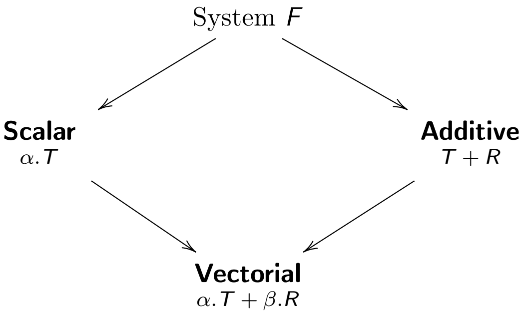
... to check for properties of quantum processes

... or whatever application needing the structure of the vector in normal form

... understand what it means "linear combination of types"

... a Curry-Howard approach to defining Fuzzy/Quantum/Probabilistic logics from Fuzzy/Quantum/Probabilistic programming languages.

Plan



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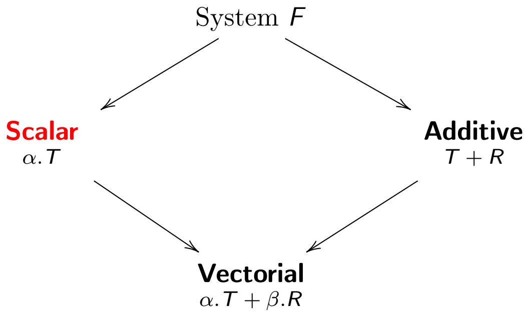

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A polymorphic type system *tracking scalars*:

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Gives the "amount" of terms $\rightarrow$ Barycentric restrictions ($\sum \alpha_i = 1$)

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Definition (Weight function (to check barycentricity))

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Contribution: [Arrighi,Díaz-Caro'09]

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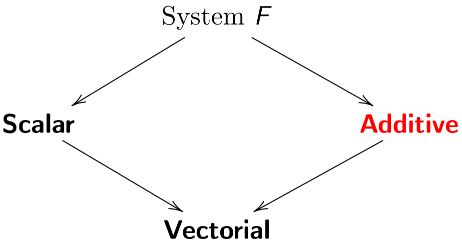

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Also we set up an inverse translation showing that it is non-trivial

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Subject reduction ✓

Strong normalisation (using the one from System F_p) ✓

Contribution: `[Díaz-Caro,Petit'10]`

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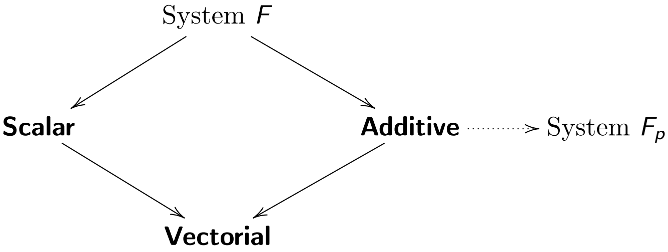
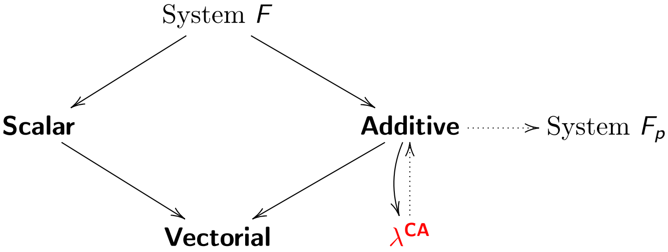


Plan



System F
Scalar
Additive
System F_p
Vectorial
λ^{CA}

The *Complete Additive* System (λ^{CA})

Extending sums to the whole calculus (with positive reals scalars)

$$\frac{\Gamma \vdash \mathbf{t} : T}{\Gamma \vdash \alpha.\mathbf{t} : \lfloor\alpha\rfloor.T} = \underbrace{T + \cdots + T}_{\lfloor\alpha\rfloor}$$

- More general than Additive
- Less complex than Vectorial
- "Amounts" approximated

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- More general than Additive
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- "Amounts" approximated

If $\vdash \mathbf{t} : T$, then $\vdash (0.9).\mathbf{t} + (1.1).\mathbf{t} : T$

$(0.9).\mathbf{t} + (1.1).\mathbf{t} \to 2.\mathbf{t}$ and $\vdash 2.\mathbf{t} : 2.T$

The *Complete Additive* System (λ^{CA})

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Weak subject reduction: $\mathbf{t} \to \mathbf{r}$, $\Gamma \vdash \mathbf{t} : T \Rightarrow \Gamma \vdash \mathbf{r} : R$ with $T \preceq R$

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Abstract interpretation (theorem)

$$\begin{array}{ccccc}
\lambda^{\mathrm{CA}} & \xrightarrow{\tau} & \lambda_{add} & \xrightarrow{[\cdot]} & F_p \\
\downarrow & & \downarrow_a\ (\preceq) & & \downarrow_F\ (\preceq) \\
\lambda^{\mathrm{CA}} & \xrightarrow[\tau]{} & \lambda_{add} & \xrightarrow[{[\cdot]}]{} & F_p
\end{array}$$

The *Complete Additive* System (λ^{CA})

Extending sums to the whole calculus (with positive reals scalars)

$$\frac{\Gamma \vdash \mathbf{t} : T}{\Gamma \vdash \alpha.\mathbf{t} : \lfloor\alpha\rfloor.T = \underbrace{T + \cdots + T}_{\lfloor\alpha\rfloor}}$$

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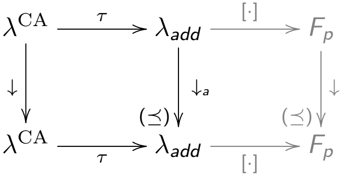
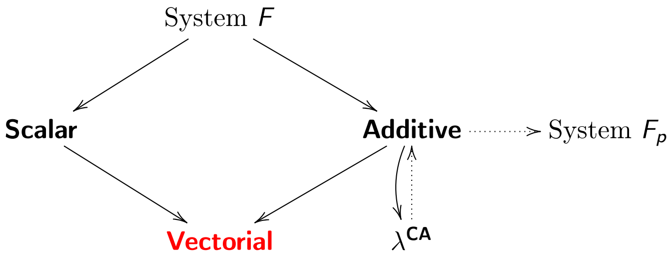


Strong normalisation (using *Additive*)

Contribution: `[Buiras,Díaz-Caro,Jaskelioff'11]`

Plan


System F
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The *Vectorial* system

Types:

$$T, R, S := U \mid T + R \mid \alpha.T$$
$$U, V, W := X \mid U \to T \mid \forall X.U$$

(U, V, W reflect the basis terms)

Equivalences:

$$\begin{array}{rcl} 1.T & \equiv & T \\ \alpha.(\beta.T) & \equiv & (\alpha \times \beta).T \\ \alpha.T + \alpha.R & \equiv & \alpha.(T + R) \\ \alpha.T + \beta.T & \equiv & (\alpha + \beta).T \\ T + R & \equiv & R + T \\ T + (R + S) & \equiv & (T + R) + S \end{array}$$

(reflect the vectorial spaces axioms)

Typing rules

$$\frac{}{\Gamma, x : U \vdash x : U}\ ax \qquad \frac{\Gamma \vdash \mathbf{t} : T}{\Gamma \vdash 0 : 0.T}\ 0_I \qquad \frac{\Gamma \vdash \mathbf{t} : T}{\Gamma \vdash \alpha.\mathbf{t} : \alpha.T}\ s_I$$

$$\frac{\Gamma \vdash \mathbf{t}\colon \sum_{i=1}^{n} \alpha_i.\forall \vec{X}.(U \to T_i) \qquad \Gamma \vdash \mathbf{r}\colon \sum_{j=1}^{m} \beta_j.V_j \qquad \begin{array}{c}\forall V_j, \exists \vec{W}_j\ / \\ U[\vec{W}_j/\vec{X}] = V_j\end{array}}{\Gamma \vdash (\mathbf{t})\ \mathbf{r}\colon \sum_{i=1}^{n}\sum_{j=1}^{m} \alpha_i \times \beta_j.T_i[\vec{W}_j/\vec{X}]}\ \to_E$$

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Strong normalisation: Reducibility candidates ✓
Main difficulty: show that $\{\mathbf{t}_i\}_i$ SN $\Rightarrow \sum_i \mathbf{t}_i$ SN (algebraic measure)

Typing rules

$$\frac{}{\Gamma, x : U \vdash x : U}\ ax \qquad \frac{\Gamma \vdash \mathbf{t} : T}{\Gamma \vdash 0 : 0.T}\ 0_I \qquad \frac{\Gamma \vdash \mathbf{t} : T}{\Gamma \vdash \alpha.\mathbf{t} : \alpha.T}\ s_I$$

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Main difficulty: show that $\{\mathbf{t}_i\}_i$ SN $\Rightarrow \sum_i \mathbf{t}_i$ SN (algebraic measure)
Subject reduction **a challenge**

The case of the factorisation rule

System F à la Curry: a term can have different, unrelated types

$$\frac{\Gamma \vdash \mathbf{t} : T \qquad \Gamma \vdash \mathbf{t} : T'}{\Gamma \vdash \alpha.\mathbf{t} + \beta.\mathbf{t} : \alpha.T + \beta.T'}$$

However, $\alpha.\mathbf{t} + \beta.\mathbf{t} \to (\alpha + \beta).\mathbf{t}$... one of the two types must be chosen!

In general $\alpha.T + \beta.T' \neq (\alpha + \beta).T \neq (\alpha + \beta).T'$

(and since we are working in System F, there is no principal types neither)

Several possible solutions:

- Remove factorisation rule (Done. SR and SN both work)
  - $+$ in scalars not used anymore. Scalars $\Rightarrow$ Monoid
  - It works!... but it is no so expressive ("vectorial" structure lost)

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  - As soon as we add this one, we have to add many others
  - Too complex and inelegant (subject reduction by axiom)

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- Weak subject reduction
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    - if R is not the factorisation rule: $\Gamma \vdash \mathbf{r} : T$
    - if R is the factorisation rule: $\exists S \sqsubseteq T \;/\; \Gamma \vdash \mathbf{r} : S$

  where $(\alpha + \beta).T \sqsubseteq \alpha.T + \beta.T'$ if $\exists \mathbf{t} \;/\; \Gamma \vdash \mathbf{t} : T$ and $\Gamma \vdash \mathbf{t} : T'$

Contribution: `[Arrighi,Díaz-Caro,Valiron'11]`

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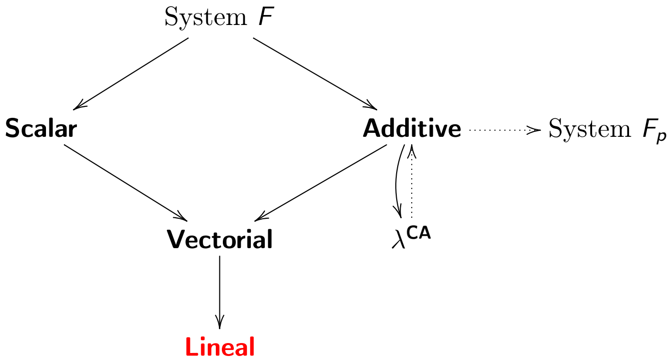
  Contribution: `[Arrighi,Díaz-Caro,Valiron'11]`
- Church style
  - Seems to be the *natural* solution: the type is part of the term, if the types are different, the terms are different (no factorisation rule)

Plan


System F
Scalar
Additive
System F_p
Vectorial
λ^{CA}
Lineal

The system *Lineal*

Types:

$$T, R, S := U \mid T + R \mid \alpha.T$$
$$U, V, W := X \mid U \to T \mid \forall X.U \mid U@(\textstyle\sum_i V_i)$$

(U, V, W reflect the basis terms)

Equivalences:

$$\begin{array}{rcl}
1.T & \equiv & T \\
\alpha.(\beta.T) & \equiv & (\alpha \times \beta).T \\
\alpha.T + \alpha.R & \equiv & \alpha.(T + R) \\
\alpha.T + \beta.T & \equiv & (\alpha + \beta).T \\
T + R & \equiv & R + T \\
T + (R + S) & \equiv & (T + R) + S \\
(\forall X.U)@V & \equiv & U[V/X]
\end{array}$$

(reflect the vectorial spaces axioms)

Typing rules

$$\frac{}{\Gamma, x : U \vdash x : U}\ ax \qquad \frac{\Gamma \vdash \mathbf{t} : T}{\Gamma \vdash 0 : 0.T}\ 0_I \qquad \frac{\Gamma \vdash \mathbf{t} : T}{\Gamma \vdash \alpha.\mathbf{t} : \alpha.T}\ s_I$$

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Subject reduction ✓

Strong normalisation (using Vectorial) ✓

Most important properties of *Lineal*

Theorem

If $\Gamma \vdash \boldsymbol{t} \colon \sum_i \alpha_i.U_i$ *then* $\boldsymbol{t} \to^* \sum_i \alpha_i.\boldsymbol{b}_i$ *where* $\Gamma \vdash \boldsymbol{b}_i : U_i$

(where U_i is not a type abstraction or application)

Theorem

If $\boldsymbol{t} \downarrow = \sum_i \alpha_i.\boldsymbol{b}_i$ *then* $\Gamma \vdash \boldsymbol{t} \colon \sum_i \alpha_i.U_i + 0.T,$ *where* $\Gamma \vdash \boldsymbol{b}_i : U_i$

Confluence as a side effect

Confluence

In the original untyped setting: “confluence by restrictions”:

$$Y_{\mathbf{b}} = (\lambda x.(\mathbf{b} + (x)x))\ \lambda x.(\mathbf{b} + (x)x)$$

$$Y_{\mathbf{b}} \to \mathbf{b} + Y_{\mathbf{b}} \to \mathbf{b} + \mathbf{b} + Y_{\mathbf{b}} \to \dots$$

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$$\begin{array}{ccc} Y_{\mathbf{b}} + (-1).Y_{\mathbf{b}} & \longrightarrow (1-1).Y_{\mathbf{b}} & \longrightarrow^* 0 \\ \downarrow & & \\ \mathbf{b} + Y_{\mathbf{b}} + (-1).Y_{\mathbf{b}} & & \\ \downarrow_* & & \\ \mathbf{b} & & \end{array}$$

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Solution in the untyped setting:

$\alpha.\mathbf{t} + \beta.\mathbf{t} \to (\alpha + \beta).\mathbf{t}$

only if $\mathbf{t}$ is closed-normal

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Solution in the untyped setting:
$\alpha.\mathbf{t} + \beta.\mathbf{t} \to (\alpha + \beta).\mathbf{t}$
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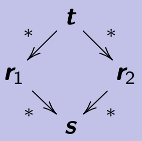
In the typed setting: **Strong normalisation solves the problem**

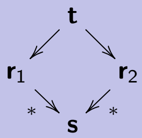
Theorem (Confluence)

$\forall \boldsymbol{t} \;/\; \Gamma \vdash \boldsymbol{t} : T$



Proof.

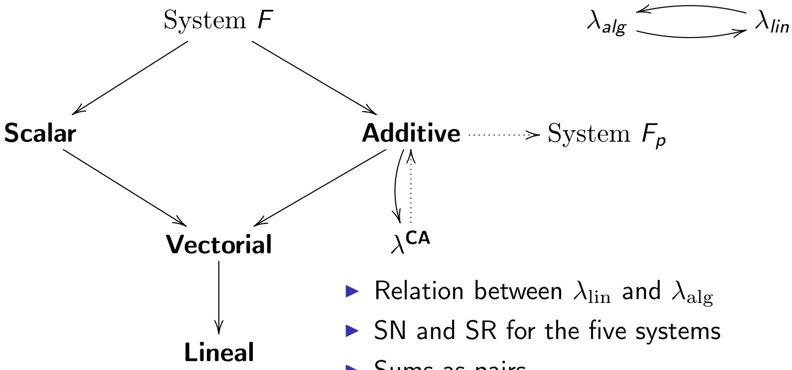
1) **local confluence**:



- Algebraic fragment: Coq proof `[Valiron'10]`
- Beta-reduction: Straightforward extension
- Commutation: Induction

2) Local confluence + Strong normalisation $\Rightarrow$ Confluence □

Contributions



- Relation between $\lambda_{\rm lin}$ and $\lambda_{\rm alg}$
- SN and SR for the five systems
- Sums as pairs
- Types $\leftrightarrow$ vectorial structure of terms
- Extra: No cloning theorem

Papers
Díaz-Caro,Perdrix,Tasson,Valiron HOR'10 (journal version in preparation)
Arrighi,Díaz-Caro QPL'09 (journal version submitted)
Díaz-Caro,Petit (in preparation)
Buiras,Díaz-Caro,Jaskelioff LSFA'11
Arrighi,Díaz-Caro,Valiron DCM'11 (journal version in preparation)

Future work

- Invariability of models of λ_{alg} through the CPS simulation
- Differential λ-calculus $\leftrightarrow$ Linear-algebraic λ-calculus
- Algebraic Linearity $\leftrightarrow$ Linear logic resources
- Quantum language (orthogonality issues)
- Relations with Probabilistic/Quantum/Fuzzy Logics