Departamento de Computación, Facultad de Ciencias Exactas y Naturales, UBA

Azar y Autómatas

Clase 6: Autómatas finitos y secuencias normales

Normality as incompressibility by finite automata

The definition of normality can be expressed as a notion of incompressibility by finite automata with output also known as transducers.

We focus on transducers that operate in real-time, that is, they process exactly one input alphabet symbol per transition. We consider non-deterministic transducers.

Non-deterministic real-time finite automata with output

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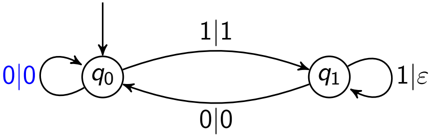

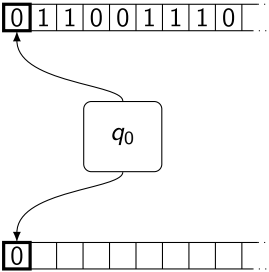

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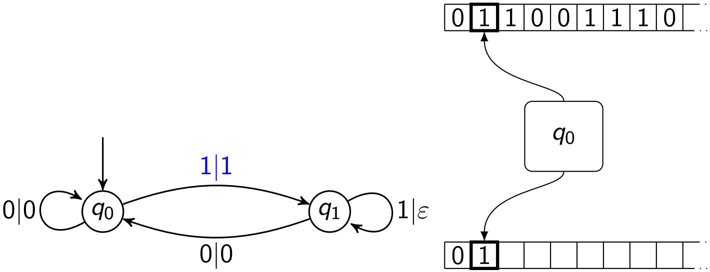

The transducer transforms rows of 1s into a single 1.

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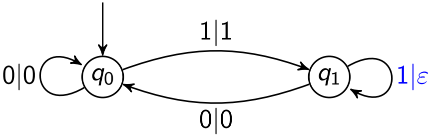

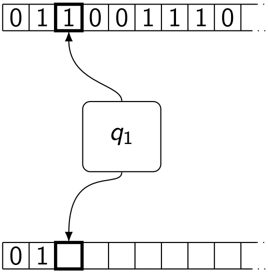

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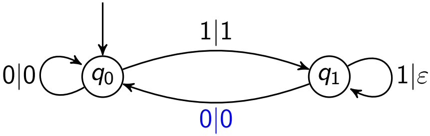

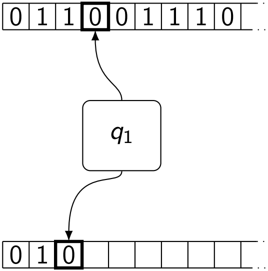

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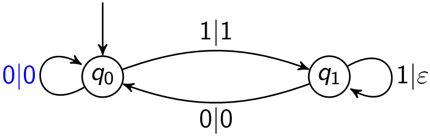

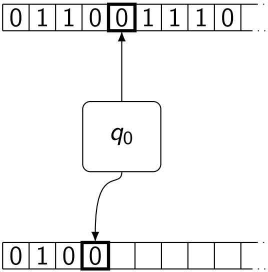

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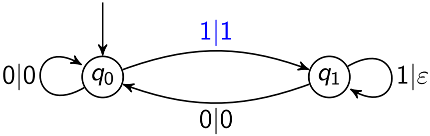

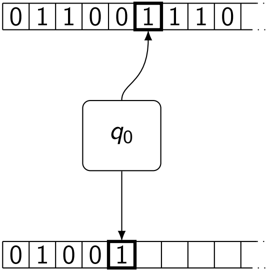

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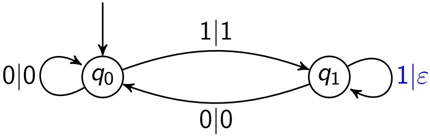

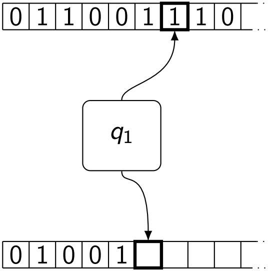

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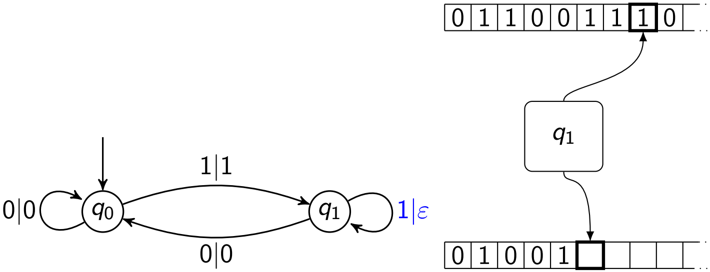

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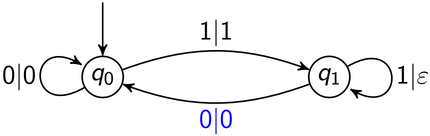

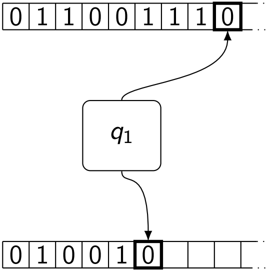

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For two infinite words $x \in A^\omega$ and $y \in B^\omega$, we write $\mathcal{T}(x, y)$ whenever there is an accepting run $q_0 \xrightarrow{x|y} \infty$ in $\mathcal{T}$.

$\mathcal{T}$ is bounded-to-one if the function $y \mapsto |\{x : \mathcal{T}(x, y)\}|$ is bounded.

Compressibility by finite automata

An infinite word $x = a_1 a_2 a_3 \cdots$ is compressible if there is a bounded-to-one non-deterministic transducer with an accepting run $q_0 \xrightarrow{a_1 | v_1} q_1 \xrightarrow{a_2 | v_2} q_2 \xrightarrow{a_3 | v_3} q_3 \cdots$ satisfying

$$\liminf_{n \to \infty} \frac{|v_1 v_2 \cdots v_n|}{n} \frac{\log |B|}{\log |A|} < 1.$$

Example of a compressible sequence

The sequence $x = 01010101010101\ldots$ is compressible because there is a one-to-one automaton $\mathcal{T}$ that maps $0101 \to 0$ and for every v such that $|v| = 4$ and $v \neq 0101$ $v \to 1v$. Then,

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$$\liminf_{n\to\infty} \qquad \frac{|1010| + \left(\sum_{j=2}^{n} 4 + (j-2)/3 + 1\right)}{\sum_{j=1}^{n} j + 1}$$

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Normality as incompressibility by finite automata

Theorem 1

An infinite word is normal if and only if it is not compressible by any bounded-to-one non-deterministic transducer.

Normal implies not compressible

Lemma 2

Let ℓ be a positive integer and let $u_1, u_2, u_3, \ldots$ be words of length ℓ over the alphabet A such that $x = u_1 u_2 u_3 \cdots$ is simply normal to length ℓ. Consider a bounded-to-one transducer and let an accepting run with input x

$$q_0 \xrightarrow{u_1|v_1} q_1 \xrightarrow{u_2|v_2} q_2 \xrightarrow{u_3|v_3} q_3 \cdots$$

Assume there is a real $\varepsilon > 0$ and a set $U \subseteq A^\ell$ of at least $(1-\varepsilon)|A|^\ell$ words such that $u_i \in U$ implies $|v_i| \geq \ell(1-\varepsilon)$. Then,

$$\liminf_{n\to\infty} \frac{|v_1 v_2 \cdots v_n|}{n\ell} \geq (1-\varepsilon)^3.$$

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&\geq (1 - \varepsilon)^3 n\ell. \qquad \square
\end{aligned}$$

Normal implies not compressible

Lemma 3

If x is normal then x can not be compressed by any bounded-to-one real-time transducer.

Proof of Lemma 3

Fix a normal infinite word $x = a_1 a_2 a_3 \cdots$, a real $\varepsilon > 0$, a bounded-to-one non-deterministic $\mathcal{T} = \langle Q, A, B, \delta, q_0, F \rangle$ and an accepting run

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where $a \in A$ and $v \in B^*$.

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It suffices to show that there is ℓ and U such that Lemma 2 applies to this arbitrary choice of ε, $\mathcal{T}$ and accepting run.

Proof of Lemma 3

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For each $u \in A^*$ let h_u be its minimum output length

$$h_u = \min\{|v| : \exists i, j, 0 \leq i \leq j, q_i \xrightarrow{u|v} q_j\}$$

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Then,

$$|U_\ell| \geq |A|^\ell - |Q|^2 t |B|^{(1-\varepsilon)\ell+1}.$$

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For each length ℓ, pair of states p, q in the run and each word v define

$$U'(\ell, p, q, v) = \{u \in A^\ell : p \xrightarrow{u|v} q\}.$$

Thus, for each $u \in U'(\ell, p, q, v)$ we can write

$$q_0 \xrightarrow{x|y} \infty = q_0 \xrightarrow{u_0|v_0} p \xrightarrow{u|v} q \xrightarrow{x_0|y_0} \infty$$

Let t be such that T is t-to-one. Observe that $|U'(\ell, p, q, v)| \leq t$.

For each $u \in A^*$ let h_u be its minimum output length

$$h_u = \min\{|v| : \exists i, j, 0 \leq i \leq j, q_i \xrightarrow{u|v} q_j\}$$

Let U_ℓ be the set of words of length ℓ that yield large output

$$U_\ell = \{u \in A^\ell : h_u \geq (1 - \varepsilon)\ell\}$$

Then,

$$|U_\ell| \geq |A|^\ell - |Q|^2 t |B|^{(1-\varepsilon)\ell+1}.$$

Then, there is ℓ large enough such that $|U_\ell| > |A|^\ell(1 - \varepsilon)$ and apply Lemma 2 with $U = U_\ell$ to the considered run. □

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Lemma 4

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This is stronger than we need (non-deterministic, bounded-to-one)

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There exists then an increasing sequence $(n_i)_{i\geq 0}$ of integers such that for each word u of length k

$$f_u = \lim_{i\to\infty} \frac{\|x[1\ldots n_i]\|_u}{n_i/k}$$

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Note that $\sum_{u\in A^k} f_u = 1$.

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Notice that $\sum_{w \in A^{km}} f_w = 1$.
Now put A^{km} in a one to one correspondence with a prefix free set in B^*:
for each $w \in A^{km}$ let $v_w \in B^*$ such that

$$|v_w| \le \left\lceil \frac{-\log f_w}{\log |B|} \right\rceil .$$

Proof of Lemma 4

We construct a deterministic transducer $\mathcal{T}_m = \langle Q_m, A, B, \delta_m, I, F_m \rangle$,

$$\begin{aligned} Q_m &= A^{<km} \\ I &= \{\lambda\} \\ F_m &= Q_m \\ \delta_m &= \{w \xrightarrow{a|\lambda} wa : |wa| < km\} \cup \{w \xrightarrow{a|v_{wa}} \lambda : |wa| = km\}. \end{aligned}$$

The transducer $\mathcal{T}_m$ always comes back to its initial state λ after reading km symbols.

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&\leq \frac{|z|}{km} + \sum_{u \in A^{k}} \|z\|_u (-\log f_u)/\log |B|.
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Since at least one number f_u is not equal to $1/|A|^k$,

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Then, for m chosen large enough, we obtain that $\mathcal{T}_m$ compresses x. □

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Theorem 5

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