Linearity in the non-deterministic call-by-value setting

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$$(\lambda x.\mathbf{t})(\mathbf{u} + \mathbf{s}) \to \mathbf{t}[(\mathbf{u} + \mathbf{s})/x] \qquad \text{(CBN)}$$

$$(\lambda x.\mathbf{t})(\mathbf{u} + \mathbf{s}) \to (\lambda x.\mathbf{t})\mathbf{u} + (\lambda x.\mathbf{t})\mathbf{s} \qquad \text{(CBV)}$$

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$\mathbf{f}(x+y) = \mathbf{f}(x)+\mathbf{f}(y)$

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We want to understand this 'linearity'

Outline

- The untyped calculus
- The *Additive* type system capturing the CBV behaviour of $+$
- Logical interpretation: translation into System F with pairs

The untyped CBV non-deterministic λ-calculus

$$\mathbf{t},\mathbf{u},\mathbf{s} ::= \mathbf{v} \mid \mathbf{t}\mathbf{u} \mid \mathbf{t}+\mathbf{u} \mid \mathbf{0}$$
$$\mathbf{v} ::= x \mid \lambda x.\mathbf{t}$$

Intuitions

$\mathbf{t}+\mathbf{u}$ = non-deterministic superposition between $\mathbf{t}$ and $\mathbf{u}$
$\mathbf{0}$ = impossible computation

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Because non of them reduces ($\mathbf{0}$ not a value) and there is an *impossible* computation

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Finally $\mathbf{t} + \mathbf{u} = \mathbf{u} + \mathbf{t}$

*Running **t** or **u** non-deterministically, is the same as running **u** or **t** non-deterministically*

Also $\mathbf{t} + (\mathbf{u} + \mathbf{s}) = (\mathbf{t} + \mathbf{u}) + \mathbf{s}$

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β_v-reduction

$$(\lambda x.\mathbf{t})\mathbf{v} \to \mathbf{t}[\mathbf{v}/x]$$

Distributivity rules

$$(\mathbf{t} + \mathbf{u})\mathbf{s} \to \mathbf{ts} + \mathbf{us}$$
$$\mathbf{t}(\mathbf{u} + \mathbf{s}) \to \mathbf{tu} + \mathbf{ts}$$

Zero rules

$$\mathbf{t} + \mathbf{0} \to \mathbf{t}$$
$$\mathbf{0t} \to \mathbf{0}$$
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with $+$ associative and commutative

Remark: in the algebraic case, $\mathbf{0}$ is the sum of zero terms

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But $(T + R) \to S$ does not exists

there is no function taking a non-deterministic superposition as argument

Recall $\mathbf{t}(\mathbf{v}_1 + \mathbf{v}_2) \to \mathbf{t}\mathbf{v}_1 + \mathbf{t}\mathbf{v}_2$

If $\mathbf{v}_1 : T$ and $\mathbf{v}_2 : R$ then $\mathbf{t}$ needs to be both $T \to S_1$ and $R \to S_2$

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Polymorphism & *unit types* "U" (atomic types w.r.t. $+$)

Arrows: $U\to T$

The *Additive* type system (cont.)

Examples

Concrete example

$$\mathbf{v}_1 : U_1 \qquad ; \qquad \mathbf{v}_2 : U_2 \qquad ; \qquad I : \forall X.X \to X$$

Hence $\quad I(\mathbf{v}_1 + \mathbf{v}_2) \to I\mathbf{v}_1 + I\mathbf{v}_2 \quad$ has type $\quad U_1 + U_2$

A more generic example

$$\mathbf{v}_1 : U[W_1/X] \qquad ; \qquad \mathbf{v}_2 : U[W_2/X] \qquad ; \qquad \mathbf{t} : \forall X.U \to T$$

Hence $\quad \mathbf{t}(\mathbf{v}_1 + \mathbf{v}_2) \to \mathbf{t}\mathbf{v}_1 + \mathbf{t}\mathbf{v}_2 \quad$ has type $\quad T[W_1/X] + T[W_2/X]$

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$$\mathbf{t} : U \to T \qquad ; \qquad \mathbf{u} : V \to R$$

$$(\mathbf{t} + \mathbf{u})\mathbf{v} \to \mathbf{tv} + \mathbf{uv}$$

Hence U and V needs to "polymorph" to the type of $\mathbf{v}$

$$\mathbf{v} : U[W_1/X] = V[W_2/X] \qquad \mathbf{t} : \forall X.(U \to T) \qquad \mathbf{u} : \forall X.(V \to R)$$

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The *Additive* type system (cont.)

Arrow elimination

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combined...

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generalising...

$$\frac{\Gamma \vdash \mathbf{t} : \sum_{i=1}^{n} \forall \vec{X}.(U \to T_i) \qquad \Gamma \vdash \mathbf{u} : \sum_{j=1}^{m} U[\vec{W}_j/\vec{X}]}{\Gamma \vdash \mathbf{tu} : \sum_{i=1}^{n}\sum_{j=1}^{m} T_i[\vec{W}_j/\vec{X}]}$$

The *Additive* type system (cont.)

Examples

$$V_1 = U[W_1/X] \quad ; \quad V_2 = U[W_2/X]$$

$$\frac{\Gamma \vdash \lambda x.\mathbf{t} + \lambda y.\mathbf{u} : \forall X.(U \to T) + \forall X.(U \to R) \qquad \Gamma \vdash \mathbf{v}_1 + \mathbf{v}_2 : V_1 + V_2}{\Gamma \vdash (\lambda x.\mathbf{t} + \lambda y.\mathbf{u})(\mathbf{v}_1 + \mathbf{v}_2) : T[W_1/X] + T[W_2/X] + R[W_1/X] + R[W_2/X]}$$

$$\downarrow$$

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Simpler example

$$\frac{\Gamma \vdash \lambda x.x : \forall X.X \to X \qquad \Gamma \vdash \mathbf{v_1} + \mathbf{v_2} : U + V}{\Gamma \vdash (\lambda x.x)(\mathbf{v_1} + \mathbf{v_2}) : U + V}$$

without simultaneous arrow/forall elimination it is not possible to type it!

The *Additive* type system (cont.)

Summarising

$$T, R, S := U \mid T + R \mid \bar{0} \qquad \text{general types}$$
$$U, V, W := X \mid U \to T \mid \forall X.U \qquad \text{unit types}$$

$$T + \bar{0} \equiv T \quad ; \quad T + R \equiv R + T \quad ; \quad T + (R + S) \equiv (T + R) + S$$

$$\frac{}{\Gamma, x : U \vdash x : U}\ ax \qquad \frac{}{\Gamma \vdash \mathbf{0} : \bar{0}}\ ax_{\bar{0}} \qquad \frac{\Gamma \vdash \mathbf{t} : T \qquad \Gamma \vdash \mathbf{u} : R}{\Gamma \vdash \mathbf{t} + \mathbf{u} : T + R}\ +_I$$

$$\frac{\Gamma \vdash \mathbf{t} : \sum_{i=1}^{n} \forall \vec{X}.(U \to T_i) \qquad \Gamma \vdash \mathbf{u} : \sum_{j=1}^{m} U[\vec{W}_j/\vec{X}]}{\Gamma \vdash \mathbf{tu} : \sum_{i=1}^{n} \sum_{j=1}^{m} T_i[\vec{W}_j/\vec{X}]}\ \to_E \qquad \frac{\Gamma, x : U \vdash \mathbf{t} : T}{\Gamma \vdash \lambda x.\mathbf{t} : U \to T}\ \to_I$$

$$\frac{\Gamma \vdash \mathbf{t} : \forall X.U}{\Gamma \vdash \mathbf{t} : U[V/X]}\ \forall_E \qquad \frac{\Gamma \vdash \mathbf{t} : U \qquad X \notin FV(\Gamma)}{\Gamma \vdash \mathbf{t} : \forall X.U}\ \forall_I \qquad \frac{\Gamma \vdash \mathbf{t} : T \qquad T \equiv R}{\Gamma \vdash \mathbf{t} : R}\ \equiv$$

- Strong normalisation ✓
- Subject reduction ✓

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System F with pairs

$$t, u ::= x \mid \lambda x.t \mid tu \mid \star \mid \langle t, u \rangle \mid \pi_1(t) \mid \pi_2(t)$$
$$A, B ::= X \mid A \Rightarrow B \mid \forall X.A \mid \mathbf{1} \mid A \times B$$

$$(\lambda x.t)u \to t[u/x] \qquad \pi_1(\langle t_1, t_2 \rangle) \to t_1 \qquad \pi_2(\langle t_1, t_2 \rangle) \to t_2$$

Additive		**System F with pairs**
X	$\rightsquigarrow$	X
$U \to T$	$\rightsquigarrow$	$\lvert U \rvert \Rightarrow \lvert T \rvert$
$\forall X.U$	$\rightsquigarrow$	$\forall X.\lvert U \rvert$
$\bar{0}$	$\rightsquigarrow$	$\mathbf{1}$
$T + S$	$\rightsquigarrow$	$\lvert T \rvert \times \lvert S \rvert$

Sums as Pairs

$$+\ ,\ \bar{0} \qquad \rightsquigarrow \qquad \times\ ,\ \mathbf{1}$$

$$\begin{array}{cc} T+S \equiv S+T & A\times B \neq B\times A \\ T+(S+R) \equiv (T+S)+R & A\times(B\times C) \neq (A\times B)\times C \\ T+\bar{0} \equiv T & A\times \mathbf{1} \neq A \end{array}$$

Sums as Pairs

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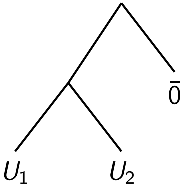
$T + S \equiv S + T$	$A \times B \neq B \times A$
$T + (S + R) \equiv (T + S) + R$	$A \times (B \times C) \neq (A \times B) \times C$
$T + \bar{0} \equiv T$	$A \times \mathbf{1} \neq A$

$$T, R, S ::= U \mid T + R \mid \bar{0}$$
$$U, V, W ::= X \mid U \to T \mid \forall X.U$$

Type $\rightsquigarrow$ Binary tree (leaf: U or $\bar{0}$)

Example:
$T = (U_1 + U_2)\ +\ \bar{0}$

$\mathcal{T}[\mathtt{r} \mapsto \bar{0}, \mathtt{lr} \mapsto U_2, \mathtt{ll} \mapsto U_1]$

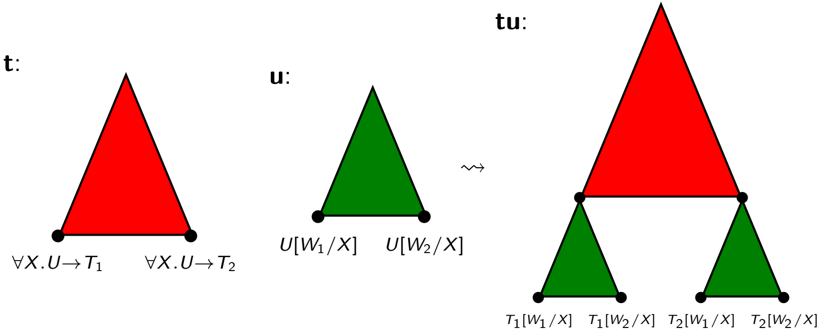


We keep structured sum types

Structured Arrow-elimination

$$\frac{\Gamma \vdash \mathbf{t} : \sum_{i=1}^{n} \forall \vec{X}.(U \to T_i) \qquad \Gamma \vdash \mathbf{u} : \sum_{j=1}^{m} U[\vec{W}_j/\vec{X}]}{\Gamma \vdash \mathbf{tu} : \sum_{i=1}^{n}\sum_{j=1}^{m} T_i[\vec{W}_j/\vec{X}]}$$

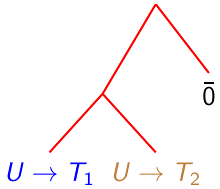
$$\frac{\Gamma \vdash \mathbf{t} : \mathcal{T}[\ell \mapsto \forall \vec{X}.(U \to T_\ell)] \qquad \Gamma \vdash \mathbf{u} : \mathcal{T}'[\ell' \mapsto U[\vec{W}_{\ell'}/\vec{X}]]}{\Gamma \vdash \mathbf{tu} : \mathcal{T} \circ \mathcal{T}'[\ell\ell' \mapsto T_\ell[\vec{W}_{\ell'}/\vec{X}]]}$$



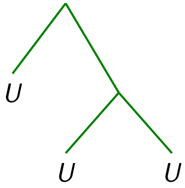
Structured Arrow-elimination

An example

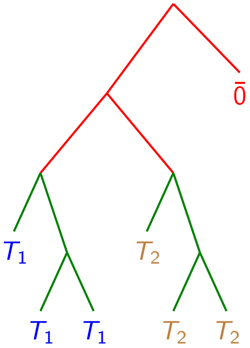
$\mathbf{t} = (\mathbf{t}_1 + \mathbf{t}_2) + 0$



$\mathbf{u} = \mathbf{u}_1 + (\mathbf{u}_2 + \mathbf{u}_3)$



$$
\begin{aligned}
\mathbf{tu} \to^* & \ \mathbf{t}_1\big(\mathbf{u}_1 + (\mathbf{u}_2 + \mathbf{u}_3)\big) + \\
& \ \mathbf{t}_2\big(\mathbf{u}_1 + (\mathbf{u}_2 + \mathbf{u}_3)\big) + 0 \\
\to^* & \ \mathbf{t}_1\mathbf{u}_1 + (\mathbf{t}_1\mathbf{u}_2 + \mathbf{t}_1\mathbf{u}_3) + \\
& \ \mathbf{t}_2\mathbf{u}_1 + (\mathbf{t}_2\mathbf{u}_2 + \mathbf{t}_2\mathbf{u}_3) + 0
\end{aligned}
$$



Equivalence in System F

What about associativity, commutativity and neutral element in system F with pairs?

Lemma

$$T \equiv T' \qquad \textit{implies} \qquad |T| \leftrightarrow |T'|$$

Where $A \leftrightarrow B$ means

$$\vdash_{\mathsf{F}} \varepsilon_{A,B} : A \Rightarrow B \qquad \text{and} \qquad \vdash_{\mathsf{F}} \varepsilon_{B,A} : B \Rightarrow A$$

for some terms $\varepsilon_{A,B}, \varepsilon_{B,A}$ s.t.

$$\varepsilon_{A,B} \circ \varepsilon_{B,A} \approx id_A \qquad \text{and} \qquad \varepsilon_{B,A} \circ \varepsilon_{A,B} \approx id_B$$

Translation of terms

What happens with the distributivity?

$$\mathbf{t} + \mathbf{u} \quad \rightsquigarrow \quad \langle [\mathbf{t}], [\mathbf{u}] \rangle$$

$(\mathbf{t}_1 + \mathbf{t}_2)(\mathbf{u}_1 + \mathbf{u}_2) \to^*$ $\mathbf{t}_1\mathbf{u}_1 + \mathbf{t}_1\mathbf{u}_2 + \mathbf{t}_2\mathbf{u}_1 + \mathbf{t}_2\mathbf{u}_2$ but $\langle t_1, t_2 \rangle \langle r_1, r_2 \rangle \not\to$ $\langle \langle t_1 r_1, t_1 r_2 \rangle, \langle t_2 r_1, t_2 r_2 \rangle \rangle$

No distributivity in System F with pairs

Translation of terms

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No distributivity in System F with pairs

Main ideas:

- The sum is distributed during the translation of application

$$[\mathbf{tr}] = \langle\ \langle [\mathbf{t}_1][\mathbf{u}_1], [\mathbf{t}_1][\mathbf{u}_2] \rangle\ ,\ \langle [\mathbf{t}_2][\mathbf{u}_1], [\mathbf{t}_2][\mathbf{u}_2] \rangle\ \rangle$$

if $\mathbf{t} = \mathbf{t}_1 + \mathbf{t}_2$ and $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$

Translation of terms

What happens with the distributivity?

$$\mathbf{t} + \mathbf{u} \quad \rightsquigarrow \quad \langle [\mathbf{t}], [\mathbf{u}] \rangle$$

$(\mathbf{t}_1 + \mathbf{t}_2)(\mathbf{u}_1 + \mathbf{u}_2) \to^*$ $\mathbf{t}_1\mathbf{u}_1 + \mathbf{t}_1\mathbf{u}_2 + \mathbf{t}_2\mathbf{u}_1 + \mathbf{t}_2\mathbf{u}_2$ but $\langle t_1, t_2 \rangle \langle r_1, r_2 \rangle \not\to$ $\langle \langle t_1 r_1, t_1 r_2 \rangle, \langle t_2 r_1, t_2 r_2 \rangle \rangle$

No distributivity in System F with pairs

Main ideas:

- The sum is distributed during the translation of application
  $$[\mathbf{tr}] = \langle \ \langle [\mathbf{t}_1][\mathbf{u}_1], [\mathbf{t}_1][\mathbf{u}_2] \rangle \ , \ \langle [\mathbf{t}_2][\mathbf{u}_1], [\mathbf{t}_2][\mathbf{u}_2] \rangle \ \rangle$$
  if $\mathbf{t} = \mathbf{t}_1 + \mathbf{t}_2$ and $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$
- The "sum structure" of a term is known thanks to its type
  $\Gamma \vdash \mathbf{t} : (T_1 + T_2) + T_3 \quad \rightsquigarrow \quad \mathbf{t} \sim (\mathbf{t}_1 + \mathbf{t}_2) + \mathbf{t}_3$
  with $\Gamma \vdash \mathbf{t}_i : T_i$

Translation of terms

$$\Gamma \vdash \mathbf{t} : T \quad \rightsquigarrow \quad |\Gamma| \vdash_{\mathsf{F}} [\mathbf{t}]_{\mathcal{D}} : |T|$$

$$\Gamma, x : T \vdash x : T \quad \rightsquigarrow \quad [x]_{\mathcal{D}} = x$$

$$\Gamma \vdash \mathbf{0} : \bar{0} \quad \rightsquigarrow \quad [\mathbf{0}]_{\mathcal{D}} = \star$$

Translation of terms

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$$\frac{\Gamma \vdash \mathbf{t} : T \quad T \equiv T'}{\Gamma \vdash \mathbf{t} : T'} \quad \rightsquigarrow \quad [\mathbf{t}]_{\mathcal{D}} = \varepsilon_{|T|,|T'|}[\mathbf{t}]_{\mathcal{D}'}$$

Translation of terms

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$$\frac{\Gamma \vdash \mathbf{t} : T \quad \Gamma \vdash \mathbf{u} : S}{\Gamma \vdash \mathbf{t} + \mathbf{u} : T + S} \quad \rightsquigarrow \quad [\mathbf{t}+\mathbf{r}]_{\mathcal{D}} = \langle [\mathbf{t}]_{\mathcal{D}_1}, [\mathbf{u}]_{\mathcal{D}_2} \rangle$$

$$\frac{\Gamma, x : U \vdash \mathbf{t} : T}{\Gamma \vdash \lambda x.\mathbf{t} : U \to T} \quad \rightsquigarrow \quad [\lambda x.\mathbf{t}]_{\mathcal{D}} = \lambda x.[\mathbf{t}]_{\mathcal{D}'}$$

Translation of terms

$$\Gamma \vdash \mathbf{t} : T \quad \rightsquigarrow \quad |\Gamma| \vdash_{\mathsf{F}} [\mathbf{t}]_{\mathcal{D}} : |T|$$

$$\Gamma, x : T \vdash x : T \quad \rightsquigarrow \quad [x]_{\mathcal{D}} = x$$

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$$\frac{\Gamma \vdash \mathbf{t} : \mathcal{T}[\ell \mapsto \forall \vec{X}.(U \to T_\ell)] \quad \Gamma \vdash \mathbf{u} : \mathcal{T}'[\ell' \mapsto U[\vec{W}_{\ell'}/\vec{X}]]}{\Gamma \vdash \mathbf{tu} : \mathcal{T} \circ \mathcal{T}'[\ell\ell' \mapsto T_\ell[\vec{W}_{\ell'}/\vec{X}]]}$$

$$\rightsquigarrow [\mathbf{tu}]_{\mathcal{D}} = \mathcal{T} \circ \mathcal{T}'[\ell\ell' \mapsto \pi_{\overline{\ell}}([\mathbf{t}]_{\mathcal{D}_1})\pi_{\overline{\ell}'}([\mathbf{u}]_{\mathcal{D}_2})]$$

Translation of terms

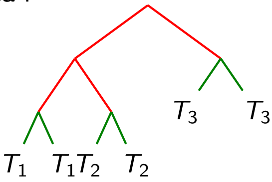
An example

$$\frac{\Gamma \vdash \mathbf{t} : \Big((U \to T_1) + (U \to T_2)\Big) \ + \ (U \to T_3) \qquad \Gamma \vdash \mathbf{u} : U + U}{\Gamma \vdash \mathbf{tu} : \Big((T_1 + T_1) + (T_2 + T_2)\Big) + (T_3 + T_3)}$$

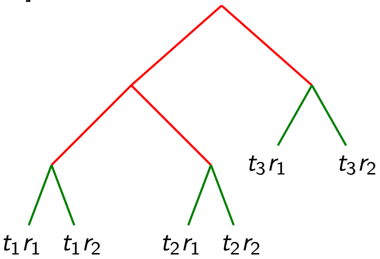
$t_1 = \pi_{11}([\mathbf{t}]); t_2 = \pi_{12}([\mathbf{t}]); t_3 = \pi_2([\mathbf{t}]);$

$r_1 = \pi_1([\mathbf{u}]); r_2 = \pi_2([\mathbf{u}]);$

$\mathbf{tu}$:



$[\mathbf{tu}]_{\mathcal{D}} =$



Correctness

Theorem (Correctness w.r.t. typing)

$\Gamma \vdash \boldsymbol{t} : T \qquad \textit{implies} \qquad |\Gamma| \vdash_F [\boldsymbol{t}]_{\mathcal{D}} : |T|$

We provide a partial inverse translation $(\!| \cdot |\!)$ and prove

Theorem (Inverse translation)

$\Gamma \vdash \boldsymbol{t} : T \qquad = \qquad (\!| |\Gamma| |\!) \vdash_F (\!| [\boldsymbol{t}]_{\mathcal{D}} |\!) : (\!| |T| |\!)$

Theorem (Reduction preservation)

$\Gamma \vdash \boldsymbol{t} : T$ *and* $\boldsymbol{t} \to \boldsymbol{t}' \qquad \textit{implies} \qquad [\boldsymbol{t}]_{\mathcal{D}} \to^{+} [\boldsymbol{t}']_{\mathcal{D}'} \qquad$ *for some* $\mathcal{D}'$

(except for $\boldsymbol{t} + \boldsymbol{0} \to \boldsymbol{t}$*)*

Summarising and concluding

- $+$ "like linear functions": $\mathbf{f}(x + y) = \mathbf{f}(x) + \mathbf{f}(y)$
- *Additive* type system tight to the behaviour of $+$
- Typed system translate to System F with pairs

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System F with pairs correspond to the non-linear fragment of IMELL

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- $+$ "like linear functions": $\mathbf{f}(x+y) = \mathbf{f}(x) + \mathbf{f}(y)$
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System F with pairs correspond to the non-linear fragment of IMELL

Linear Logic	CBV nd/alg
Force unique use of the argument e.g. $\mathbf{f}(x) = x^2$ no linear $\mathbf{f}(x) = x + 1$ linear	Ban sum terms substitutions e.g. $\mathbf{f}(x+y) \to \mathbf{f}(x) + \mathbf{f}(y)$

In the CBV non-deterministic setting (or algebraic) it is enough to treat functions as linear, even if they are not. (In LL all functions are linear).