THE DISCREPANCY OF THE CHAMPERNOWNE CONSTANT

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Abstract. A number is normal in base b if, in its base b expansion, all blocks of digits of equal length have the same asymptotic frequency. The rate at which a number approaches normality is quantified by the classical notion of discrepancy, which measures how far the scaling of the number by powers of b is from being equidistributed modulo 1. This rate is known as the discrepancy of a normal number. The Champernowne constant $c_{10} = 0.12345678910111213141516\ldots$ is the most well-known example of a normal number. In 1986, Schiffer provided the discrepancy of numbers in a family that includes the Champernowne constant. His proof relies on exponential sums. Here, we present a discrete and elementary proof specifically for the discrepancy of the Champernowne constant.

Contents

1. Normal numbers

More than a hundred years ago, Émile Borel defined the property of normality of real numbers: a real number is normal in a given integer base b if in its expansion in base b all digits have the same asymptotic frequency and furthermore, all blocks of digits of equal length have the same asymptotic frequency. Borel proved that almost all real numbers, with respect to Lebesgue's measure, are normal in all integer bases greater than or equal to 2. A nice version of this proof appears in Hardy and Wright's book [8, Theorem 148].

Date: July 17, 2024.
1991 *Mathematics Subject Classification.* Primary 11K16.
Key words and phrases. normal numbers; Champernonwne constant; discrepancy.

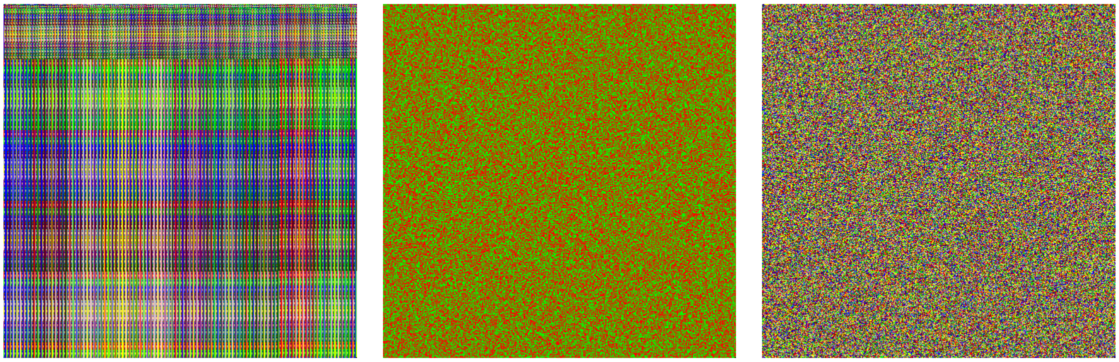


FIGURE 1. Plot of the expansion of the first 250000 digits of Champernowne constant in base 10, in base 2 and in base 6, from left to right. In each base each digit is assigned a different color, and the digits in the expansion are displayed in row-major order.

Borel would have liked to give an example of a normal number that is one of the mathematical constants such as π, or e, or $\sqrt{2}$. But so far none of these has been proved normal in any base. It remains an open problem [3, 1].

The best known example of a normal number is Champernowne constant,

$$c_{10} = 0.12345678910111213141516171819202122232425262 7...$$

Its expansion is the concatenation of all positive integers expressed in base 10 in increasing order. David Champernonwne defined it specifically to be an example of a number that is normal in base 10 [4]. He did this work in 1933 with the supervision of G.H. Hardy while he was a student at King's College, Cambridge [9]. Champernowne's proof is elementary, based on a rigorous counting.

Notice that c_{10} expressed in base b, for $b \neq 10$, is different from the Champernowne constant c_b which is the concatenation of the positive integers expressed in base b, in increasing order. Figure 1 depicts the expansion Champernowne number constant c_{10} expressed in base 10, in base 2 and in base 6. It is not known whether c_{10} is normal to any integer base other than 10.

In this note we present all the definitions and results in base 10. It is is equally possible to make the presentation for an arbitrary base b greater than or equal to 2.

For any real number $\alpha \in [0, 1)$, its expansion in base 10 is the sequence of digits $(\alpha_i)_{i\in\mathbb{N}}$ such that $\alpha_i \in \{0, .., 9\}$ and

$$\alpha = \lfloor \alpha \rfloor + \sum_{i\geq 1} \alpha_i 10^{-i}.$$

In case α is a rational number, it may have two expansions and we choose the one ending with a tail of 0s.

Definition (Ocurrences counter)**.** Let $k \in \mathbb{N}$ and let $B = (b_1 \dots b_k)$ be a block of digits $b_i \in \{0, \dots, 9\}$ of length k. Let $N \in \mathbb{N}$. We define $occ(\alpha, B, N)$ as the number of occurrences of the block B in $(\alpha_1\, \alpha_2 \dots \alpha_N)$ as k consecutive digits,

$$occ(\alpha, B, N) = \#\{i : \alpha_i \dots \alpha_{i+k-1} = b_1 \dots b_k, 1 \leq i \leq N - k + 1\}.$$

Example. For $N = 20$, $B = (131)$ and $\alpha = 0.13311321\mathbf{13131}751\mathbf{131}9\underset{\alpha_N}{\underset{\uparrow}{1}}31$ $occ(\alpha, B, N) = 3.\square$

Definition (Normal number in base 10)**.** A real number α is *normal in base 10* if for all $k \in \mathbb{N}$ and for every B block of digits of length k,

$$\lim_{N\to\infty} \frac{occ(\alpha, B, N)}{N} = \frac{1}{10^k}.$$

Example. The rational number $0.0123456789\,0123456789\,0123456789\ldots$ is not normal because although the frequency of each digit is $1/10$, the frequency of the block $(1\,1)$ is $0.\square$

2. Discrepancy estimate of normal numbers

Normality can be expressed in the theory of uniform distribution of sequences modulo 1. A number is normal in a given base b if the scaling of the number by powers of b is uniformly distributed modulo 1, [5]. The rate at which a number approaches normality in a base b is given by the classical notion of discrepancy.

The discrepancy of a sequence $(x_n)_{n\in\mathbb{N}} \subset [0, 1)$ measures how far is the sequence from being equidistributed in the unit interval.

Definition (Discrepancy of a sequence)**.** For a sequence $(x_n)_{n\in\mathbb{N}} \subset [0, 1)$ the *discrepancy* of it first N terms is

$$D\left((x_n)_{n\in\mathbb{N}}, N\right) = \sup_{0\le a<b<1} \left| \frac{\#\{n : x_n \in [a, b), 1 \le n \le N\}}{N} - (b - a) \right|.$$

For normality in base 10 we consider for each real number, the scaling of the number by increasing powers of 10. If the expansion of α in base 10 is given by $\alpha = 0.\alpha_1\,\alpha_2\,\alpha_3\ldots$, we consider the sequence $(x_n)_{n\in\mathbb{N}}$ where

$$\begin{aligned}
x_1 &= 10^0\alpha \mod 1 &&= 0.\alpha_1\,\alpha_2\,\alpha_3\ldots\\
x_2 &= 10^1\alpha \mod 1 &&= 0.\alpha_2\,\alpha_3\,\alpha_4\ldots\\
&\vdots\\
x_n &= 10^{n-1}\alpha \mod 1 &&= 0.\alpha_n\,\alpha_{n+1}\,\alpha_{n+2}\ldots
\end{aligned}$$

Definition (Discrepancy of a number for base 10)**.** For $\alpha \in [0, 1)$, its discrepancy for base 10 is

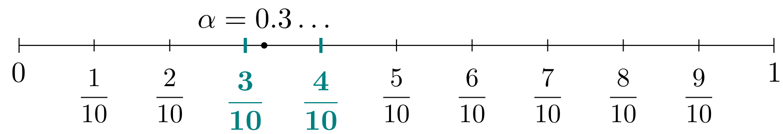
$$D(\alpha, N) = D\left((10^{n-1}\alpha \mod 1)_{n\in\mathbb{N}}, N\right).$$

Proposition. *A real number α is normal in base 10 if and only if* $\lim_{N\to\infty} D(\alpha, N) = 0$.

Example. Let $\alpha \in [0, 1)$ and let $(\alpha_i)_{i\in\mathbb{N}}$ the sequence of digits in its decimal expansion. Partition the unit interval $[0, 1)$ into intervals of size $1/10$. To ask in which of those ten intervals is α is equivalent to determine α_1,

$$\begin{array}{lll}
\alpha \in \left[0.\frac{1}{10}\right) & \text{if and only if} & \alpha_1 = 0\\
\alpha \in \left[\frac{1}{10}, \frac{2}{10}\right) & \text{if and only if} & \alpha_1 = 1\\
\alpha \in \left[\frac{2}{10}, \frac{3}{10}\right) & \text{if and only if} & \alpha_1 = 2\\
\vdots & & \\
\alpha \in \left[\frac{9}{10}, 1\right) & \text{if and only if} & \alpha_1 = 9.
\end{array}$$

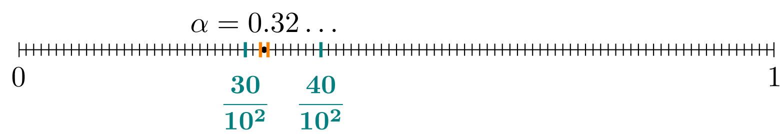
For instance,



Recall that for rational numbers having two expansions we commit to the expansion ending with an infinite tail of 0s instead of the infinite tail of 9's. This is justified because we are working with intervals that include the left point, For example, $\alpha = 0.30000 = 0.2999\ldots$ and $\alpha \in [3/10, 4/10)$, so it is justified that we chose the expansion that sets $\alpha_1 = 3$.

If instead of partitioning $[0, 1)$ in 10 intervals, we partition it in 100 intervals of size $1/10^2$, then to find out the subinterval containing α we need to determine the first two digits of its decimal expansion, α_1 and α_2. For instance,

$$\alpha \in \left[\frac{32}{10^2}, \frac{33}{10^2}\right) \text{ if and only if } \alpha_1 = 3 \text{ and } \alpha_2 = 2.$$

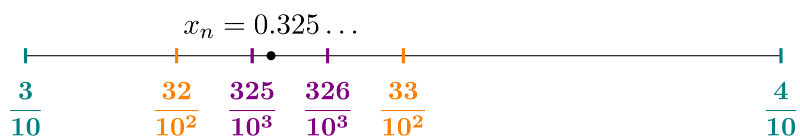


For each $n \in \{1, \ldots, N\}$, we need to identify the interval of size of size $1/10^n$ that contains $x_n = 10^{n-1}\alpha \mod 1$, For instance, partition the interval $[0, 1)$ into 1000 intervals of size $1/10^3$ and consider one of them,

$$I = \left[\frac{325}{10^3}, \frac{326}{10^3}\right).$$

Since $x_n = 10^{n-1}\alpha \mod 1 = 0.\alpha_n\,\alpha_{n+1}\,\alpha_{n+2}\ldots$, we have that

$$\begin{aligned} x_n \in I &\text{ if and only if } \alpha_n = 1 \text{ and } \alpha_{n+1} = 3 \text{ and } \alpha_{n+2} = 0 \\ &\text{ if and only if } \alpha = 0.\alpha_1 \ldots \alpha_{n-1}\,3\,2\,5\,\alpha_{n+3}\ldots. \end{aligned}$$



That is, x_n belongs to the interval I if and only if there is an occurrence of the block $B = (325)$ in the digit number n of α. Thus, if we take $k = 3$, the length of the block B, then

$$\begin{aligned} \#\{n : x_n \in I, 1 \le n \le N\} &= \#\{n : (\alpha_n\,\alpha_{n+1}\,\alpha_{n+2}) = (3\,2\,5), 1 \le n \le N\} \\ &= occ(\alpha, B, N + k - 1). \end{aligned}$$

□

We use Landau's notation to make estimates. For functions f and g over the real numbers, and g strictly positive, we write $f(x) = O(g(x))$ if there exists a positive constant C and a value x_0 such that for all $x > x_0$, $|f(x)| < Cg(x)$. And we write $f(x) = o(g(x))$ if for all positive constants C, there exists x_0 such that for every $x \ge x_0$,$|f(x)| < Cg(x)$.

Schiffer in [14] gives the discrepancy of numbers in a large family, but of Lebesgue measure zero. He proves that for any non-constant polynomial f with rational coefficients such that

$f(n) \in \mathbb{N}$ for all $n \in \mathbb{N}$, the discrepancy of the real number α whose decimal expansion is formed by the concatenation of the values of f evaluated on the positive integers,

$$\alpha = 0.f(1)f(2)f(3)\ldots$$

satisfies the following: There are two constants K_1 and K_2 such that, there are cofinitely many N for which $D(\alpha, N) < K_1/\log N$; and there are infinitely many N for which $D(\alpha, N) > K_2/\log N$. Nakai and Shiokawa in [12] generalize this result for non-constant f with real coefficients, such that $f(t) > 0$ for all $t > 0$. Schiffer's result applies to the Champernowne constant taking the polynomial $f(x) = x$. In this note we give a discrete an elementary proof of of the exact discrepancy of the Champernowne number.

How does it compare the discrepancy of the Champernowne constant to the discrepancy of other normal numbers? The minimum discrepancy achievable by a normal number is still not known. The question was posed by Korobov in 1955 [10], see also Bugeaud's book [3]. Without restricting to sequences of the form $(b^n\alpha)_{n\geq 1}$ the minimal discrepancy known: Schmidt [15] proved that there is a constant C such that for *all* sequences $(x_n)_{n\in\mathbb{N}} \subset [0, 1)$ there are infinitely many N where the discrepancy of the first N terms, is above $C \log(N)/N$. And the discrepancy of first N terms of the van der Corput sequences is $O((\log N)/N)$ so this is the minimal discrepancy that an arbitrary sequence $(x_n)_{n\geq 1}$ can have [11].

Surprisingly, almost all numbers in the sense of the Lebesgue measure are normal with the same discrepancy. Gál and Gál [7] proved an upper bound for the discrepancy of almost all numbers with respect to Lebesgue's measure. Philipp [13] gave the explicit constants, and Fukuyama [6] refined them, obtaining that for every base $b > 1$, there exists a constant K_b such that for almost every real number α,

$$\limsup_{N\to\infty} \frac{\sqrt{N}\, D\left((b^n\alpha \mod 1)_{n\geq 0}\, , N\right)}{\sqrt{\log(\log N)}} = K_b.$$

That is to say that for almost every real number α,

$$D(\alpha, N) = O\left(\frac{\sqrt{\log(\log N)}}{\sqrt{N}}\right).$$

Gál and Gál showed that the discrepancy of almost all numbers is below the law of the iterated logarithm. This, in turn, is below the of discrepancy of Schiffer's numbers. Thus, Champernowne constant approaches normality much slower than almost every number.

The set of bases to which a real number can be normal is not tied to any arithmetical properties other than multiplicative dependence (for integer bases r and s, if $r^n = s^m$ for some $m, n \in \mathbb{N}$, a number is normal in base r exactly when it is normal to base s). For any given set of bases closed under multiplicative dependence, there are real numbers that are normal to each base in the given set, but not normal to any base in its complement. In [2, Theorem 2.8] Becher and Slaman show that the discrepancy functions for multiplicatively independent bases are pairwise independent.

3. Statement of results

Here we give a discrete and elementary proof of the exact discrepancy of the Champernowne constant c_{10}. Schiffer [14] provided the discrepancy of a family of numbers that includes c_{10}. His proof relies on exponential sums. This is and alternative proof of Schiffer's result specifically for the Champernowne constant.

Theorem 1. *Let c_{10} be the Champernowne constant for base 10. Then, there is a constant $K_1 > 0$ such that there are cofinitely many N such that $D(c_{10}, N) < K_1/\log N$.*

Theorem 2. *Let c_{10} be the Champernowne constant for base 10. Then, there exists a constant $K_2 > 0$ such that for infinitely many N, $D(c_{10}, N) > K_2/\log N$. In particular, one can take $K_2 = 1/(10^3 3)$.*

Remark. Theorems 1 and 2 imply that $D(c_{10}, N) = O(1/\log N)$ and $D(c_{10}, N) \neq o(1/\log N)$. That is, the estimate cannot be improved.

Remark. Theorems 1 and 2 hold for any other integer base $b \geq 2$ with the definition of Champernowne constant c_b for that base (the concatenation of all positive integers represented in that base). The constants K_1, K_2 depend on the base b.

4. Basic Tools

In the sequel the Champernowne constant is called $c = c_{10}$,

$$c = 0.1234567891011121314151617181920212 2\ldots$$

We define three sequences.

Definition. Let $(t_i)_{i\in\mathbb{N}}$, where $t_i = i$, be the sequence of terms that concatenated yield the expansion of c.

Definition. Let $(c_i)_{i\in\mathbb{N}}$, where each $c_i \in \{0, .., 9\}$, be the decimal expansion of c.

For example, $c_{11} = 0$ and $c_{14} = 1$, because

$$c = 0.\underset{\substack{\uparrow\\ c_1}}{1}\,\overset{\substack{c_2\\ \downarrow}}{2}\;3\;4\;5\;6\;7\;8\;9\;\underset{\substack{\uparrow\\ c_{10}}}{1}\overset{\substack{c_{11}\\ \downarrow}}{0}\;11\;\underset{\substack{\uparrow\\ c_{14}}}{1}2\;13\;14\;15\;16\ldots.$$

Definition. Let $(x_n)_{n\in\mathbb{N}} = (10^{n-1}c \mod 1)_{n\in\mathbb{N}}$.

Thus, $x_1 = 0.1\,2\,3\,4\,5\ldots, x_2 = 0.2\,3\,4\,5\,6\ldots, \ldots x_n = 0.c_n\, c_{n+1}\, c_{n+2}\ldots$

Definition (Overlapping occurrences)**.** An occurrence of B in c is *overlapping* if B occurs between two or more t_i. We denote $occ_o(c, B, N)$ to the number of overlapping occurrences of B in $(c_1\, c_2 \ldots c_N)$.

Definition (Non-overlapping occurrences)**.** An occurrence of B in c *non-overlapping*, if B occurs within a single term t_i. We denote $occ_{no}(c, B, N)$ to the number of non-overlapping occurrences of B in $(c_1\, c_2 \ldots c_N)$.

Example. For $N = 36$ and $B = (1\,2)$. Then, $occ_{no}(c, B, N) = 1$, because

$$c = 0.1\;2\;3\;4\;5\;6\;7\;8\;9\;10\;11\;\mathbf{12}\;13\;14\;15\;16\;17\;18\;19\;20\;21\;22\;\underset{\substack{\uparrow\\ x_N}}{2}3\;24\ldots$$

And $occ_o(c, B, N) = 2$, because

$$c = 0.\mathbf{1}\;\mathbf{2}\;3\;4\;5\;6\;7\;8\;9\;10\;11\;12\;13\;14\;15\;16\;17\;18\;19\;20\;2\mathbf{1}\;\mathbf{2}2\;\underset{\substack{\uparrow\\ x_N}}{2}3\;24\ldots.$$

□

Remark. $occ(c, B, N) = occ_{no}(c, B, N) + occ_o(c, B, N)$.

Definition (segment s_ℓ)**.** Given $\ell \in \mathbb{N}$, s_ℓ is the concatenation of terms formed by all natural numbers of ℓ digits ordered in ascending order, that is, from $10^{\ell-1}$ to $10^\ell - 1$,

$$s_\ell = (\overbrace{10\ldots0}^{\ell \text{ digits}}, \ldots, \overbrace{9\ldots9}^{\ell \text{ digits}}).$$

Example. $s_1 = (1\,2\,3\,4\,5\,6\,7\,8\,9)$, $s_2 = (10\,11\,12\ldots98\,99)$, $s_3 = (100\,101\ldots998\,999)$. □

Definition (The numbers $v = v(N)$ and $n = n(N)$)**.** For $N \in \mathbb{N}$, let $v = v(N)$ be such that $T(v) \geq N$ and $T(v-1) < N$. Let $n = n(N)$ the number of digits of v.

Definition. For $v \in \mathbb{N}$, $T(v)$ is the number of digits in the expansion of c up to the term v.

Example. For $v = 11$, $T(v) = 13$, since there are 13 digits in $1\,2\,3\,4\,5\,6\,7\,8\,9\,10\,11$. □
Example. For $N = 14$, then $v = 12$ and $n = 2$, since $T(v) = 15 \geq N$ and $T(v-1) = 13 < N$.

$$c = 0.1\,2\,3\,4\,5\,6\,7\,8\,9\,10\,11\,\overset{v}{\underset{\underset{c_N}{\uparrow}}{12}}\,13\ldots$$

□

Definition (Lengths L and M)**.** Given N, we define $L = L(N)$ as number of digits from s_1 to s_{n-1},

$$L = \sum_{j=1}^{n-1} j \cdot 9 \cdot 10^{j-1}.$$

We define $M = M(N)$ as the number of digits within s_n to the number v.

$$M = n(v - 10^{n-1} + 1) = n\left(\sum_{i=1}^{n} v_i 10^{n-i} - 10^{n-1} + 1\right).$$

Remark. Given $n \in \mathbb{N}$, let $L = L(N), M = M(N), n = n(N)$. Then,

$$L \leq N \leq L + M = \sum_{j=1}^{n} 9j10^{j-1} = n10^n - \frac{10^n}{9} + \frac{1}{9}. \tag{1}$$

Definition. $occ_{no}(B, v)$ is the number of non-overlapping occurrences of B in all n digit terms less than or equal to v, where n is the number of digits of v.

Definition. $occ_o(B, v)$ is the number of overlapping occurrences of B in all n digit terms less than or equal to v, where n is the number of digits of v.

Definition. For $\ell \in \mathbb{N}$, $occ_{no}(B, s_\ell)$ is the number of non-overlapping occurrences of B in s_ℓ.

Definition. For$\ell \in \mathbb{N}$, $occ_o(B, s_\ell)$ is the number of overlapping occurrences in s_ℓ.

5. Theorem 1: Upper bound

We give an upper bound, for every N, of $D(c, N)$.

5.1. **Counting occurrences.** We start with the following lemma.

Lemma 1. *Given $N \in \mathbb{N}$ and B a block of length $k > 1$, then*

$$occ(c, B, N) = 10^{-k}N + O(10^{n-k}),$$

where the hidden constant in $O(10^{n-k})$ does not depend on B.

Proof. Let $N \in \mathbb{N}$ and $B = (b_1, \dots, b_k)$. We know that

$$occ(c, B, N) = occ_{no}(c, B, N) + occ_o(c, B, N).$$

We separate the proof into steps.

Step 1: We estimate $occ_{no}(c, B, N)$, the non-overlapping occurrences.

Step 1.1: For each $\ell \geq 1$ we show

$$occ_{no}(B, s_\ell) \leq 10^{\ell-k} + (\ell - k) \cdot 9 \cdot 10^{\ell-k-1}.$$

If $\ell < k$, then $occ_{no}(B, s_\ell) = 0$, since B does not fit within a block of ℓ digits.

If $\ell \geq k$: We want to count the amount of numbers of the form:

$$y = \underbrace{* \dots * B * \dots *}_{\ell \text{ digits}}$$

where the asterisk $*$ represents any possible digit. We move the position of B and count in each case:

- We count the amount of numbers of the form

$$y_0 = B \underbrace{* \dots *}_{\ell-k \text{ digits}}$$

that is, ℓ digit numbers that contain B in the first position:

If $b_1 = 0$: There are 0 numbers of the form y_0, since no natural number begins with 0.

If $b_1 \neq 0$: There are $10^{\ell-k}$ numbers of the form y_0, so we can choose the last $\ell - k$ digits between 0 and 9.

- We count the amount of numbers of the form

$$y_1 = \underbrace{*}_{1 \text{ digit}} \; B \; \underbrace{* \dots *}_{\ell-k-1 \text{ digits}}$$

that is, ℓ digit numbers that contain B in the second position: There are $9 \cdot 10 \cdot 10^{\ell-k-1}$ numbers of the form y_1.

- We count the amount of numbers of the form

$$y_2 = \underbrace{**}_{2 \text{ digits}} \; B \; \underbrace{* \dots *}_{\ell-k-2 \text{ digits}}$$

that is, ℓ digit numbers that contain B in the third position: There are $9 \cdot 10 \cdot 10^{\ell-k-2}$ numbers of the form y_2. Because:

  - 9 is the number of values that the first digit can take (between 1 and 9 since it cannot take the value 0).
  - 10 is the number of values that the second digit can take (between 0 and 9).
  - $10^{\ell-k-2}$ is the number of values that the last $\ell - k - 2$ digits can take.

• We count the amount of numbers of the form

$$y_3 = \underbrace{*\ *\ *}_{3 \text{ digits}} B \quad \underbrace{*\ldots*}_{\ell-k-3 \text{ digits}}$$

that is, ℓ digit numbers that contain B in the fourth position: There are $9 \cdot 10^2 \cdot 10^{\ell-k-3}$ numbers of the form y_3.

• Continuing like this, we arrive at the last position: We count the amount of numbers of the form

$$y_{\ell-k} = \underbrace{*\ldots*}_{\ell-k \text{ digits}} B$$

that is, ℓ digit numbers containing B in position $\ell - k + 1$: There are $9 \cdot 10^{\ell-k-1}$ numbers of the form $y_{\ell-k}$.

Overall we obtain that for each $j \in \{1, \ldots, \ell - k\}$, the amount of numbers of the form y_j is

$$9 \cdot 10^{\ell-k-1}.$$

Therefore, if $b_1 = 0$:

$$occ_{no}(B, s_\ell) = \sum_{j=1}^{\ell-k} 9 \cdot 10^{\ell-k-1} = (\ell - k) \cdot 9 \cdot 10^{\ell-k-1};$$

if $b_1 \neq 0$:

$$occ_{no}(B, s_\ell) = 10^{\ell-k} + \sum_{j=1}^{\ell-k} 9 \cdot 10^{\ ell-k-1} = 10^{\ell-k} + (\ell - k) \cdot 9 \cdot 10^{\ell-k-1}.$$

Then, for all $\ell \geq k$,

$$occ_{no}(B, s_\ell) \leq 10^{\ell-k} + (\ell - k) \cdot 9 \cdot 10^{\ell-k-1},$$

Step 1.2: We show that

$$occ_{no}(c, B, N) = \sum_{\ell=k}^{n-1} occ_{no}(B, s_\ell) + occ_{no}(B, v) + O(n).$$

To count non-overlapping occurrences of B up to the position N, that is, $occ_{no}(c, B, N)$), we count occurrences in s_ℓ for each $\ell \in \{1, \ldots, n - 1\}$ that is, $occ_{no}(B, s_\ell)$, and then count the occurrences in s_n by cutting it at v (that is $occ_{no}(B, v)$). Finally, the $O(n)$ comes from substracting the possible occurrences of B within v after the N'th digit, there could be at most $n - k$ of those occurrences.

Example. For $N = 9523$, $v = 2658$ and $n = 4$,

$$c = 0.\underbrace{1\ldots9}_{\substack{9\cdot1\\ \text{digits}}}\underbrace{10\ldots99}_{\substack{90\cdot2\\ \text{digits}}}\underbrace{100\ldots999}_{\substack{900\cdot3\\ \text{digits}}}\underbrace{1000\ldots2658}_{\substack{1658\cdot4\\ \text{digits}}}\ \overbrace{26\underset{\substack{\uparrow\\ c_N}}{5}8}^{v}$$

Let's take B to be any two-digit block, that is, $k = 2$. To count non-overlapping occurrences up to N, we have

$$occ_{no}(c, B, N) = \underbrace{occ_{no}(B, s_2)}_{\substack{\text{occurrences}\\ \text{in } s_2}} + \underbrace{occ_{no}(B, s_3)}_{\substack{\text{occurrences}\\ \text{in } s_3}} + \underbrace{occ_{no}(B, v)}_{\substack{\text{occurrences}\\ \text{in } s_n \text{ until } v}} + \underbrace{O(n)}_{\substack{\text{occurrences}\\ \text{after } c_N}}$$

because

$$c = 0.\overbrace{1\dots9}^{s_1}\overbrace{10\dots99}^{s_2}\overbrace{100\dots999}^{s_3}1000\dots2658\ \underset{\substack{\uparrow\\ c_N}}{\overbrace{2658}^{v}}.$$

Then, the procedure consists of first, counting the occurrences in s_1 (which are 0 because $k = 2$), in s_2 and s_3; then count the occurrences from 1000 to 2658; and finally, subtract possible occurrences in the last two digits of v. □

Step 1.3: Bound $occ_{no}(B, v)$. which is the amount of numbers of the form

$$y = \underbrace{*\dots * B * \dots *}_{n \text{ digits}} \qquad \text{with } y \le v.$$

If $n < k$: $occ_{no}(B, v) = 0$, because, B does not fit inside blocks of length n.

If $n \ge k$: Let $v = \sum_{i=1}^{n} v_i 10^{n-i} = v_1 \dots v_n$. Given $j \in \{0, \dots, n-k\}$, we define:

$$a_j = \sum_{i=1}^{j} v_i 10^{j-i} = v_1 \dots v_j$$

Again, we move the position of B and count in each case: Given $j \in \{0, \dots, n-k\}$, let us call $occ_{no}(B, v, j)$ the amount of numbers of the form

$$y_j = \underbrace{*\dots*}_{j \text{ digits}}\ B\ \ \underbrace{*\dots*}_{n-k-j \text{ digits}} \qquad \text{with } y_j \le v.$$

Case $j = 0$. We want to count the amount of numbers of the form

$$y_0 = B\ \ \underbrace{*\cdots*}_{n-k \text{ digits}}$$

We note that again, if $b_1 = 0$, then $occ_{no}(B, v, j) = 0$, since there are no numbers that begin with a leading zero. Now, if $b_1 \ne 0$, we must separate cases, since $occ_{no}(B, v, j)$ depend on who is B and who is v.

- If $B > v_1 \dots v_k$: $occ_{no}(B, v, j) = 0$, then $y_0 > v$.
- If $B = v_1 \dots v_k$: $occ_{no}(B, v, j) = v_{k+1} \dots v_n + 1$, then the last $n - k$ digits of y_0 we can choose between 0 and $v_{k+1} \dots v_n$.
- If $B < v_1 \dots v_k$: $occ_{no}(B, v, j) = 10^{n-k}$, then the last $n - k$ digits of y_0 can take any value from 0 up to $\underbrace{9\dots9}_{n-k \text{ digits}}$.

Therefore, $occ_{no}(B, v, 0) \le 10^{n-k}$.

Case $1 \le j \le n - k$.

$$\begin{aligned} occ_{no}(B, v, j) &= \underbrace{(a_j - 10^{j-1} + \theta_j)}_{\substack{\text{Choose the first}\\ j \text{ digits}}}\ \underbrace{10^{n-k-j}}_{\substack{\text{Choose the last}\\ n-k-j \text{ digits}}} \\ &= 10^{-k}\left(\sum_{i=1}^{j} v_i 10^{n-i} - 10^{n-1} + \theta_j 10^{n-j}\right) \text{ where } 0 \le \theta_j \le 1. \end{aligned}$$

.

Let's see why the first equality is valid. For y_j to be effectively less than or equal to v, we must choose the first j digits within $(10^{j-1},\dots,a_j)$. It depends on the values of B and v, whether we are taking a_j inclusive or exclusive.

Example. $B=(3\ 1)$, $n=4$, $v=2\mathbf{32}5$ and $j=1$. So,

$$occ_{no}(B,v,1)=\underbrace{2}_{\substack{\text{Choose thefirst digit}\\ \text{between 1 and 2}}}\cdot\underbrace{10}_{\substack{\text{Choose the}\\ \text{fourth digitbetween 0 and 9}}}=(2-10^{1-1}+\theta_1)10^{4-2-1}$$

with $\theta_1=1$. In this case, we are taking $a_j=2$ inclusive. □

Example. $B=(3\ 1)$, $n=4$, $v=2\mathbf{30}5$ and $j=1$. Then,

$$occ_{no}(B,v,1)=\underbrace{1}_{\substack{\text{The first}\\ \text{digit only}\\ \text{can be 1}}}\cdot\underbrace{10}_{\substack{\text{Choose the}\\ \text{fourth digit}\\ \text{between 0 and 9}}}=(2-10^{1-1}+\theta_1)10^{4-2-1},\ \text{with }\theta_1=0.$$

with $\theta_1=0$. In this case, we are taking $a_j=2$ exclusive. □

Example. $B=(3\ 1)$, $n=4$, $v=2\mathbf{31}5$ and $j=1$. So,

$$\begin{aligned}occ_{no}(B,v,1)&=\underbrace{10}_{\substack{\text{If the first}\\ \text{digit is 1,}\\ \text{choose the fourth}\\ \text{digit between 0 and 9}}}+\underbrace{6}_{\substack{\text{If the first}\\ \text{digit is 2,}\\ \text{choose the fourth}\\ \text{digit between 0 and 5}}}\\&=(2-10^{1-1})10^{4-2-1}+(10^{4-2-1}-4)\\&=(2-10^{1-1})10^{4-2-1}+10^{4-2-1}(1-\frac{4}{10})\\&=(2-10^{1-1}+\theta_1)10^{4-2-1},\ \text{with }\theta_1=1-\frac{4}{10}.\end{aligned}$$

□

Recall M is the number of digits from 10^{n-1} to v, that is, the number of digits in s_n up to the number v. Then, Recall L the number of digits from 1 to $10^{n-1}-1$ By (1), $L\le N\le L+M$. Indeed, $N=L+M-O(n)$ because in the worst case, N is the position of the first digit of v and we have to subtract $n-1$. We have,

$$\begin{aligned}occ_{no}(B,v)&=\sum_{j=0}^{n-k}occ_{no}(B,v,j)\\&\le 10^{n-k}+10^{-k}\left(\sum_{j=0}^{n-k}\sum_{i=1}^{j}v_i10^{n-i}-10^{n-1}+\theta_j10^{n-j}\right)\\&=10^{-k}\left(\sum_{j=0}^{n-k}\sum_{i=1}^{j}(v_i10^{n-i}-10^{n-1})+\sum_{j=0}^{n-k}\sum_{i=1}^{j}\theta_j10^{n-j}\right)+O(10^{n-k})\\&\le 10^{-k}\left(-(n-k+1)10^{n-1}+\sum_{j=0}^{n-k}\sum_{i=1}^{j}v_i10^{n-i}\right)+10^{-k}10^n\sum_{j=0}^{n-k}j\frac{1}{10^j}+O(10^{n-k})\end{aligned}$$

$$
\begin{aligned}
&= 10^{-k}\left(-(n-k+1)10^{n-1}+\sum_{j=0}^{n-k}\sum_{i=1}^{j} v_i 10^{n-i}\right)+O(10^n)+O(10^{n-k})\\
&\le 10^{-k}\left(-(n-k+1)10^{n-1}+\sum_{j=1}^{n-k} v_i 10^{n-i}(n-k-i+1)\right)+O(10^{n-k})\\
&\le 10^{-k}M+O(10^{n-k})
\end{aligned}
$$

Let's see why the last two inequalities are valid. For the previous to the last we have

$$
\begin{aligned}
\sum_{j=0}^{n-k}\sum_{i=1}^{j} v_i 10^{n-i} &= \sum_{i=1}^{0} v_i 10^{n-i} + \sum_{i=1}^{1} v_i 10^{n-i} + \ldots \sum_{i=1}^{n-k} v_i 10^{n-i}\\
&= (v_1 10^{n-1}) + (v_1 10^{n-1} + v_2 10^{n-2}) + \ldots + (v_1 10^{n-1} + \ldots + v_{n-k}10^{n-(n-k)})\\
&= \sum_{i=1}^{n-k} v_i 10^{n-i}(n-k-i+1)
\end{aligned}
$$

For the last we have,

$$
\begin{aligned}
-(n-k+1)10^{n-1}+\sum_{j=1}^{n-k} v_i 10^{n-i}(n-k-i+1) &\le -n10^{n-1}+10^{n-1}(k-1)+n\sum_{i=1}^{n} v_i 10^{n-i}\\
&\le M+O(10^n).
\end{aligned}
$$

We conclude,

$$
\begin{aligned}
occ_{no}(c,B,N) &= \sum_{\ell=k}^{n-1} occ_{no}(B,s_\ell)+occ_{no}(B,v)-O(n)\\
&\le \sum_{\ell=k}^{n-1} 10^{\ell-k}+(\ell-k)\cdot 9\cdot 10^{\ell-k-1}+10^{-k}M+O(10^{n-k}),
\end{aligned}
\tag{2}
$$

where the hidden constant inside $O(10^{n-k})$ does not depend on B.

Step 2: We bound $occ_o(c,B,N)$, the number of overlapping occurrences.
If $k=1$, there are no overlapping occurrences of B, so we assume $k>1$. Observe that

$$
occ_o(c,B,N) \le \sum_{\ell=1}^{n} occ_o(B,s_\ell)+O(n).
$$

The worst case for N is realized in the last digit of v when v is the last number in s_n,

$$
v = 10^n - 1 = \underbrace{9\ldots 9}_{n \text{ digits}},
$$

so should add each $occ_o(B,s_\ell)$ up to $\ell=n$. The $O(n)$ comes from summing all the overlapping occurrences that could appear between two (or more) s_ℓ. At most there are kn of those occurrences, that is k occurrences for each s_ℓ). We can bound them by a constant that does not depend on the choice of B.

Step 2.1: We bound $occ_o(B, s_\ell)$.

Case $\ell \geq k$. For blocks of length $\ell \geq k$, there can only be overlapping occurrences two blocks, and no more. If x is a number of ℓ digits, and overlapping occurrence of B between x and $x+1$ can happen only if the last digits of x are $(b_1, \ldots, b_{k-j})$ and the first digits of $x+1$ are $(b_{k-j+1}, \ldots, b_k)$ for some $j \in \{1, \ldots, k-1\}$. Therefore, x must be of the form

$$x = b_{k-j+1} \ldots b_k \underbrace{* \ldots *}_{\ell-k \text{ digits}} b_1 \ldots b_{k-j}.$$

Since there are $\ell - k$ free digits, x can at most take $10^{\ell-k}$ values. Then, for each $j \in \{1, \ldots, k-1\}$, B can occur overlapping blocks of length ℓ at most $10^{\ell-k}$ times. Therefore,

$$occ_o(B, s_\ell) \leq \sum_{j=1}^{k-1} 10^{\ell-k} = (k-1)10^{\ell-k}.$$

Example. For $\ell = 6$, $B = (1\,2\,3\,4)$, $k = 4$. Then, the overlapping occurrences of B are:

x	$x+1$
$2\,3\,4 * * \mathbf{1}$	$\mathbf{2\,3\,4} * * 2$
$3\,4 * * \mathbf{1\,2}$	$\mathbf{3\,4} * * 1\,3$
$4 * * \mathbf{1\,2\,3}$	$\mathbf{4} * * 1\,2\,4$

Then, $occ_o(B, s_\ell) = 3 \cdot 10^2$. In this example equality applies because the block B consists of all different digits. But, if B has repeated digits, we could be counting the same occurrence repeatedly. That's why we get a bound for $occ_o(B, s_\ell)$ and not an equality. □

Example. For $\ell = 6$, $B = (1\,1\,1\,1)$, $k = 4$, the overlapping occurrences of B are:

x	$x+1$
$1\,1\,1 * * \mathbf{1}$	$\mathbf{1\,1\,1} * * 2$
$1\,1 * * \mathbf{1\,1}$	$\mathbf{1\,1} * * 1\,2$
$1 * * \mathbf{1\,1\,1}$	$\mathbf{1} * * 1\,1\,2$

Then, $occ_o(B, s_\ell) < 3 \cdot 10^2$. The equality would imply that numbers such as $x = 111011$ be counted twice of (once in the first row and once once in the second row). □

Case $\ell < k$. To simplify the work, and as it is sufficient for the bound we are looking for, we bound all overlapping occurrences B from s_1 to s_{k-1} in terms of the number of digits from s_1 to s_{k-1}, that is, the number of digits in

$$1\,2 \ldots 10\,11 \ldots 100 \ldots 999 \ldots \underbrace{10^{k-1} - 1}_{\substack{=9\ldots9 \\ \text{(It has } k-1 \text{ digits)}}}.$$

Then,

$$\sum_{\ell=1}^{k-1} occ_o(B, s_\ell) \leq \sum_{i=1}^{k-1} \underbrace{9 \cdot 10^{i-1}}_{\substack{\text{Amount of} \\ \text{numbers in } s_i}} \cdot \underbrace{i}_{\substack{\text{Amount} \\ \text{of} \\ \text{digits}}}.$$

Therefore, we bound the overlapping occurrences, concluding Step 2:

$$occ_o(c, B, N) \leq \sum_{\ell=k}^{n} 10^{\ell-k}(k-1) + \sum_{j=1}^{k-1} 9 \cdot 10^{j-1} \cdot j \tag{3}$$

Using the bounds (2) (from Step 1) and (3) (from Step 2) we obtain,

$$\begin{aligned}
occ(c,B,N) &= occ_{no}(c,B,N) + occ_o(c,B,N)\\
&\leq \sum_{\ell=k}^{n-1} 10^{\ell-k} + 9(\ell-k)10^{\ell-k-1} + 10^{n-k} + 10^{-k}M + O(10^{n-k})\\
&\quad + \sum_{\ell=k}^{n} 10^{\ell-k}(k-1) + \sum_{j=1}^{k-1} 9\cdot 10^{j-1}\cdot j\\
&= \sum_{\ell=k}^{n-1} 10^{\ell-k} + 9(\ell-k)10^{\ell-k-1} + 10^{-k}M\\
&\quad + \sum_{\ell=k}^{n} 10^{\ell-k}(k-1) + O(10^{n-k}).
\end{aligned}$$

Then,

$$\begin{aligned}
occ(c,B,N) &\leq \sum_{\ell=k}^{n-1} 10^{\ell-k-1}(10+9\ell-9k+10k-10) + 10^{n-k}(k-1) + 10^{-k}M + O(10^{n-k})\\
&= 10^{-k}\sum_{\ell=k}^{n-1} 10^{\ell-1}\cdot 9\cdot\ell + 10^{-k}\cdot k\cdot\frac{1}{10}\sum_{\ell=k}^{n-1}10^{\ell} O(10^{n-k}) + 10^{-k}M + O(10^{n-k})\\
&\leq 10^{-k}L + 10^{-k}\cdot k\cdot\frac{1}{10}\left(\frac{1-10^n}{-9} - \frac{1-10^k}{-9}\right) + O(10^{n-k}) + 10^{-k}M + O(10^{n-k})\\
&= 10^{-k}L + O(10^{n-k}) + 10^{-k}M + O(10^{n-k})\\
&= 10^{-k}N + O(10^{n-k}). \qquad \square
\end{aligned}$$

5.2. **Proof of Theorem 1.** We need to show that there exists $N_0 \in \mathbb{N}$ and $C > 0$ such that for all $N \geq N_0$, $D(c,N) \leq C\dfrac{1}{\log N}$. We introduce notation. For $0 \leq a < b < 1$ and $N \in \mathbb{N}$,

$$D(c,a,b,N) = \frac{\#\{j \in \{1,\ldots,N\} : x_j \in [0,b)\}}{N} - (b-a).$$

Then,

$$\sup_{0\leq a<b<1} |D(c,a,b,N)| = D(c,N),$$

To bound the discrepancy of c we should take supremum of $|D(c,a,b,N)|$ over all intervals $[a,b) \subseteq [0,1)$. Let us see that it suffices to consider intervals of the form $[0,b)$. Suppose we have proved it for every interval of the form $[0,b)$, that is, for every $N \gg 1$,

$$\sup_{0\leq b<1} |D(c,0,b,N)| = \sup_{0<b<1}\left|\frac{\#\{j \in \{1,\ldots,N\} : x_j \in [0,b)\}}{N} - b\right| \leq \frac{K}{\log N}.$$

For every $N \gg 1$, $|D(c,a,b,N)|$ is equal to

$$\left|\frac{\#\{j \in \{1,\ldots,N\} : x_n \in [0,b)\} - \#\{j \in \{1,\ldots,N\} : x_j \in [0,a)\}}{N} - b + a\right|.$$

Then,

$$\begin{aligned}|D(c,a,b,N)| &\leq \left|\frac{\#\{j\in\{1,\ldots,N\}: x_n\in[0,b)\}}{N}-b\right|+\left|\frac{\#\{j\in\{1,\ldots,N\}: x_j\in[0,a)\}}{N}-a\right|\\ &\leq 2\sup_{0\leq y<1}|D(c,0,y,N)|\\ &\leq 2\frac{K}{\log N}\\ &=\frac{C}{\log N}.\end{aligned}$$

We now prove that for every $N \gg 1$, for all $b\in[0,1)$, $\sup_{0\leq b<1}|D(c,0,b,N)|\leq \frac{K}{\log N}$, where the constant K does not depend on b. It is enough to see that $|D(c,0,b,N)| = O\left(\frac{1}{\log N}\right)$, for all $b\in[0,1)$, where the hidden constant in Landau's O does not depend on b. We divide the proof in three steps.

Step 1: Let $k\in\mathbb{N}$ and $\alpha\in[0,1)$, $\alpha = 0.\alpha_1\ldots\alpha_k = \sum_{i=1}^{k}\alpha_i 10^{-i}$. Let $N\in\mathbb{N}$. We show

$$|D(c,\alpha,\alpha+10^{-k},N)| = O\left(\frac{10^{-k}}{\log N}\right),$$

where the constant does not depend on α. For $B=(\alpha_1,\ldots,\alpha_k)$ we have,

$$\#\{j\in\{1,\ldots,N\}: x_j\in[\alpha,\alpha+10^{-k})\} = occ(c,B,1,N)+O(k).$$

The term $O(k)$ comes from counting the possible occurrences that could occur after the digit c_N and up to the digit c_{N+k-1}. Using Lemma 1 for the second equality we obtain,

$$\begin{aligned}|D(c,\alpha,\alpha+10^{-k},N)| &= \left|\frac{1}{N}occ(c,B,N)-10^{-k}+O\left(\frac{k}{N}\right)\right|\\ &= \left|\frac{1}{N}10^{-k}N+O\left(\frac{1}{N}10^{n-k}\right)-10^{-k}+O\left(\frac{k}{N}\right)\right|\\ &= \left|10^{-k}+O\left(\frac{1}{\log N}10^{-k}\right)-10^{-k}+O\left(\frac{k}{N}\right)\right| \qquad (4)\\ &= O\left(\frac{1}{\log N}10^{-k}\right),\end{aligned}$$

for N large enough (since k is fixed). By Lemma 1, the hidden constant in $O\left(\frac{10^{-k}}{\log N}\right)$ does not depend on α.

Recall that N is the position at which we find v, v is a term with n digits, L is number of digits from s_1 to s_{n-1}, and M is the number of digits within s_n to the number v. Let us see why equality (4) holds. We prove that for for sufficiently large N

$$10^n/N < 2/\log N,$$

Using $L \leq N$ we have,

$$\frac{2N}{10^n} \geq \frac{2}{10^n}\left((n-1)10^{n-1} - \frac{10^{n-1}}{9} + \frac{1}{9}\right) \geq 2\left((n-1) - \frac{1}{9}\right)\frac{1}{10} \geq 2\left(n - 1 - \frac{1}{9}\right) \geq n + \log n.$$

And using $N \leq L + M$ we have

$$\log N < n + \log n,$$

So,

$$\log N < n + \log n < 2N/10^n.$$

We obtain, $|D(c, \alpha, \alpha + 10^{-k}, N)| = O\left(\frac{10^{-k}}{\log N}\right)$.

Step 2: Let $h \in \mathbb{N}$ and $\gamma \in [0, 1)$, $\gamma = 0.\gamma_1 \ldots \gamma_h = \sum_{i=1}^{h} \gamma_i 10^{-i}$. We show

$$|D(c, 0, \gamma, N)| = O\left(\frac{1}{\log N}\right),$$

whose constant does not depend on γ. This is the desired result just for intervals whose extremes γ are numbers with finite decimal expansion. Let $k \in \{1, \ldots, h\}$ and $j \in \{0, \ldots, \gamma_k\}$. We define

$$\lambda_{k,j} = \sum_{i=1}^{k-1} \gamma_i 10^{-i} + j 10^{-k} = 0.\gamma_1 \ldots \gamma_{k-1}\, j.$$

Then, it holds that

$$\lambda_{k,j+1} = \sum_{i=1}^{k-1} \gamma_i 10^{-i} + (j+1)10^{-k} = \lambda_{k,j} + 10^{-k}.$$

We observe

$$\begin{array}{llll}
\lambda_{1,0} = 0\,, & \lambda_{1,1} = 0.1\,, & \ldots & \lambda_{1,\gamma_1} = 0.\gamma_1 \\
\lambda_{2,0} = 0.\gamma_1\,, & \lambda_{2,1} = 0.\gamma_1 1\,, & \ldots & \lambda_{2,\gamma_2} = 0.\gamma_1\ gamma_2 \\
\vdots & \vdots & & \vdots \\
\lambda_{h,0} = 0.\gamma_1 \ldots \gamma_{h-1}\,, & \lambda_{h,1} = \gamma_1 \ldots \gamma_{h-1} 1\,, & \ldots & \lambda_{h,\gamma_h} = 0.\gamma_1 \ldots \gamma_h = \gamma.
\end{array}$$

Then,

$$\begin{aligned}
\sum_{k=1}^{h} \sum_{j=0}^{\gamma_k - 1} D(c, \lambda_{k,j}, \lambda_{k,j+1}, N) &= \frac{1}{N} \sum_{n=1}^{N} \sum_{k=1}^{h} \sum_{j=0}^{\gamma_k - 1} \chi_{[\lambda_{k,j}, \lambda_{k,j+1})}(x_n) - \sum_{k=1}^{h} \sum_{j=0}^{\gamma_k - 1} 10^{-k} \\
&= \frac{1}{N} \sum_{n=1}^{N} \chi_{[0,\gamma)}(x_n) - \sum_{k=1}^{h} \gamma_k 10^{-k} \\
&= \frac{1}{N} \sum_{n=1}^{N} \chi_{[0,\gamma)}(x_n) - \gamma \\
&= D(c, 0, \gamma, N).
\end{aligned}$$

Hence,

$$\begin{aligned}
|D(c,0,\gamma,N)| &\leq \sum_{k=1}^{h}\sum_{j=0}^{\gamma_k-1} |D(c,\lambda_{k,j},\lambda_{k,j+1},N)| \\
&= \sum_{k=1}^{h}\sum_{j=0}^{\gamma_k-1} D(c,\lambda_{k,j},\lambda_{k,j}+10^{-k},N) \\
&= \sum_{k=1}^{h} O\left(\frac{10^{-k}}{\log N}\right) \qquad \text{by step 1, taking } N \text{ sufficiently large} \\
&= O\left(\frac{1}{\log N}\right)
\end{aligned}$$

where the hidden constant in $O\left(\frac{1}{\log N}\right)$ does not depend on γ, nor on h, since

$$\sum_{k=1}^{h} O\left(\frac{10^{-k}}{\log N}\right) \leq O\left(\frac{1}{\log N}\right)\sum_{k=1}^{\infty} 10^{-k} \leq O\left(\frac{1}{\log N}\right).$$

Step 3: Let $\beta \in [0,1)$. We prove that $|D(c,0,\beta,N)| = O\left(\frac{1}{\log N}\right)$. Let $N \in \mathbb{N}$. Let's take $h = [\log(\log N))]$. If β has a finite decimal expansion, then we are in the case of Step 2 and the proof is complete. Otherwise, $\beta = 0.\beta_1\beta_2\ldots\beta_h\beta_{h+1}\ldots$. Let $\alpha,\gamma \in [0,1)$ be such that

$$\begin{aligned}
&\alpha \leq \beta \leq \gamma, \\
&\gamma - \alpha = 10^{-h}, \\
&\alpha 10^h \in \mathbb{N}, \\
&\gamma 10^h \in \mathbb{N}.
\end{aligned}$$

That is to say, $\alpha = 0.\beta_1\ldots\beta_h$ and $\gamma = \alpha + 10^{-h}$. So,

$$\begin{aligned}
D(c,0,\beta,N) &= \frac{1}{N}\sum_{n=1}^{N} \chi_{[0,\beta)}(x_n) - \beta \\
&\leq \frac{1}{N}\sum_{n=1}^{N} \chi_{[0,\beta)}(x_n) - \gamma + \gamma - \alpha = D(c,0,\gamma,N) + 10^{-h}.
\end{aligned}$$

Similarly,

$$D(c,0,\beta,N) \geq \frac{1}{N}\sum_{n=1}^{N} \chi_{[0,\beta)}(x_n) - \alpha + \alpha - \gamma = D(c,0,\alpha,N) - 10^{-h}.$$

Then, $D(c,0,\alpha,N) - 10^{-h} \leq D(c,0,\beta,N) \leq D(c,0,\gamma,N) + 10^{-h}$. Therefore,

$$\begin{aligned}
|D(c,0,\beta,N)| &\leq \max\{|D(c,0,\alpha,N) - 10^{-h}|, |D(c,0,\gamma,N) + 10^{-h}|\} \\
&\leq \max\{|D(c,0,\alpha,N)|, |D(c,0,\gamma,N)|\} + 10^{-h} \\
&= O\left(\frac{1}{\log N}\right) + 10^{-h} \qquad \text{by step 2}
\end{aligned}$$

$$= O\left(\frac{1}{\log N}\right).$$

The proof of Theorem 1 is complete. □

6. Theorem 2: Lower Bound

We prove a lower bound for the discrepancy $D(c, N)$. We need to find out how much the number of occurrences of the blocks of equal length can differ. It suffices to find only two *witnessing blocks* one whose occurrences are in excess and the other in defect.

6.1. **Witnessing Blocks.** The statement of Lemma 2 is due to Schiffer [14, Lemma 2]. This is our version of the proof.

Lemma 2. *Let $\alpha \in [0, 1)$. Let B_1 and B_2 be blocks of equal length k. Suppose there exists a constant $C > 0$ such that for infinitelly many $N \in \mathbb{N}$,*

$$|occ(\alpha, B_1, N) - occ(\alpha, B_2, N)| > C\frac{N}{\log(N)}.$$

Then, there exists a constant $K > 0$ such that for infinitely many $N \in \mathbb{N}$,

$$D(N, \alpha) > \frac{K}{\log(N)}.$$

Furthermore, it holds for the constant $K = C/3$.

Proof. Let $B_1 = (b_1 \dots b_k)$, $B_2 = (d_1 \dots d_k)$ and $\alpha = 0.\alpha_1\,\alpha_2\,\alpha_3 \dots \in [0, 1)$. Let

$$\beta_1 = 0.b_1 \dots b_k = \textstyle\sum_{i=1}^k b_i 10^{-i} \quad \text{and} \quad \beta_2 = 0.d_1 \dots d_k = \sum_{i=1}^k d_i 10^{-i}$$
$$I_1 = [\beta_1,\ \beta_1 + 10^{-k}) \subseteq [0, 1) \quad \text{and} \quad I_2 = [\beta_2,\ \beta_2 + 10^{-k}) \subseteq [0, 1).$$

Observe that $\alpha \in I_1$ if and only if $(\alpha_1 \dots \alpha_k) = (b_1, \dots, b_k)$. And $\alpha \in I_2$ if and only if $(\alpha_1 \dots \alpha_k) = (d_1, \dots, d_k)$. Let $N \in \mathbb{N}$ be such that $N + k - 1$ satisfies the hypothesis of the statement,

$$|occ(\alpha, B_1, N + k - 1) - occ(\alpha, B_2, N + k - 1)| > C\frac{N + k - 1}{\log(N + k - 1)}.$$

So,

$$\begin{aligned}
D(\alpha, N) &= D((10^{n-1}\alpha \mod 1, N)_{n\in\mathbb{N}})\\
&= \sup_{0\le a<b<1}\left|\frac{\#\{n \in \{1, \dots, N\} : 10^{n-1}\alpha \mod 1 \in [a, b)\}}{N} - (b - a)\right|\\
&\ge \max_{i=1,2}\left|\frac{\#\{n \in \{1, \dots, N\} : 10^{n-1}\alpha \mod 1 \in I_i\}}{N} - 10^{-k}\right|\\
&= \max_{i=1,2}\left|\frac{occ(\alpha, B_i, N + k - 1)}{N} - 10^{-k}\right| \qquad (5)\\
&\ge \frac{|occ(\alpha, B_1, N + k - 1) - occ(\alpha, B_2, N + k - 1)|}{2N} \qquad (6)\\
&> \frac{C(N + k - 1)}{2N\log(N + k - 1)}\\
&\ge \frac{C}{2\log(N + k - 1)}.
\end{aligned}$$

To see inequality (5) notice that $10^{n-1}\alpha \mod 1 \in I_1$, if and only if $\alpha_n = b_1, \ldots, \alpha_{n+k-1} = b_k$, and this happens exactly when B_1 occurs in $(\alpha_n, \ldots, \alpha_{n+k-1})$. The same holds for B_2. Observe that when $n = 1$, we consider the occurrences in $(\alpha_1 \ldots \alpha_k)$, and when $n = N$, we consider the occurrences in $(\alpha_N, \ldots, \alpha_{N+k-1})$.

To see inequality (6) notice that

$$
\begin{aligned}
&\frac{|occ(\alpha, B_1, N+k-1) - occ(\alpha, B_2, N+k-1)|}{2N} = \\
&\quad = \frac{1}{2}\left|\frac{occ(\alpha, B_1, N+k-1)}{N} - 10^{-k} \ - \left(\frac{occ(\alpha, B_2, N+k-1)}{N} - 10^{-k}\right)\right| \\
&\quad \leq \frac{1}{2}\left(\left|\frac{occ(\alpha, B_1, N+k-1)}{N} - 10^{-k}\right| \qquad + \left|\frac{occ(\alpha, B_2, N+k-1)}{N} - 10^{-k}\right|\right) \\
&\quad \leq \frac{1}{2} 2 \max_{i=1,2}\left|\frac{occ(\alpha, B_i, N+k-1)}{N} - 10^{-k}\right|.
\end{aligned}
$$

We obtained that $D(\alpha, N) \geq C/(1\log(N+k-1))$. To finish the proof we can take $K = C/3$ because, since k is fixed, for sufficiently large N,

$$\frac{C}{2\log(N+k-1)} \geq \frac{C}{3\log(N)}.$$

□

6.2. **Proof of the Theorem 2.** We use Lemma 2. Let B_1 and B_2 be of equal length $k \geq 2$, $B_2 = (0 \ldots 0)$ the block of all zeros, and $B_1 = (1\,1 * \ldots *)$, where the asterisk $*$ represents any digit between 0 and 9.

Example. Let $k = 2$, $B_1 = (1\,1)$ and $B_2 = (0\,0)$, Let's see that B_1 has occurrences in excess and B_2 in defect. Notice that B_1 has overlapping occurrences in the expansion of c bit B_2 does not, because B_2 is the block of all zeros and there aren't terms t_i beginning with a 0. For the non-overlapping occurrences of each block we must count. Let's count of the non-overlapping occurrences of B_1 and B_2 in $(1 \ldots 999) = (s_1\, s_2\, s_3)$; that is, we want to calculate for i =1.2,

$$\sum_{\ell=1}^{3} occ_{no}(B_i, s_\ell).$$

We count up to $\ell = 3$, but the procedure is similar for every ℓ.

Case of B_1*:*

- $occ_{no}(B_1, s_1) = 0$, because $k = 2 > 1$.
- $occ_{no}(B_1, s_2) = 1$, because the only occurrence of B_1 in s_2 is $(1\,1)$.
- $occ_{no}(B_1, s_3) = 19$, because there are ten occurrences of the form $(1\,1\,*)$ and nine of the form $(*\,1\,1)$.

Thus, $\displaystyle\sum_{\ell=1}^{3} occ_{no}(B_1, s_\ell) = 20$.

Case of B_2*:*

- $occ_{no}(B_1, s_1) = 0$, because $k > 1$.
- $occ_{no}(B_1, s_2) = 0$, because the term $(0\,0)$ does not appear in the c.

- $occ_{no}(B_1, s_3) = 9$ because there are nine occurrences of the form $(*\,0\,0)$, but none of the form $(0\,0\,*)$.

Thus, $\displaystyle\sum_{\ell=1}^{3} occ_{no}(B_1, s_\ell) = 9$.

□

In order to use Lemma 2, we must see that there exists a constant C such that for infinitely many $N \in \mathbb{N}$,

$$|occ(c, B_1, N) - occ(c, B_2, N)| > C\frac{N}{\log(N)},$$

where B_1 and B_2 are the witnessing blocks. Let $N \in \mathbb{N}$. Observe that $|occ(c, B_1, N) - occ(c, B_2, N)|$ is equal to

$$|occ_{no}(c, B_1, N) + occ_o(c, B_1, N) - (occ_{no}(c, B_2, N) + occ_o(c, B_2, N))|\,.$$

We first calculate $occ_{no}(c, B_1, N) - occ_{no}(c, B_2, N)$. For $\ell \in \mathbb{N}$, we calculate $occ_{no}(B_1, s_\ell) - occ_{no}(B_2, s_\ell)$. Recall in Lemma 1 we define $occ_{no}(B_i, s_\ell)$ as the number of non-overlapping occurrences of B_i in

$$s_\ell = (10^{\ell-1}, \dots, 10^\ell - 1) = (\underbrace{10\dots0}_{\ell \text{ digits}}, \ \dots\ , \underbrace{9\dots9}_{\ell \text{ digits}}).$$

If $\ell < k$: $occ_{no}(B_i, s_\ell) = 0$, for $i = 1, 2$.
If $\ell \geq k$:
$occ_{no}(B_1, s_\ell) = 10^{\ell-k} + (\ell - k) \cdot 9 \cdot 10^{\ell-k-1}$, because the first digit of B_1 is not 0.
$occ_{no}(B_2, s_\ell) = (\ell - k) \cdot 9 \cdot 10^{\ell-k-1}$, because the first digit of B_2 is 0.
Therefore,

$$occ_{no}(B_1, s_\ell) - occ_{no}(B_2, s_\ell) = 10^{\ell-k}, \ \text{ for all } \ell \geq k. \tag{7}$$

Remark. We are using that the first digit of B_2 is zero and the first digit of B_1 is not zero.

Let $v = v(N)$ and $n = n(N)$. As in the proof of Lemma 1, for $i = 1, 2$ it holds that

$$occ_{no}(B_i, c, N) = \sum_{\ell=k}^{n-1} occ_{no}(B_i, s_\ell) + occ_{no}(B_i, v) - O(n)$$

We must calculate $occ_{no}(B_1, v) - occ_{no}(B_2, v)$.
If $n < k$: $occ_{no}(B_i, v) = 0$.
If $n \geq k$: As in the proof of Lemma 1 let $v = v_1 \dots v_n$, and for each $j \in \{0, \dots, n-k\}$, let $a_j = v_1 \dots v_j$. Let $occ_{no}(B, v, j)$ be the amount of numbers of the form

$$y_j = \underbrace{*\dots*}_{j \text{ digits}} \ B \ \underbrace{*\dots*}_{n-k-j \text{ digits}} \quad \text{with } y_j \leq v.$$

Case $j = 0$. We count the amount of numbers of the form

$$y_0 = B_i \ \underbrace{*\cdots*}_{n-k \text{ digits}} \quad \text{con } y_0 \leq v.$$

Since the first digit of B_2 is zero,

$$occ_{no}(B_2, v, 0) = 0.$$

Again by proof of Lemma 1,

$$occ_{no}(B_1, v, 0) \leq 10^{n-k}.$$

Case $1 \leq j \leq n-k$. The amount of numbers of the form y_j with the first j digits less than a_j , then $y_j < v$, and then $occ_{no}(B_1, v, j) = occ_{no}(B_2, v, j)$. If the first j digits of y_j are greater than a_j, then $y_j > v$ so they do not add up to $occ_{no}(B_1, v, j)$ nor to $occ_{no}(B_2, v, j)$. Finally, if the first j digits of y_j are equal to a_j. There, the number of non-overlapping occurrences of B_1 could be different from that of B_2. We analyze this case. Let $\Delta_j(B_i)$ be the amount of numbers of the form

$$y_j = v_1 \dots v_j \; B_i \; \underbrace{* \dots *}_{n-k-j \text{ digits}}, \quad \text{with } y_j \leq v.$$

- If $B_i > v_{j+1} \dots v_{j+k}$: $\Delta_j(B_i) = 0$, then in that case $y_j > v$.
- If $B_i = v_{j+1} \dots v_{j+k}$: $\Delta_j(B_i) = v_{j+k+1} \dots v_n + 1$, then the last $n-k-j$ they can take any value from 0 to $v_{j+k+1} \dots v_n$.
- If $B_i < v_{j+1} \dots v_{j+k}$: $\Delta_j(B_i) = 10^{n-k-j}$, then the last $n-k-j$ can take any value from 0 to $\underbrace{9 \dots 9}_{n-j-k}$.

Therefore, $\Delta_j(B_i) \leq 10^{n-k-j}$, for each $j \in \{1, \dots, n-k\}$. So, for N large enough so that $n \geq k$,

$$\begin{aligned} \text{(8)} \qquad occ_{no}(B_1, v) - occ_{no}(B_2, v) &= occ_{no}(B_1, v, 0) + \sum_{j=1}^{n-k} (occ_{no}(B_1, v, j) - occ_{no}(B_2, v, j)) \\ &= occ_{no}(B_1, v, 0) + \sum_{j=1}^{n-k} (\Delta_j(B_1) - \Delta_j(B_2)). \end{aligned}$$

Remark. Since $B_2 \leq B_1$, then $0 \leq \Delta_j(B_1) \leq \Delta_j(B_2)$. Hence,

$$v_1 \dots v_j \; B_1 \underbrace{* \cdots *}_{\substack{n-k-j \\ \text{digits}}} \leq v \quad \text{implies} \quad v_1 \dots v_j \; : B_2 \underbrace{* \cdots *}_{\substack{n-k-j \\ \text{digits}}} \leq v.$$

Hence,

$$\text{(9)} \qquad 0 \leq \Delta_j(B_1) - \Delta_j(B_2) \leq 10^{n-k-j}.$$

We conclude that for all N large enough so that $n \geq k$,

$$\begin{aligned} \text{(10)} \qquad & occ_{no}(c, B_1, N) - occ_{no}(c, B_2, N) = \\ &= \sum_{\ell=k}^{n-1} occ_{no}(B_1, s_\ell) + occ_{no}(B_1, v) - O(n) - \left(\sum_{ell=k}^{n-1} occ_{no}(B_2, s_\ell) + occ_{no}(B_2, v) - O(n) \right) \\ &= \sum_{\ell=k}^{n-1} occ_{no}(B_1, s_\ell) - occ_{no}(B_2, s_\ell) + occ_{no}(B_1, v) - \; occ_{no}(B_2, v) + O(n) \\ &= \sum_{\ell=k}^{n-1} 10^{\ell-k} + occ_{no}(B_1, v) - occ_{no}(B_2, v) + O(n) \qquad \text{using (7)} \end{aligned}$$

$$= \sum_{\ell=k}^{n-1} 10^{\ell-k} + occ_{no}(B,v,0) + \sum_{j=1}^{n-k} \Delta_j(B_1) - \Delta_j(B_2) + O(n) \qquad \text{using (8)}$$
$$\geq \sum_{\ell=k}^{n-1} 10^{\ell-k} + \sum_{j=1}^{n-k} \Delta_j(B_1) - \Delta_j(B_2) + O(n)$$
$$= \sum_{\ell=k}^{n-1} 10^{\ell-k} - \sum_{j=1}^{n-k} \Delta_j(B_2) - \Delta_j(B_1) + O(n)$$
$$\geq \sum_{\ell=k}^{n-1} 10^{\ell-k} - \sum_{j=1}^{n-k} 10^{n-j-k} + O(n) \qquad \text{using (9).}$$

We now count the overlapping occurrences.

Remark. Since B_2 is the block of all zeros, it has no overlapping occurrences (since no number starts with leading zeros). This explains the choice of B_2.

We count the overlapping occurrences of B_1. It is enough for us to count some of them, enough so as to verify the hypothesis of Lemma 2.

Recall that $B_1 = (1\,1\,\underbrace{*\cdots*}_{k-2 \text{ digits}}) = (b_1 b_2 \dots b_k)$. Then, we define for each $\ell \in \mathbb{N}$,

$$P_\ell = \{m \in \mathbb{N} : m = \underbrace{b_2 b_3 \dots b_k \overbrace{*\cdots*}^{\ell-k \text{ digits}} b_1}_{ell \text{ digits}} = b_2 10^{\ell-1} + b_3 10^{\ell-2} + \dots + b_k 10^{\ell-k+1} + \dots + b_1\}$$

That is, P_ℓ is the set numbers of ℓ digits, whose first $k-1$ digits are $(b_2\, b_3 \dots b_k)$, and its last digit is b_1 .

If $\ell < k$: $\#P_\ell = 0$.

If $\ell \geq k$: $\#P_\ell = 10^{\ell-k}$, then there are $\ell - k$ free digits.

Remark. We are using that $b_2 \neq 0$, otherwise we would be looking for $m \in \mathbb{N}$ that starts with a leading zero.

Observe that for every element of P_ℓ, there is an occurrence overlapping B_1 between two ℓ digit numbers. Then,

$$(11) \qquad occ_o(B_1, s_\ell) \geq 10^{\ell-k}, \quad \forall \ell \geq k$$

where, remember, $occ_o(B_1, s_\ell)$ was the number of overlapping occurrences B_1 in s_ℓ.

Remark. We are using:

- $k \neq 1$, otherwise there would be no occurrences around B_1.
- $b_1 \neq 9$, because if not, it could happen that $m = b_2\, 9 \dots 9$ and then $m+1$ does not have b_2 as its first digit, and therefore, it does not produce an overlapping.

Therefore, for all N large enough so that $n \geq k$,

$$occ(c, B_1, N) - occ(c, B_2, N) =$$
$$= occ_{no}(c, B_1, N) + occ_o(c, B_1, N)) - (occ_{no}(c, B_2, N) + occ_o(c, B_2, N))$$
$$= occ_{no}(c, B_1, N) - occ_{no}(c, B_2, N) + occ_o(c, B_1, N) - \underbrace{occ_o(B_2, c, 1, N)}_{=0}$$

$$\geq \sum_{\ell=k}^{n-1} 10^{\ell-k} - \sum_{j=1}^{n-k} 10^{n-j-k} + O(n) + occ_o(c, B_1, 1, N) \qquad \text{using (10)}$$

$$\geq \sum_{\ell=k}^{n-1} 10^{\ell-k} - \sum_{j=1}^{n-k} 10^{n-j-k} + O(n) + \sum_{l=1}^{n-1} occ_o(B_1, s_\ell)$$

$$\geq \sum_{\ell=k}^{n-1} 10^{\ell-k} - \sum_{j=1}^{n-k} 10^{n-j-k} + O(n) + \sum_{\ell=k}^{n-1} 10^{\ell-k} \qquad \text{using (11)}$$

$$= 2\sum_{\ell=k}^{n-1} 10^{\ell-k} - \sum_{m=k}^{n-1} 10^{m-k} + O(n) \qquad \text{change of variable: } m = n-j$$

$$= \sum_{\ell=k}^{n-1} 10^{\ell-k} + O(n)$$

$$\geq \frac{1}{10^{k+1}} 10^n$$

$$\geq \frac{1}{10^{k+1}} \frac{N}{\log(N)}. \tag{12}$$

To see the last inequality observe that, since n is the number of digits of v,

$$\sum_{j=1}^{n-1} \underbrace{j}_{\substack{\text{Number of}\\ \text{digits of}\\ \text{each number}}} \cdot \underbrace{9 \cdot 10^{j-1}}_{\substack{\text{Quantity}\\ \text{of numbers}\\ \text{in } s_j}} \leq N \leq \sum_{j=1}^{n} j \cdot 9 \cdot 10^{j-1}.$$

Furthermore, for all $r \in \mathbb{N}$, $\sum_{j=1}^{r} j \cdot 9 \cdot 10^{j-1} = 10^r r - \frac{10^r}{9} + \frac{1}{9}$. Then,

$$N \geq \sum_{j=1}^{n-1} j \cdot 9 \cdot 10^{j-1} = 10^{n-1} n - 1 - \frac{10^{n-1}}{9} + \frac{1}{9} > 10^{n-1}\left(n - 1 - \frac{1}{9}\right)$$

Therefore,

$$\log(N) \geq n - 1 + \log\left(n - \frac{10}{9}\right).$$

And

$$\log(N) 10^n \geq \left(n - 1 + \log(n - \frac{10}{9})\right) 10^n \geq \left(10^n n - \frac{10^n}{9} + \frac{1}{9}\right) = \sum_{j=1}^{n} j \cdot 9 \cdot 10^{j-1} \geq N.$$

Finally, we make the constant K explicit. We just proved in (12) that

$$occ(c, B_1, N) - occ(c, B_2, N) \geq C \frac{N}{\log(N)}$$

with $C = \dfrac{1}{10^{k+1}}$. Since since the smallest possible value of k that we can take is $k = 2$, and by Lemma 2 we can take $K = C/3$, we obtain

$$K = \frac{C}{3} = \frac{1}{10^{k+1}3} = \frac{1}{10^3 3}.$$

The statement of Theorem 2 asks the bound for infinitely many N. Since we gave the lower bound for all N sufficiently large so that $n(N) \geq k$, we gave it for for cofinitely many N, a stronger result. The proof of Theorem 2 is complete. □

Acknowledgements

This work was supported by grant UBACYT 20020220100065BA from Universidad de Buenos Aires.

References

[1] V. Becher and O. Carton. Normal numbers and computer science. In V. Berthé and M. Rigo, editors, *Sequences, Groups, and Number Theory*, Trends in Mathematics Series. Birkhäuser/Springer, 2018.
[2] V. Becher and T. Slaman. On the normality of numbers to different bases. *Journal of the London Mathematical Society*, 90(2):472–494, 2014.
[3] Y. Bugeaud. *Distribution modulo one and Diophantine approximation*, volume 193 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2012.
[4] D. G. Champernowne. The construction of decimals normal in the scale of ten. *Journal of the London Mathematical Society*, 8:254–260, 1933.
[5] D.D.Wall. *Normal numbers*. PhD thesis, University of California Berkeley, 1949. Ph.D.Thesis.
[6] K. Fukuyama. The law of the iterated logarithm for discrepancies of $\{\theta \text{ nx}\}$. *Acta Mathematica Hungarica*, 118:155–170, 2008.
[7] I. Gál and L. Gál. The discrepancy of the sequence $(2^n x)$. *Indagationes Mathematicae*, 26:129–143, 1964.
[8] G. H. Hardy and E. M. Wright. *An introduction to the theory of numbers*. Oxford, at the Clarendon Press,, 1954. 3rd ed.
[9] G.H. Hardy. Letter from G.H. Hardy to A.M. Turing, late 1930s. Turing Digital Archive AMT/D/5 -1 ALS, G.H. Hardy. [late 1930s].
[10] N. Korobov. Numbers with bounded quotient and their applications to questions of diophantine approximation. *Izv. Akad. Nauk SSSR Ser. Mat.*, 19:361–380, 1955.
[11] L. Kuipers and H. Niederreiter. *Uniform distribution of sequences*. Pure and Applied Mathematics. Wiley-Interscience [John Wiley & Sons], New York-London-Sydney, 1974.
[12] Y. Nakai and I. Shiokawa. Discrepancy estimates for a class of normal numbers. *Acta Arithmetica*, 62(3):271–284, 1992.
[13] W. Philipp. Limit theorems for lacunary series and uniform distribution mod 1. *Acta Arithmetica*, 26(3):241–251, 1975.
[14] J. Schiffer. Discrepancy of normal numbers. *Acta Arithmetica*, 47(2):175–186, 1986.
[15] W. Schmidt. Irregularities of distribution VII. *Acta Arithmetica*, 21:45–50, 1972.

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