Proof Normalisation in a Logic Identifying Isomorphic Propositions

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4th International Conference on Formal Structures for Computation and Deduction (FSCD'19)
June 24-30, 2019. Dortmund, Germany

Type isomorphisms

Definition

$$A \equiv B \quad \Leftrightarrow \quad \exists \left\{ \begin{array}{l} \textbf{prog}_1 : A \Rightarrow B \\ \textbf{prog}_2 : B \Rightarrow A \end{array} \right\} \bigg/ \left\{ \begin{array}{l} \textbf{prog}_2 \circ \textbf{prog}_1 = Id_A \\ \textbf{prof}_1 \circ \textbf{prog}_2 = Id_B \end{array} \right\}$$

Example

$$(A \wedge B) \equiv (B \wedge A)$$

$\textbf{swap}_{AB} : (A \wedge B) \Rightarrow (B \wedge A)$ $\textbf{swap}_{AB}\ \langle x, y \rangle = \langle y, x \rangle$	$\textbf{swap}_{BA} : (B \wedge A) \Rightarrow (A \wedge B)$ $\textbf{swap}_{BA}\ \langle y, x \rangle = \langle x, y \rangle$

$\textbf{swap}_{BA}\ \textbf{swap}_{AB}\ \langle a, b \rangle = \langle a, b \rangle$ y $\textbf{swap}_{AB}\ \textbf{swap}_{BA}\ \langle b, a \rangle = \langle b, a \rangle$

Type isomorphisms

Characterization of them

Simply types with pairs

[Bruce, Di Cosmo, Longo
MSCS 2(2), 231–247, 1992]

- $(A \wedge B) \equiv (B \wedge A)$
- $((A \wedge B) \wedge C) \equiv (A \wedge (B \wedge C))$
- $(A \wedge B) \Rightarrow C \equiv A \Rightarrow B \Rightarrow C$
- $A \Rightarrow (B \wedge C) \equiv (A \Rightarrow B) \wedge (A \Rightarrow C)$

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$$\mathbf{assoc} : ((A \wedge B) \wedge C) \Rightarrow (A \wedge (B \wedge C))$$
$$\mathbf{assoc}\ \langle x, y \rangle = \langle fst\ \ x, \langle snd\ \ x, y \rangle \rangle$$

$$\mathbf{assoc}' : (A \wedge (B \wedge C)) \Rightarrow ((A \wedge B) \wedge C)$$
$$\mathbf{assoc}'\ \langle x, y \rangle = \langle \langle x, fst\ \ y \rangle, snd\ \ y \rangle$$

$$\mathbf{assoc}'\ \mathbf{assoc}\ \langle \langle a, b \rangle, c \rangle = \langle \langle a, b \rangle, c \rangle$$

$$\mathbf{assoc}\ \mathbf{assoc}'\ \langle a, \langle b, c \rangle \rangle = \langle a, \langle b, c \rangle \rangle$$

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$$\textbf{curry} : ((A \wedge B) \Rightarrow C) \Rightarrow A \Rightarrow B \Rightarrow C$$
$$\textbf{curry}\ f\ x\ y = f\ \langle x, y \rangle$$

$$\textbf{uncurry} : (A \Rightarrow B \Rightarrow C) \Rightarrow (A \wedge B) \Rightarrow C$$
$$\textbf{uncurry}\ g\ x = g\ (\mathit{fst}\ x)\ (\mathit{snd}\ x)$$

$\textbf{uncurry curry}\ f = f$ y $\textbf{curry uncurry}\ g = g$

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$$\textbf{pairf} : (A \Rightarrow (B \wedge C)) \Rightarrow ((A \Rightarrow B) \wedge (A \Rightarrow C))$$

$$\textbf{pairf}\ f = \textbf{let}\ g\ x = \ \mathit{fst}\ (f\ \ x)\ \textbf{in}\ \ \textbf{let}\ h\ x = \ \mathit{snd}\ (f\ \ x)\ \textbf{in}\ \ \langle g, h \rangle$$

$$\textbf{fpair} : ((A \Rightarrow B) \wedge (A \Rightarrow C)) \Rightarrow A \Rightarrow (B \wedge C)$$

$$\textbf{fpair}\ f\ \ x = \langle (\mathit{fst}\ \ f)\ \ x, (\mathit{snd}\ \ f)\ \ x \rangle$$

$$\textbf{fpair pairf}\ f = f \qquad \text{y} \qquad \textbf{pairf fpair}\ g = g$$

The goal

We want to go further:

$$(A \equiv B) \Rightarrow (t : A \Leftrightarrow t : B)$$

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If r is a proof of $(A \Rightarrow B) \wedge (A \Rightarrow C)$, r is also a proof showing that $A \Rightarrow (B \wedge C)$ is true

$$\frac{(A \Rightarrow B) \wedge (A \Rightarrow C) \quad A}{B \wedge C}$$

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r is also a proof showing that
$A \Rightarrow (B \wedge C)$ is true**

$$\frac{(A \Rightarrow B) \wedge (A \Rightarrow C) \quad A}{B \wedge C}$$

$$\langle \lambda x^A.r, \lambda x^A.s \rangle \rightleftarrows \lambda x^A.\langle r, s \rangle$$

The setting

- Simply types with conjunction and implication

$$A, B, C ::= \tau \mid A \Rightarrow B \mid A \wedge B$$

- An equivalence relation between types based on the known isomorphisms[1]

1. $A \wedge B \equiv B \wedge A$ (comm)
2. $A \wedge (B \wedge C) \equiv (A \wedge B) \wedge C$ (aso)
3. $(A \wedge B) \Rightarrow C \equiv A \Rightarrow B \Rightarrow C$ (curry)
4. $A \Rightarrow (B \wedge C) \equiv (A \Rightarrow B) \wedge (A \Rightarrow C)$ (distrib)

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We want

$$[A{\equiv}B]\,\frac{\Gamma \vdash r : A}{\Gamma \vdash r : B}$$

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Commutative and associative conjunction

$$\frac{\Gamma \vdash r : A \quad \Gamma \vdash s : B}{\Gamma \vdash \langle r, s \rangle : A \wedge B}\ (\wedge_i)$$

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$$A \wedge (B \wedge C) \equiv (A \wedge B) \wedge C$$

Then

$$\langle r, s \rangle \leftrightarrows \langle s, r \rangle$$
$$\langle r, \langle s, t \rangle \rangle \leftrightarrows \langle \langle r, s \rangle, t \rangle$$

Commutative and associative conjunction

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Then

$$\begin{aligned} A \wedge B &\equiv B \wedge A \\ A \wedge (B \wedge C) &\equiv (A \wedge B) \wedge C \\ \langle r, s \rangle &\leftrightarrows \langle s, r \rangle \\ \langle r, \langle s, t \rangle \rangle &\leftrightarrows \langle \langle r, s \rangle, t \rangle \end{aligned}$$

What about the elimination?

$$\frac{\Gamma \vdash \langle r, s \rangle : A \wedge B}{\Gamma \vdash \pi_1 \langle r, s \rangle : A} \ (\wedge_e)$$

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Workaround: Church-style – Projection with respect to type

If $r : A$ then $\pi_A \langle r, s \rangle \to r$

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Non determinism

If $r : A$, $s : A$ then $\pi_A \langle r, s \rangle \to r$, $\pi_A \langle r, s \rangle \to s$

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If $\begin{array}{l} r : A \\ s : A \end{array}$ then $\begin{array}{l} \pi_A \langle r, s \rangle \to r \\ \pi_A \langle r, s \rangle \to s \end{array}$

Not a big deal
both r and s are valid proofs of A

Commutative and associative conjunction

$$\frac{\Gamma\vdash r:A \quad \Gamma\vdash s:B}{\Gamma\vdash \; r\times s \; : A\wedge B}\ (\wedge_i)$$

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Curryfication

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induces

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$$(\lambda x^A.r)\ s \quad \to \quad r[s/x]$$

Example

$$\underbrace{(\lambda x^{\tau \wedge \tau}.x)}_{\substack{(\tau \wedge \tau) \Rightarrow (\tau \wedge \tau) \\ \tau \Rightarrow \tau \Rightarrow (\tau \wedge \tau)}}$$

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Example

$$\underbrace{(\lambda x^{\tau \wedge \tau}.x)}_{\substack{(\tau \wedge \tau) \Rightarrow (\tau \wedge \tau) \\ \tau \Rightarrow \tau \Rightarrow (\tau \wedge \tau)}} r^\tau s^\tau \quad \rightleftarrows \quad (\lambda x^{\tau \wedge \tau}.x)(r^\tau \times s^\tau) \to r^\tau \times s^\tau$$

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If $s : A$, $\quad (\lambda x^A.r)\ s \quad \to \quad r[s/x]$

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Other possible choices:

$$\lambda x^{A\wedge B}.t \quad \leftrightarrows \quad \lambda y^A.\lambda z^B.t[y \times z/x]$$

$$\lambda x^A.\lambda y^B.t \quad \leftrightarrows \quad \lambda z^{A\wedge B}.t[\pi_A(z)/x, \pi_B(z)/y]$$

Distributing implication over conjunction

$$A \Rightarrow (B \wedge C) \quad \equiv \quad (A \Rightarrow B) \wedge (A \Rightarrow C)$$

induces

$$\lambda x^A.r \times s \leftrightarrows (\lambda x^A.r) \times (\lambda x^A.s)$$

Distributing implication over conjunction

$$A \Rightarrow (B \wedge C) \quad \equiv \quad (A \Rightarrow B) \wedge (A \Rightarrow C)$$

induces

$$\lambda x^A.r \times s \leftrightarrows (\lambda x^A.r) \times (\lambda x^A.s) \qquad \text{and} \qquad \lambda x^A.\pi_B(r) \leftrightarrows \pi_{A \Rightarrow B}(\lambda x^A.r)$$

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Example

$$\dfrac{\dfrac{\vdash \lambda x^{A\wedge B}.x : (A \wedge B) \Rightarrow (A \wedge B)}{\vdash \lambda x^{A\wedge B}.x : ((A \wedge B) \Rightarrow A) \wedge ((A \wedge B) \Rightarrow B)}\ (\equiv)}{\vdash \pi_{(A\wedge B)\Rightarrow A}(\lambda x^{A\wedge B}.x) : (A \wedge B) \Rightarrow A}\ (\wedge_e)$$

$$\pi_{(A\wedge B)\Rightarrow A}(\lambda x^{A\wedge B}.x) \leftrightarrows \lambda x^{A\wedge B}.\pi_A(x)$$

Distributing implication over conjunction

Other possibilities

$$A \Rightarrow (B \wedge C) \quad \equiv \quad (A \Rightarrow B) \wedge (A \Rightarrow C)$$

$$\begin{array}{rcl} \lambda x^A.r \times s & \leftrightarrows & (\lambda x^A.r) \times (\lambda x^A.s) \\ \lambda x^A.\pi_B(r) & \leftrightarrows & \pi_{A \Rightarrow B}(\lambda x^A.r) \end{array} \qquad \begin{array}{rcl} \Rightarrow_i, \wedge_i & \leftrightarrows & \wedge_i, \Rightarrow_i \\ \Rightarrow_i, \wedge_e & \leftrightarrows & \wedge_e, \Rightarrow_i \end{array}$$

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* if $r : A \Rightarrow (B \wedge C)$

Normalization

Counterexample

$$
\begin{aligned}
\delta &= \lambda x.\pi_{\tau \Rightarrow \tau}(x)\pi_{\tau}(x) \\
\delta' &= \delta((zy) \times y) \\
\Omega &= \delta((zy) \times \delta')
\end{aligned}
$$

Normalization

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$$\delta' = \delta((zy) \times y)$$
$$\Omega = \delta((zy) \times \delta')$$

$$\begin{aligned}
\Omega &\to^{*}_{\rightleftarrows} \pi_{\tau\Rightarrow\tau}((z \times (\delta(zy)))y)\delta' \\
&\rightleftarrows \pi_{\tau\Rightarrow\tau\Rightarrow\tau}(z \times (\delta(zy)))y\delta' \\
&\rightleftarrows \pi_{\tau\Rightarrow\tau\Rightarrow\tau}(z \times (\delta(zy)))(y \times \delta') \\
&\rightleftarrows \pi_{\tau\Rightarrow\tau\Rightarrow\tau}(z \times (\delta(zy)))(\delta' \times y) \\
&\rightleftarrows \pi_{\tau\Rightarrow\tau\Rightarrow\tau}(z \times (\delta(zy)))\delta' y \\
&\rightleftarrows \pi_{\tau\Rightarrow\tau}((z \times (\delta(zy)))\delta')y \\
&\rightleftarrows^{*} \pi_{\tau\Rightarrow\tau}((z\delta') \times \Omega)y
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&\rightleftarrows \pi_{\tau\Rightarrow\tau}((z\times(\delta(zy)))\delta')y \\
&\rightleftarrows^* \pi_{\tau\Rightarrow\tau}((z\delta')\times\Omega)y
\end{aligned}$$

$\pi_{A\Rightarrow B}(r)s \rightleftarrows \pi_B(rs)$ Problematic rule

Normalization

We had too many rules

Working set: System I

$$r \times s \rightleftarrows s \times r \qquad \text{(comm)}$$

$$(r \times s) \times t \rightleftarrows r \times (s \times t) \qquad \text{(asso)}$$

$$\lambda x^A.(r \times s) \rightleftarrows \lambda x^A.r \times \lambda x^A.s \qquad (\text{dist}_\lambda)$$

$$(r \times s)t \rightleftarrows rt \times st \qquad (\text{dist}_{\text{app}})$$

$$rst \rightleftarrows r(s \times t) \qquad \text{(curry)}$$

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Theorem (Strong normalization)

System I is strongly normalizing

Proof. highlights
No neutral terms: $(r \times s)t \rightleftarrows rt \times st$
We use elimination contexts: $K := [] \mid Kr \mid \pi_A(K)$
A term r is reductible if $\forall K$ such that $K[t] : \tau$, $K[t] \in \mathcal{SN}$.

Progression and consistency

No progression:

$$\text{Let } s : B, \quad \underbrace{(\lambda x^A.\lambda y^B.r)}_{\substack{A\Rightarrow B\Rightarrow C\\ B\Rightarrow A\Rightarrow C}} s \quad \text{is in normal form}$$

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$$\text{Let } s : B, \quad \underbrace{(\lambda x^A.\lambda y^B.r)}_{\substack{A \Rightarrow B \Rightarrow C \\ B \Rightarrow A \Rightarrow C}} s \quad \text{is in normal form}$$

Theorem (Consistency of System I)

There is no closed normal term of type τ.

Progression and consistency

No progression:

$$\text{Let } s : B, \quad \underbrace{(\lambda x^A.\lambda y^B.r)}_{\substack{A\Rightarrow B\Rightarrow C\\ B\Rightarrow A\Rightarrow C}} s \quad \text{is in normal form}$$

Theorem (Consistency of System I)

There is no closed normal term of type τ.

Future work (in progress)
η-expansion and surjective pairing

$$\begin{aligned}(\lambda x^A.\lambda y^B.r)s &\to_\eta \lambda z^A.(\lambda x^A.\lambda y^B.r)sz\\ &\rightleftarrows^* \lambda z^A.(\lambda x^A.\lambda y^B.r)zs\\ &\to \lambda z^A.((\lambda y^B.r[z/x])s)\end{aligned}$$

Summarizing

What have we done?

We defined System I, where isomorphic propositions have the same proofs

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Why?

If $A \equiv B$, a **proof** of A
should be indistinguishable of a proof of B

$\dfrac{A \quad B}{A \wedge B}$ and $\dfrac{B \quad A}{B \wedge A}$ are the same!

If $A \equiv B$, a **function** defined over A
can used directly as B

If $f\ \langle a, b\rangle$ is valid, it should also be $f\ a\ b$, or even $f\ a$.