Irrationality Exponent, Hausdorff Dimension and Effectivization

Verónica Becher, Jan Reimann and Theodore A. Slaman

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Abstract

We generalize the classical theorem by Jarník and Besicovitch on the irrationality exponents of real numbers and Hausdorff dimension. Let a be any real number greater than or equal to 2 and let b be any non-negative real less than or equal to $2/a$. We show that there is a Cantor-like set with Hausdorff dimension equal to b such that, with respect to its uniform measure, almost all real numbers have irrationality exponent equal to a. We give an analogous result relating the irrationality exponent and the effective Hausdorff dimension of individual real numbers. We prove that there is a Cantor-like set such that, with respect to its uniform measure, almost all elements in the set have effective Hausdorff dimension equal to b and irrationality exponent equal to a. In each case, we obtain the desired set as a distinguished path in a tree of Cantor sets.

The irrationality exponent a of a real number x reflects how well x can be approximated by rational numbers. Precisely, it is the supremum of the set of real numbers z for which the inequality

$$0 < \left| x - \frac{p}{q} \right| < \frac{1}{q^z}$$

is satisfied by an infinite number of integer pairs (p, q) with $q > 0$. Rational numbers have irrationality exponent equal to 1. An immediate consequence of Dirichlet's Approximation Theorem (see Bugeaud, 2004, Chapter 1) is that every irrational number has irrationality exponent greater than or equal to 2. It follows directly that almost all irrational numbers (with respect to Lebesgue measure) have irrationality exponent equal to 2. For every a greater than 2 or equal to infinity, there is a real number x with irrationality exponent equal to a, (Jarník, 1929).

The sets of real numbers with irrationality exponent a thin as a increases. This is made precise by calculating their dimensions. For a set of real numbers X and a non-negative real number s the s-dimensional Hausdorff measure of X is defined by

$$\inf \left\{ \sum_{j \geq 1} r_j^s : \text{there is a cover of } X \text{ by balls with radii } (r_j : j \geq 1) \right\}.$$

The Hausdorff dimension of a set X is the infimum of the set of non-negative reals s such that the s-dimensional Hausdorff measure of X is zero. By results of Jarník (1929) and Besicovitch

(1934), $2/a$ is the exact Hausdorff dimension of the set of real numbers x such that x has irrationality exponent equal to a.

The theory of computability defines the effective versions of the classical notions. A computable function from non-negative integers to non-negative integers is one which can be effectively calculated by some algorithm. The definition extends to functions from one countable set to another, by fixing enumerations of those sets. A real number x is computable if there is a computable sequence of rational numbers $(r_j)_{j\geq 0}$ such that $|x - r_j| < 2^{-j}$ for $j \geq 0$. A set is computably enumerable if it is the range of a computable function with domain the set of non-negaive integers.

Cai and Hartmanis (1994) considered effectively presented properties which are related to Hausdorff dimension. Lutz (2000) formulated a definition of effective Hausdorff dimension for individual sequences in terms of computable martingales. Reimann and Stephan (2005) reformulated the notion in terms of computably enumerable covers, as follows. Let $\mathbb{W}$ be the set of finite binary sequences (sequences of 0s and 1s), and we write $\mathbb{N}$ for the set of non-negative integers. A set X of real numbers has effective s-dimensional Hausdorff measure zero, if there exists a computably enumerable set $C \subseteq \mathbb{N} \times \mathbb{W}$ such that for every $n \in \mathbb{N}$, $C_n = \{w \in \mathbb{W} : (n, w) \in C\}$ satisfies that for every x in X there is a length ℓ such that the sequence of first ℓ digits in the base-2 expansion of x is in C_n, and

$$\sum_{w \in C_n} 2^{-\mathit{length}(w)s} < 2^{-n}.$$

The effective Hausdorff dimension of a set X if real number is the infimum of the set of non-negative reals s such that the effective s-dimensional Hausdorff measure of X is zero. The effective Hausdorff dimension of an individual real number x is the effective Hausdorff dimension of the singleton set $\{x\}$. Note that the effective Hausdorff dimension of x can be greater than 0: x has effective Hausdorff dimension greater than or equal to s if for all $t < s$, x avoids every effectively presented countable intersection of open sets (namely, every effective G_δ set) of t-dimensional Hausdorff measure zero. The effective notion reflects the classical one in that the set $\{x : x \text{ has effective Hausdorff dimension equal to } t\}$ has Hausdorff dimension t.

The notion of effective Hausdorff dimension can also be defined in terms of computable approximation. Intuitively, the Kolmogorov complexity of a finite sequence is the length of the shortest computer program that outputs that sequence. Precisely, consider a computable function h from finite binary sequences to finite binary sequences such that the domain of h is an antichain. Define the h-complexity of τ to be the length of the shortest σ such that $h(\sigma) = \tau$. There is a universal computable function u with the property that for every such h there is a constant c such that for all τ, the h-complexity of τ is less than the u-complexity of τ plus c. Fix a universal u and define the prefix-free Kolmogorov complexity of τ be its u-complexity. In these terms, the effective Hausdorff dimension of a real number x is the infimum of the set of t such that there is a c for which there are infinitely many ℓ such that the prefix-free Kolmogorov complexity of the first ℓ digits in the base-2 expansion of x is less than $t \cdot \ell - c$. See Downey and Hirschfeldt (2010) for a thorough presentation.

Effective Hausdorff dimension was introduced by Lutz (2000) to add computability to the notion of Hausdorff dimension, in the same way that the theory of algorithmic randomness adds computability to Lebesgue measure. But, we could also view the effective Hausdorff dimension of a real number x as a counterpart of its irrationality exponent. Where the irrationality exponent of x reflects how well it can be approximated by rational numbers, the effective Hausdorff dimension of a real x reflects how well it can be approximated by computable numbers. The connection is more than an analogy.

Except for rational numbers all real numbers have irrationality exponent greater than or equal to 2. This means that for each irrational number x, the supremum of the set $\{z :$ there are infinitely many rationals p/q such that $|x - p/q| < 1/q^z\}$ is greater than or equal to 2. On the other hand, most real numbers have effective Hausdorff dimension 1 and are algorithmically random, which means that the initial segments of their expansions can not be described by concise algorithms. Thus, for any such x, all rationals p/q provide at most the first $2\log(q)$ digits of the base-2 expansion of x (take p and q integers, such that such that $0 < p < q$, and describe each of them with $\log q$ digits.). Consequently, for a rational p/q is impossible that $|x - p/q|$ be much less than $1/q^2$. It follows that for most real numbers the irrationality exponent is just equal to 2. In case x is a Liouville number x, its irrationality exponent is infinite, so for every n there is a rational p/q such that $|x - p/q| < 1/q^n$. Thus, $2\log(q)$ digts can describe the first $n\log(q)$ bits of x. Therefore, each Liouville number has effective dimension 0 (Staiger, 2002).

In general, if x has irrationality exponent a then the effective Hausdorff dimension of x is less than or equal to $2/a$. By inspection of the Jarník-Besicovich proofs, $2/a$ is also the effective Hausdorff dimension of the set of real numbers with irrationality exponent equal to a.

In this note we prove the following results.

Theorem 1. *Let a be a real number greater than or equal to 2. For every real number $b \in [0, 2/a]$ there is a Cantor-like set E with Hausdorff dimension equal to b such that, for the uniform measure on E, almost all real numbers have irrationality exponent equal to a.*

Theorem 2. *Let a and b be real numbers such that $a \geq 2$ and $b \in [0, 2/a]$. There is a Cantor-like set E such that, for the uniform measure on E, almost all real numbers in E have irrationality exponent equal to a and effective Hausdorff dimension equal to b.*

A classic result due to Besicovitch (1952) ensures that, for any real number $s \geq 0$, any closed subset of the real number of infinite s-dimensional Hausdorff measure has a subset of finite, non-zero s-dimensional Hausdorff measure. Therefore, it follows directly from this theorem that the set of reals of irrationality exponent greater than or equal to a has a subset of Hausdorff dimension b, for any $0 \leq b \leq 2/a$. However, the proof, which uses binary net measures (see for example Falconer, 1986), does not preserve the Cantor set structure of Jarník's fractal, and hence it also does not provide a nice measure concentrated on a set of reals of irrationality exponent a. Both properties, on the other hand, are features of our construction.

1 Jarník's Fractal and Its Variations

Let a be a real number greater than 2. As mentioned earlier, Jarník (1929) and Besicovitch (1934) independently established that the set of real numbers with irrationality exponent greater than or equal to a has Hausdorff dimension $2/a$. Jarník exhibited a Cantor-like set $E^J(a)$ such that every element of $E^J(a)$ has irrationality exponent greater than or equal to a and such that, for every d greater than a, the set of real numbers with irrationality exponent d is null for the uniform measure μ_J on $E^J(a)$. The latter condition followed by application of the mass distribution principle on $E^J(a)$:

Lemma 3 (Mass Distribution Principle, cf. Falconer, 2003)**.** *Let μ be a measure on E, a subset of the real numbers, and let a be a positive real number. If $\mu(E) > 0$ and there are positive constants c and δ such that for every interval I with $|I| < \delta$, $\mu(I) < c|I|^a$, then the Hausdorff dimension of E is greater than or equal to a.*

For b given so that $0 \leq b < 2/a$, we will define a subset E of $E^J(a)$ with dimension b. E will also be a Cantor-like set and have its own uniform measure μ. Using μ_J as a guide, we will ensure that for any d greater than a the set of real numbers with irrationality exponent d is null with respect to μ. Further, we shall arrange that μ has the mass distribution property for exponent b.

We fix some notation to be applied in the course of our eventual construction. For a positive integer p,

$$G_p(a) = \left\{ x \in \left(\frac{1}{p^a}, 1 - \frac{1}{p^a} \right) : \exists q \in \mathbb{N}, \left| \frac{q}{p} - x \right| \leq \frac{1}{p^a} \right\}.$$

For M a sufficiently large positive integer, and p_1 and p_2 primes such that $M < p_1 < p_2 < 2M$, the sets $G_{p_1}(a)$ and $G_{p_2}(a)$ are disjoint. In fact, the distance between any point in $G_{p_1}(a)$ and any point in $G_{p_2}(a)$ is greater than or equal to

$$\frac{1}{4M^2} - \frac{2}{M^a} \geq \frac{1}{8M^2}.$$

For such M, the set

$$K_M(a) = \bigcup_{\substack{p \text{ prime} \\ M<p<2M}} G_p(a)$$

is the disjoint union of the intervals composing the sets $G_p(a)$. So $K_M(a)$ is made up of intervals of length less than or equal to $2/M^a$ which are separated by gaps of length at least $1/(8M^2)$.

We obtain Jarník's fractal by choosing a sequence $\vec{m} = (m_i : i \geq 1)$, which is sufficiently fast growing in a sense to be determined below. We let $E^J_0(\vec{m}, a) = [0, 1]$ and for $k = 1, 2, \ldots$ let $E^J_k(\vec{m}, a)$ be the union of those intervals of $K_{m_k}(a)$ that are completely contained in $E^J_{k-1}(\vec{m}, a)$. By discarding a negligible number of intervals, we arrange that all intervals from $E^J_{k-1}(\vec{m}, a)$ are split into the same number of intervals in $E^J_k(\vec{m}, a)$. Let i_k be the number of intervals from $E^J_k(\vec{m}, a)$ which are contained in a single interval of $E^J_{k-1}(\vec{m}, a)$. Let $E^J(\vec{m}, a) = \cap_{k\geq 1} E^J_k(\vec{m}, a)$. We define a mass distribution $\mu_J(\vec{m}, a)$ on $E^J(\vec{m}, a)$ by assigning a mass of $1/(i_1 \times \ldots \times i_k)$ to each of the $i_1 \times \ldots \times i_k$ many k-level intervals in $E^J_k(\vec{m}, a)$. Then $\mu_J(\vec{m}, a)$ has the mass distribution property for exponent $2/a$, (see Falconer, 2003, Chapter 10).

There are two ways by which $E^J(\vec{m}, a)$ can be thinned to a subset of lower Hausdorff dimension. First, we could use fewer numbers between m_k and $2m_k$ when we define $E^J_k(\vec{m}, a)$. However, even if we choose only one denominator at each level, the resulting set has dimension $1/a$ (see Falconer, 2003, Example 4.7). To obtain a dimension smaller than $1/a$, not only do we choose just one denominator at each level, but we also choose only the intervals centered on a uniformly spaced subset of the rational numbers with that denominator.

There is a further variation on Jarník's construction and the above thinned version of it which allows for approximating a and b. To express it we introduce the following definition.

Definition 4. The sequences of real numbers $\vec{a}$ and $\vec{b}$ are *appropriate* when $\vec{a} = (a_k : k \in \mathbb{N})$ is non-decreasing with limit a greater than or equal to 2 and $\vec{b} = (b_k : k \in \mathbb{N})$ is strictly increasing with limit b less than or equal to $2/a$ such that if $1/a < b$ then $1/a < b_1$.

For appropriate $\vec{a}$ and $\vec{b}$, we can modify Jarník's fractal to accomodate the specification of a in the limit by substituting a_k in place of a in the definition of E^J_k. That is, we let $E^J_k(\vec{m}, \vec{a})$ be the collection of intervals of $K_{m_k}(a_k)$ which are completely contained in $E^J_{k-1}(\vec{m}, \vec{a})$, adjusted by removing intervals so that every interval in $E^J_{k-1}(\vec{m}, \vec{a})$ contains the same number of intervals in $E^J_k(\vec{m}, \vec{a})$. By construction, the intervals in $E^J_k(\vec{m}, \vec{a})$ are of the form $\left[\frac{q}{p} - \frac{1}{p^{a_k}}, \frac{q}{p} + \frac{1}{p^{a_k}}\right]$. It follows that every real number in $E^J(\vec{m}, \vec{a})$ has irrationality exponent greater than or equal to a. Further, when the sequence m grows sufficiently quickly, the uniform measure $\mu_J(\vec{m}, \vec{a})$ on $E^J(\vec{m}, \vec{a})$ has the mass distribution property for exponent $2/a$. It follows that $\mu_J(\vec{m}, \vec{a})$-almost every real number has irrationality exponent exactly equal to a. Similarly, we can modify the way that we thin $E^J(\vec{m}, \vec{a})$ to reduce dimension from $2/a$ to b. The construction is not sensitive on this point and using b_k to determine how to thin at step k results in a fractal of dimension b.

1.1 Irrationality exponent a, Hausdorff Dimension b and $0 < b \leq 1/a$

Definition 5 (Family of fractals $\mathcal{E}(\vec{q}, \vec{m}, \vec{a})$)**.** Let $\vec{m}$ be an increasing sequence of positive integers; let $\vec{q}$ be a sequence of integers; let $\vec{h}$ be a sequence of integers such that for each k, $h_k \in [0, q_k)$; and let $\vec{a}$ be a non-decreasing sequence of real numbers greater than or equal to 2 with limit a.

- Let $\mathcal{E}_1(\vec{h}, \vec{q}, \vec{m}, \vec{a})$ be $[0, 1]$.
- Given $\mathcal{E}_{k-1}(\vec{h}, \vec{q}, \vec{m}, \vec{a})$, let $\mathcal{E}_k(\vec{h}, \vec{q}, \vec{m}, \vec{a})$ be the collection of intervals in $G_{m_k}(a_k)$ which are completely contained in intervals from $\mathcal{E}_{k-1}(\vec{h}, \vec{q}, \vec{m}, \vec{a})$ and which are of the form $\left[\frac{r}{m_k} - \frac{1}{m_k^{a_k}}, \frac{r}{m_k} + \frac{1}{m_k^{a_k}}\right]$ such that $r \equiv h_k \mod q_k$. As usual, discard a negligible number of intervals so that each interval in $\mathcal{E}_{k-1}(\vec{h}, \vec{q}, \vec{m}, \vec{a})$ has the same number of subintervals in $\mathcal{E}_k(\vec{h}, \vec{q}, \vec{m}, \vec{a})$. Further, ensure that this number of subintervals is independent of $\vec{h}$.

Let $\mathcal{E}(\vec{q}, \vec{m}, \vec{a})$ be the family of fractals obtained by considering all possible sequences $\vec{h}$.

By construction, if $\vec{h}$ and $\vec{g}$ have the same first k values, then for all $j \leq k$, $\mathcal{E}_j(\vec{h}, \vec{q}, \vec{m}, \vec{a})$ is equal to $\mathcal{E}_j(\vec{g}, \vec{q}, \vec{m}, \vec{a})$. So, $\mathcal{E}(\vec{q}, \vec{m}, \vec{a})$ is actually a finitely-branching tree of fractals.

Lemma 6. *Suppose that $\vec{a}$ and $\vec{b}$ are appropriate sequences of reals with limits a and b such that $a \geq 2$ and $b \leq 1/a$. There is a function f, computable from $\vec{a}$ and $\vec{b}$, such that for any sequence of integers $\vec{m} = (m_k : k \in \mathbb{N})$ for which for all k, $m_{k+1} \geq f(k, m_k)$, there is a sequence of integers $\vec{q}$, such that for all $E \in \mathcal{E}(\vec{q}, \vec{m}, \vec{a})$ and for μ the uniform measure on E, the following conditions hold.*

- *For all k greater than 2, all intervals I such that $|I| \leq \frac{q_{k-1}}{m_{k-1}} - \frac{2}{m_{k-1}^a}$,*

$$\mu(I) < |I|^{b_k}.$$

- *For all integers k,*

$$m_k^{a_k b_{k+1}-1} \leq \frac{1}{q_k} \leq m_k^{a_k b_{k+2}-1}.$$

Further, we can compute q_k from $(a_1, \dots, a_k)$, $(b_1, \dots, b_{k+2})$ and $(m_1, \dots, m_k)$.

Proof. We consider $E \in \mathcal{E}(\vec{q}, \vec{m}, \vec{a})$ and μ defined on E as above from $\vec{a}$, an increasing sequence $\vec{m} = (m_k : k \geq 1)$ and $\vec{q} = (q_k : k \geq 1)$. For a given interval I, we estimate $\mu(I)$ and we deduce a sufficient growth rate on $\vec{m}$ and appropriate values for $\vec{q}$ in terms of $\vec{m}$ so as to to ensure the desired inequality $\mu(I) < |I|^{b_k}$, for I as specified. The existence of f follows by observing that these functions are computable from $\vec{a}$ and $\vec{b}$.

We follow a modified version of the proof of Jarník's Theorem as presented in Falconer (2003). We take m_1 to be larger than 3×2^a and sufficiently large so that $1/m_1 > 2m_1^{-a}$.

We let $E_0 = [0, 1]$ and for $k \geq 1$ let E_k consist of those intervals of $G_{m_k}(a_k)$ that are completely contained in E_{k-1} and are of the form $\left[\frac{r}{m_k} - \frac{1}{m_k^{a_k}}, \frac{r}{m_k} + \frac{1}{m_k^{a_k}}\right]$ such that $r \equiv h_k \mod q_k$. Thus, the intervals of E_k are of length $2/m_k^{a_k}$ and are separated by gaps of length at least

$$g_k = \frac{q_k}{m_k} - \frac{2}{m_k^{a_k}}.$$

Let i_k be the number of intervals of E_k contained in a single interval of E_{k-1}. By construction $i_1 = m_1/q_1$ and for every $k > 1$,

$$i_k \geq \frac{1}{2} \frac{2}{(m_{k-1})^{a_{k-1}}} \frac{1}{q_k} m_k = \frac{m_k}{(m_{k-1})^{a_{k-1}} q_k}, \tag{1}$$

which represents half of the product of the length of an interval in E_{k-1} and the number of intervals with centers p/m_k, where p is an integer with fixed residue modulo q_k. This estimate applies provided that q_k is less than m_k and m_k is sufficiently large with respect to the value of m_{k-1}.

Now, we suppose that S is a subinterval of $[0, 1]$ of length $|S| \leq g_1$ and we estimate $\mu(S)$. Let k be the integer such that $g_k \leq |S| < g_{k-1}$. The number of k-level intervals that intersect S is

- at most i_k, since S intersects at most one $(k-1)$-level interval,
- at most $2 + |S|m_k/q_k \leq 4|S|m_k/q_k$, by an estimate similar to that for the lower bound on i_k.

Each k-level interval has measure $1/(i_1 \times \ldots \times i_k)$. We have that

$$\begin{aligned} \mu(S) &\leq \frac{\min(4|S|m_k/q_k, i_k)}{i_1 \times \ldots \times i_k} \qquad (2) \\ &\leq \frac{(4|S|m_k/q_k)^{b_k} \ i_k^{1-b_k}}{i_1 \times \ldots \times i_k} \\ &= \frac{4^{b_k} m_k^{b_k}}{(i_1 \times \ldots \times i_{k-1}) \ i_k^{b_k} q_k^{b_k}} |S|^{b_k} \end{aligned}$$

$$
\begin{aligned}
&\leq 1\frac{m_1^{a_1} q_2}{m_2}\ \frac{m_2^{a_2} q_3}{m_3}\cdots\frac{m_{k-2}^{a_{k-2}} q_{k-1}}{m_{k-1}}\ \frac{4^{b_k} m_k^{b_k}}{i_k^{b_k} q_k^{b_k}}\ |S|^{b_k} \\
&\leq \frac{m_1^{a_1} q_2}{m_2}\ \frac{m_2^{a_2} q_3}{m_3}\cdots\frac{m_{k-2}^{a_{k-2}} q_{k-1}}{m_{k-1}}\ \left(\frac{m_{k-1}^{a_{k-1}} q_k}{m_k}\right)^{b_k} \frac{m_k^{b_k}}{q_k^{b_k}} 4^{b_k}\ |S|^{b_k} \\
&= (q_2 \cdots q_{k-1})\big(m_1^{a_1} m_2^{a_2-1} \ldots m_{k-2}^{a_{k-2}-1}\big) 4^{b_k} m_{k-1}^{a_{k-1}b_k-1} |S|^{b_k}. \qquad (3)
\end{aligned}
$$

We want to ensure that for all such S, $\mu(S) < |S|^{b_k}$, from which we may infer that $\mu(S) < |S|^s$ for all s in $[0, b_k]$. Thus, it suffices to show that there is a suitable growth function f for the sequence $\vec{m}$, and there is a suitable sequence of values for $\vec{q}$ such that

$$
(q_2 \cdots q_{k-2} q_{k-1})\big(m_1^{a_1} m_2^{a_2-1} \ldots m_{k-2}^{a_{k-2}-1}\big) 4^{b_k} m_{k-1}^{a_{k-1}b_k-1} < 1.
$$

Equivalently,

$$
(q_2 \cdots q_{k-2})\big(m_1^{a_1} m_2^{a_2-1} \ldots m_{k-2}^{a_{k-2}-1}\big) 4^{b_k} m_{k-1}^{a_{k-1}b_k-1} < \frac{1}{q_{k-1}}.
$$

If we let C be the term that does not depend on m_{k-1} or on q_{k-1}, we can satisfy the first claim of the Lemma by satisfying the requirement

$$
C m_{k-1}^{a_{k-1}b_k-1} < \frac{1}{q_{k-1}}.
$$

Since C is greater than 1, this requirement on m_{k-1} and q_{k-1} also ensures part of the second claim of the lemma, that $m_{k-1}^{a_{k-1}b_k-1} < 1/q_{k-1}$. Since $b_k < b \leq 1/a \leq 1/a_{k-1}$, by making m_{k-1} sufficiently large and letting q_{k-1} take the largest value such that $C m_{k-1}^{a_{k-1}b_k-1} < 1/q_{k-1}$, we may assume that

$$
C m_{k-1}^{a_{k-1}b_k-1} > \frac{1}{2q_{k-1}}.
$$

Equivalently, we may assume that

$$
2C m_{k-1}^{a_{k-1}b_k-1} > \frac{1}{q_{k-1}}.
$$

The second clause in the second claim of the Lemma is that $1/q_{k-1} < m_{k-1}^{a_{k-1}b_{k+1}-1}$. Since $\vec{b}$ is strictly increasing, by choosing m_{k-1} to be sufficiently large, we may ensure that

$$
2C < m_{k-1}^{a_{k-1}(b_{k+1}-b_k)}
$$

and so $m_{k-1}^{a_{k-1}b_k-1} < 1/q_{k-1} < m_{k-1}^{a_{k-1}b_{k+1}-1}$, as required. Further, by appropriateness of $\vec{a}$ and $\vec{b}$, b_k is less than or equal to $1/a_{k-1}$, the value of $1/q_{k-1}$ can be made arbitrarily small by choosing m_{k-1} to be sufficiently large, so the above estimates apply.

Note: the labels (1), (2) and (3) will be used in the proof of Lemma 9. □

Lemma 7. *Suppose that $\vec{a}$ and $\vec{b}$ are appropriate sequences with limits a and b such that $a \geq 2$ and $0 \leq b \leq 1/a$. Let f be as in Lemma 6, $\vec{m}$ be such that for all k, $m_{k+1} \geq f(k, m_k)$, and $\vec{q}$ be defined from these sequences as in Lemma 6. For every E in $\mathcal{E}(\vec{q}, \vec{m}, \vec{a})$, E has Hausdorff dimension b.*

Proof. Let E be an element of $\mathcal{E}(\vec{q}, \vec{m}, \vec{a})$ and let μ be the uniform measure on E. By Lemma 6, for every $\beta < b$, for all sufficiently small intervals I, $\mu(I) < |I|^\beta$. Consequently, by application of the Mass Distribution Principle, the Hausdorff dimension of E is greater than or equal to b. Next, consider a real number β greater than b. For each $k \geq 1$, there are at most m_k/q_k intervals in E_k, each with radius $1/m_k^{a_k}$. Then

$$\sum_{j \leq m_k/q_k} \left(\frac{2}{m_k^{a_k}}\right)^\beta \leq m_k \cdot \frac{2}{q_k} \cdot m_k^{-a_k\beta} \leq m_k \cdot m_k^{a_k b_{k+2}-1} \cdot m_k^{-a_k\beta} \leq \frac{2}{m_k^{\beta - b_{k+2}}} \leq \frac{2}{m_k^{\beta-b}},$$

which goes to 0 as k goes to infinity. It follows that E has Hausdorff dimension less than or equal to b, as required to complete the verification of the Lemma. □

1.2 Irrationality exponent a, Hausdorff Dimension b and $1/a \leq b \leq 2/a$

Definition 8 (Family of fractals $\mathcal{F}(\vec{q}, \vec{m}, \vec{a})$)**.** Let $\vec{m}$ be an increasing sequence of positive integers; let $\vec{q}$ be a sequence of integers such that for each k, q_k is between 1 and the cardinality of the set of prime numbers in $[m_k, 2m_k)$; let $\vec{H}$ be a sequence of subsets of the set of prime numbers in $[m_k, 2m_k)$ such that for each k, H_k has exactly q_k elements; and let $\vec{a}$ be a non-decreasing sequence of real numbers greater than or equal to 2 with limit a.

- Let $\mathcal{F}_1(\vec{H}, \vec{q}, \vec{m}, \vec{a})$ be $[0, 1]$.
- Given $\mathcal{F}_{k-1}(\vec{H}, \vec{q}, \vec{m}, \vec{a})$, let $\mathcal{F}_k(\vec{H}, \vec{q}, \vec{m}, \vec{a})$ be the collection of intervals in

  $$\bigcup_{p \in H_k} G_p(a_k),$$

  which are completely contained in intervals from $\mathcal{F}_{k-1}(\vec{H}, \vec{q}, \vec{m}, \vec{a})$.
- Let s denote the function that maps k to the ratio given by q_k, the number of retained denominators, divided by the number possible denominators, which is the number of primes in $[m_k, 2m_k)$. Discard a negligible number of intervals so that each interval in $\mathcal{F}_{k-1}(\vec{H}, \vec{q}, \vec{m}, \vec{a})$ has the same number of subintervals in $\mathcal{F}_k(\vec{H}, \vec{q}, \vec{m}, \vec{a})$, and further, so that this number of subintervals depends only on $k, \vec{q}, \vec{m}, \vec{a}$ and not on $\vec{H}$.

Let $\mathcal{F}(\vec{q}, \vec{m}, \vec{a})$ be the family of fractals obtained by considering all possible sequences $\vec{H}$.

Lemma 9. *Suppose that $\vec{a}$ and $\vec{b}$ are appropriate sequences with limits a and b such that $a > 2$ and $1/a \leq b \leq 2/a$. There is a function f, computable from $\vec{a}$ and $\vec{b}$, such that for any sequence $\vec{m} = (m_k : k \in \mathbb{N})$ for which for all k, $m_{k+1} \geq f(k, m_k)$, there is a sequence $\vec{q}$, such that for all $E \in \mathcal{F}(\vec{q}, \vec{m}, \vec{a})$ and for μ the uniform measure on E, the following conditions hold:*

- *For all $k > 2$, all intervals I such that* $|I| \leq \dfrac{1}{4m_{k-1}^2} - \dfrac{2}{m_{k-1}^a}$,

  $$\mu(I) < |I|^{b_k}.$$

- *For all integers k,*

  $$\log(m_k) m_k^{a_k b_{k+1}-2} \leq \frac{1}{q_k} \leq \log(m_k) m_k^{a_k b_{k+2}-2}.$$

Further, we can exhibit such an $\vec{q}$ for which q_k is uniformly computable from $(a_1, \dots, a_k)$, $(b_1, \dots, b_k)$ and $(m_1, \dots, m_{k-1})$.

Proof. The proof of Lemma 9 has the same structure as the proof of Lemma 6, with fundamental difference as follows. Lemma 6 refers to $\mathcal{E}$, the tree of subfractals of $E^J(\vec{m}, \vec{a})$, where the subfractals are obtained by recursion during which at step k only $1/q_k$ of the m_k-many eligible intervals in $G_{m_k}(a_k)$ are used. Lemma 9 refers to $\mathcal{F}$, the tree of subfractals of $E^J(\vec{m}, \vec{a})$, where the subfractals are obtained by recursion during which at step k for only $1/q_k$ of the eligible denominators m, all of the intervals in $G_m(a_k)$ are used. The set of eligible denominators is the set of prime numbers between m_k and $2m_k$. By choosing m_0 large enough, the prime number theorem implies that this set of eligible denominators has between $m_k/2\log(m_k)$ and $2m_k/\log(m_k)$ many elements and each contributes at least m_k many eligible intervals. Now, we give an abbreviated account to indicate how this difference propagates through the proof.

We consider $E \in \mathcal{F}(\vec{q}, \vec{m}, \vec{a})$ and μ the uniform measure on E. We let i_k be the number of intervals in E_k contained in a single interval of E_{k-1}. Now, we have the following version of Inequality (1),

$$i_k \geq \frac{1}{2} \frac{2}{(2m_{k-1})^{a_{k-1}}} \frac{1}{q_k} \frac{m_k^2}{2\log(m_k)} > \frac{m_k^2}{2^{a+1}\,(m_{k-1})^{a_{k-1}}\ \log(m_k)\, q_k}.$$

The intervals in E_k are separated by gaps of length at least $g_k = 1/4m_k^2 - (2/m_k^{a_k})$, because given two intervals with centers c_1/d_1 and c_2/d_2, the gap between them is

$$\left|\frac{c_1 d_2 - c_2 d_1}{d_1 d_2}\right| - \left(\frac{1}{c^{a_k}} + \frac{1}{d^{a_k}}\right).$$

The numerator in the first fraction is at least 1 and the denominator is no greater than $(2m_k)^2$. The denominators in the second term are at least m_k^{a-k}.

Now we suppose that S is a subinterval of $[0, 1]$ of length $|S| \leq g_1$ and we estimate $\mu(S)$. Let k be the integer such that $g_k \leq |S| < g_{k-1}$. The number of k-level intervals that intersect S is

- at most i_k, since S intersects at most one $(k-1)$-level interval,
- at most $2 + |S| 4m_k^2/(q_k \log(m_k))$ which is less than $8|S|m_k^2/(q_k \log(m_k))$. This is because there are at most $2m_k/\log m_k$ primes between m_k and $2m_k$ and each prime contributes at most $2m_k$ many intervals in $[0, 1]$. And we keep $1/q_k$ of these.

As before, each k-level interval has measure $1/(i_1 \times \ldots \times i_k)$. Then, we have a version of Inequality (2).

$$\mu(S) \leq \frac{\min\left(\frac{8|S|m_k^2}{q_k \log(m_k)}, i_k\right)}{i_1 \times \ldots \times i_k}.$$

Upon manipulation as before, we have a version of Inequality (3).

$$\mu(S) \leq 8^{b_k} 2^{k(a+1)} (q_2 \cdots q_{k-1})\, (\log(m_2) \cdots \log(m_{k-1})\, (m_1^{a_1} m_2^{a_2 - 2} \cdots m_{k-2}^{a_{k-2} - 2}) m_{k-1}^{a_{k-1} b_k - 2} |S|^{b_k}.$$

To ensure that $\mu(S) < |S|^{b_k}$, we must find a suitable growth function f for the sequence $\vec{m}$ so that there is a suitable sequence of values for $\vec{q}$ such that

$$8^{b_k} 2^{k(a+1)} (q_2 \cdots q_{k-1})\, (\log(m_2) \cdots \log(m_{k-1}))\, (m_1^{a_1} m_2^{a_2 - 2} \cdots m_{k-2}^{a_{k-2} - 2}) m_{k-1}^{a_{k-1} b_k - 2} < 1.$$

Equivalently,

$$8^{b_k}2^{k(a+1)}(q_2\cdots q_{k-2})\,(\log(m_2)\cdots\log(m_{k-1}))\,(m_1^{a_1}m_2^{a_2-2}\cdots m_{k-2}^{a_{k-2}-2})m_{k-1}^{a_{k-1}b_k-2} < 1/q_{k-1}.$$

Let C be the term that does not depend on m_{k-1} or q_{k-1}. We can satisfy the first claim of Lemma 9 by satisfying the requirement

$$C\log(m_{k-1})m_{k-1}^{a_{k-1}b_k-2} < 1/q_{k-1}.$$

Since C is greater than 1, this requirement on m_{k-1} and on q_{k-1} also ensures part of the second claim of the lemma, that $\log(m_{k-1})m_{k-1}^{a_{k-1}b_k-2} < 1/q_{k-1}$. Since $b_k \leq 2/a < 2/a_{k-1}$, $a_{k-1}b_k - 2$ is negative. By making m_{k-1} sufficiently large and letting q_{k-1} take the largest value such that $C\log(m_{k-1})m_{k-1}^{a_{k-1}b_k-2} < 1/q_{k-1}$, we may assume that

$$C\log(m_{k-1})m_{k-1}^{a_{k-1}b_k-2} > 1/2q_{k-1}.$$

The second clause in the second claim of the lemma is that $1/q_{k-1} < m_{k-1}^{a_{k-1}b_{k+1}-2}$. Since $\vec{b}$ is strictly increasing, by choosing m_{k-1} to be sufficiently large, we may ensure that

$$2C < \log(m_{k-1})m_{k-1}^{a_{k-1}(b_{k+1}-b_k)}$$

and so $\log(m_{k-1})m_{k-1}^{a_{k-1}b_k-1} < 1/q_{k-1} < \log(m_{k-1})m_{k-1}^{a_{k-1}b_{k+1}-2}$, as required. Further, since b_k is less than or equal to $1/a_{k-1}$, the value of $1/q_{k-1}$ can be made arbitrarily small by choosing m_{k-1} to be sufficiently large, so the above estimates apply. □

Lemma 10. *Suppose that $\vec{a}$ and $\vec{b}$ are appropriate sequences of real numbers with limits a and b such that $a > 2$, $1/a \leq b \leq 2/a$ and $1/a_1 \leq b_1$. Let f be as in Lemma 9, $\vec{m}$ be such that for all k, $m_{k+1} \geq f(k, m_k)$, and $\vec{q}$ be defined from these sequences as in Lemma 9. For every E in $\mathcal{F}(\vec{q},\vec{m},\vec{a})$, E has Hausdorff dimension b.*

Proof. Let E be an element of $\mathcal{F}(\vec{q},\vec{m},\vec{a})$ and let μ be the uniform measure on E. By Lemma 9, for every $\beta < b$, for all sufficiently small intervals I, $\mu(I) < |I|^\beta$. Consequently, by application of the Mass Distribution Principle, the Hausdorff dimension of E is greater than or equal to b. Next, consider a real number β greater than b. For each $k \geq 1$, there are at most $4m_k^2/(\log(m_k)q_k)$ intervals in E_k, each with radius less than or equal to $1/m_k^{a_k}$. Then

$$\begin{aligned}
\sum_{j\leq 4m_k^2/(\log(m_k)q_k)} \left(\frac{2}{m_k^{a_k}}\right)^\beta &\leq \frac{4m_k^2}{\log(m_k)}\cdot\frac{1}{q_k}\cdot 2^\beta m_k^{-a_k\beta} \\
&\leq \frac{4m_k^2}{\log(m_k)}\cdot\log(m_k)m_k^{a_kb_{k+2}-2}\cdot 2^\beta m_k^{-a_k\beta} \\
&\leq \frac{8}{m_k^{a_k(\beta-b_{k+2})}} \\
&\leq \frac{8}{m_k^{\beta-b}},
\end{aligned}$$

which goes to 0 as k goes to infinity. It follows that E has Hausdorff dimension less than or equal to b, as required to complete the verification of the Lemma. □

1.3 Irrationality Exponent 2, Hausdorff Dimension b, and $0 < b \leq 1$

The next case to consider is that in which the desired irrationality exponent is equal to 2 and the desired effective Hausdorff dimension is equal to $b \in (0, 1]$. In fact, we need only consider the case of $b < 1$, since almost every real number with respect to Lebesgue measure has both irrationality exponent equal to 2 and effective Hausdorff dimension equal to 1.

Definition 11 (Family of fractals $\mathcal{G}(\vec{q}, \vec{m})$)**.** Let $\vec{m}$ be an increasing sequence of positive integers such that for all k, m_k divides m_{k-1} evenly; let $\vec{q}$ be a sequence of integers; and let $\vec{h}$ be a sequence of integers such that for each k, $h_k \in [0, q_k)$.

- Let $\mathcal{G}_1(\vec{h}, \vec{q}, \vec{m})$ be $[0, 1]$.
- Given $\mathcal{G}_{k-1}(\vec{h}, \vec{q}, \vec{m})$, let $\mathcal{G}_k(\vec{h}, \vec{q}, \vec{m})$ be the collection of intervals which are completely contained in intervals from $\mathcal{G}_{k-1}(\vec{h}, \vec{q}, \vec{m})$ and which are of the form $\left[\frac{r}{m_k}, \frac{r}{m_k} + \frac{1}{m_k}\right]$ such that $r \equiv h_k \mod q_k$. As usual, discard a negligible number of intervals so that each interval in $\mathcal{G}_{k-1}(\vec{h}, \vec{q}, \vec{m})$ has the same number of subintervals in $\mathcal{G}_k(\vec{h}, \vec{q}, \vec{m})$. Further, ensure that this number of subintervals is independent of $\vec{h}$.

Let $\mathcal{G}(\vec{q}, \vec{m})$ be the family of fractals obtained by considering all possible sequences $\vec{h}$.

In the following Lemma 12 is parallel to Lemmas 6 and 9. Likewise, the next Lemma 13 is parallel to Lemmas 7 and 10. In fact, the estimates here are simpler than the earlier ones. We leave the proofs for the curious reader.

Lemma 12. *Suppose $\vec{b}$ is strictly increasing sequence of real numbers with limit b such that $0 < b < 1$. There is a function f, computable from $\vec{b}$, such that for any sequence $\vec{m} = (m_k : k \in \mathbb{N})$ for which for all k, $m_{k+1} \geq f(k, m_k)$, there is a sequence $\vec{q}$, such that for all $E \in \mathcal{G}(\vec{q}, \vec{m})$ and for μ the uniform measure on E, the following conditions hold.*

- *For all k greater than 2, all intervals I such that $|I| \leq \frac{q_{k-1} - 1}{m_{k-1}}$,*
$$\mu(I) < |I|^{b_k}.$$
- *For all integers k,*
$$2m_k^{2b_{k+1}-1} \leq \frac{1}{q_k} \leq m_k^{2b_{k+2}-1}.$$

Further, we can compute q_k from $(b_1, \ldots, b_{k+2})$ and $(m_1, \ldots, m_k)$.

Lemma 13. *Suppose $\vec{b}$ is strictly increasing sequence of real numbers with limit b such that $0 < b < 1$. Let f be as in Lemma 12, $\vec{m}$ be such that for all k, $m_{k+1} \geq f(k, m_k)$, and $\vec{q}$ be defined from these sequences as in Lemma 12. For every E in $\mathcal{G}(\vec{q}, \vec{m})$, G has Hausdorff dimension b.*

2 Hausdorff Dimension and Irrationality Exponent

Recall the statement of Theorem 1.

Theorem 1. *Let a be a real number greater than or equal to 2. For every real number $b \in [0, 2/a]$ there is a Cantor-like set E with Hausdorff dimension equal to b such that, for the uniform measure on E, almost all real numbers have irrationality exponent equal to a.*

In the proof we will use the following definition.

Definition 14.

$$\begin{aligned} B(d_1, d_2, a^*) &= \bigcup \left\{ \left[\frac{p}{q} - \frac{1}{q^{a^*}}, \frac{p}{q} + \frac{1}{q^{a^*}} \right] : p, q \in \mathbb{N} \text{ and } d_1 \leq q \leq d_2 \right\} \\ B(d, \infty, a^*) &= \bigcup \left\{ \left[\frac{p}{q} - \frac{1}{q^{a^*}}, \frac{p}{q} + \frac{1}{q^{a^*}} \right] : p, q \in \mathbb{N} \text{ and } d \leq q \right\} \end{aligned}$$

Proof of Theorem 1. If $b = 0$ the desired set E is quite trivial: for every a greater than or equal to 2, including $a = \infty$, there is a real number x such that a is the irrationality exponent of x. Let $E = \{x\}$ and note that E has Hausdorff dimension equal to 0 and the uniform measure on E concentrates on real numbers of irrationality exponent a.

Assume that $b > 0$. Let $\vec{a}$ be the constant sequence with values a and let $\vec{b}$ be a strictly increasing sequence of positive rational numbers with limit b. Thus, $\vec{a}$ and $\vec{b}$ are appropriate. The desired set E will be an element of $\mathcal{E}(\vec{q}, \vec{m}, \vec{a})$, $\mathcal{F}(\vec{q}, \vec{m}, \vec{a})$ or $\mathcal{G}(\vec{q}, \vec{m})$, for $\vec{m}$ and $\vec{q}$ constructed according to Lemma 7, 10 or 12, depending on whether $a > 2$ and $b \in (0, 1/a)$, or $a > 2$ and $b \in [1/a, 2/a)$, or $a = 2$ and $b \in (0, 1)$, respectively. Since the first claim of the Theorem follows from Lemmas 7, 10 or 13, we need only check the second claim. We give a full account of the case $a > 2$ and $0 < b \leq 1/a$. We leave it to the reader to note that the same argument applies in the other cases.

Suppose that $b \in (0, 1/a)$ and let f and $\vec{q}$ be the functions obtained in Lemma 6. Let $\vec{m}$ be the sequence defined by letting m_1 be sufficiently large in the sense of Lemma 6 and letting m_k be the least m such that m is greater than $f(k, m_{k-1})$. Let $E^J(\vec{m}, \vec{a})$ be the Jarník-fractal determined from $\vec{m}$ and $\vec{a}$. Let $E^J_k(\vec{m}, \vec{a})$ denote the set of intervals used in the definition of $E^J(\vec{m}, \vec{a})$ at step k. Let g_k denote the minimum gap between two intervals in $E^J_k(\vec{m}, \vec{a})$. We define the sequence k_s, E_{k_s} and d_s by recursion on s. Let $k_1 = 1$, let $E_1 = E^J_1(\vec{m}, \vec{a}) = [0, 1]$, and let d_1 be the least integer d such that $\mu_J(B(d_1, \infty, a + 1/2))$ is less than $\mu_J(E_1)/2^1 = 1/2$. Now, suppose that k_s, E_{k_s}, and d_s are defined so that E_{k_s} is an initial segment of the levels of an element of $\mathcal{E}(\vec{q}, \vec{m}, \vec{a})$ and so that

$$\mu_J\big(B\big(d_s, \infty, a + \frac{1}{2^s}\big) \cap E_{k_s}\big) < \frac{\mu_J(E_{k_s})}{2^s}.$$

Let d_{s+1} be the least d such that

$$\mu_J\big(B\big(d, \infty, a + \frac{1}{2^{s+1}}\big)\big) < \mu_J(E_{k_s})/(4 \cdot 2^{s+1}).$$

Let c be the number of intervals in $B(d_s, d_{s+1}, a + 1/2^s)$. Let k_{s+1} be the least k such that the union of $2c$ many of the intervals in $E^J_k(\vec{m}, \vec{a})$ has measure less than $\mu_J(E_{k_s})/(4 \cdot 2^s)$, and let C be the collection of intervals in $E^J_{k_{s+1}}$ which contain at least one endpoint of an interval in $B(d_s, d_{s+1}, a + 1/2^s)$.

Consider the set F_{s+1} of extensions of the branch in $\mathcal{E}(\vec{q},\vec{m},\vec{a})$ with endpoint E_{k_s} to branches of length k_{s+1}. Each element F_{s+1} specifies a set of intervals S at level k_{s+1}. Distinct elements in F_{s+1} have empty intersection and identical μ_J-measure. At most one-fourth of the elements S in F_{s+1} can be such that

$$\mu_J\big(B\big(d_s,\infty,a+\frac{1}{2^s}\big)\cap S\big)\geq 4\cdot\frac{\mu_J(S)}{2^s}.$$

Similarly, at most one-fourth of the S in F_{s+1} can be such that

$$\mu_J\big(B\big(d_{s+1},\infty,a+\frac{1}{2^{s+1}}\big)\cap S\big)\geq 4\cdot\frac{\mu_J(S)}{4\cdot 2^{s+1}}=\frac{\mu_J(S)}{2^{s+1}},$$

and at most one-fourth of the S in F_{s+1} can have

$$\mu_J(C\cap S)\geq 4\cdot\frac{\mu_J(S)}{4\cdot 2^s}=\frac{\mu_J(S)}{2^s}.$$

Choose one element S of F_{s+1} which does not belong to any of these fourths and define E through its first k_{s+1} levels so as to agree with that element. Note that

$$\mu_J\big(B\big(d_{s+1},\infty,a+\frac{1}{2^{s+1}}\big)\cap E_{k_{s+1}}\big)\leq\frac{\mu_J(E_{k_{s+1}})}{2^{s+1}},$$

which was the induction assumption on E_{k_s}. Further note that at most $4/2^s$ of the intervals in E_{k_s} can be contained in $\cup B(d_s,d_{s+1},a+1/2^s)$ and at most $1/2^s$ of the intervals in E_{k_s} can contain an endpoint of an interval in $B(d_s,d_{s+1},a+1/2^s)$. Thus, the uniform measure μ on any element of $\mathcal{E}(\vec{q},\vec{m},\vec{a})$ extending the branch up to $E_{k_{s+1}}$ assigns $B(d_s,d_{s+1},a+1/2^s)$ measure less than $5/2^s$.

Let E be the set defined as above and for each k, let E_k denote level-k of E, and let μ denote the uniform measure on E. The first claim of the theorem, that the Hausdorff dimension of E is equal to b, follows from Lemma 7. For the second claim, every element of E^J, and hence of E, has irrationality exponent greater than or equal to a. So, it is sufficient to show that for every positive ϵ, there is an s such that $\mu(B(d_s,\infty,a+\epsilon))<\epsilon$. Let s be sufficiently large so that $5/2^{s-1}<\epsilon$. Then, since

$$\begin{aligned}B(d_s,\infty,a+\epsilon)&=\bigcup_{t\geq s}B(d_t,d_{t+1},a+\epsilon)\\&\subseteq\bigcup_{t\geq s}B\big(d_t,d_{t+1},a+\frac{1}{2^s}\big)\\&\subseteq\bigcup_{t\geq s}B\big(d_t,d_{t+1},a+\frac{1}{2^t}\big),\end{aligned}$$

we have

$$\begin{aligned}\mu(B(d_s,\infty,a+\epsilon))&\leq\sum_{t\geq s}\mu\big(B\big(d_t,d_{t+1},a+\frac{1}{2^t}\big)\big)\\&\leq\sum_{t\geq s}\frac{5}{2^t}\\&\leq\frac{5}{2^{s-1}}.\end{aligned}$$

The desired result follows. □

3 Effective Hausdorff Dimension and Irrationality Exponent

Let us recall the statement of Theorem 2.

Theorem 2. *Let a and b be real numbers such that $a \geq 2$ and $b \in [0, 2/a]$. There is a Cantor-like set E such that, for the uniform measure on E, almost all real numbers in E have irrationality exponent equal to a and effective Hausdorff dimension equal to b.*

Proof of Theorem 2. As in our discussion of Theorem 1, we will consider the case $a > 2$ and $b \in [0, 1/a]$ in detail. We leave it to the reader to note that with straightforward modifications the argument applies to the other two cases. For $b = 0$, the argument reduces to finding a singleton set E, so the tree of subfractals $\mathcal{E}$ is just the tree of elements of E^J. For $a > 2$ and $b \in [1/a, 2/a]$ $\mathcal{F}$ replaces $\mathcal{E}$. For $a = 2$ and $b \in (0, 1]$, $\mathcal{G}$ replaces $\mathcal{E}$.

Our proof follows the outline of the proof of Theorem 1. That is, we will produce a version of E^J and E so that the uniform measure μ on E concentrates on real numbers with irrationality exponent a and so that every element of E has effective Hausdorff dimension b. However, since we are not assuming that a and b are computable real numbers, we must work with rational approximations when ensuring the condition on effective Hausdorff dimension. This change to add effectiveness to the representation of dimension leads us to weaken our conclusions elsewhere: we must settle for showing that x has irrationality exponent greater than or equal to a by showing that for every $a^* < a$ there are infinitely many p and q such that $|p/q - x| < 1/q^{a^*}$. A similar modification of Jarník's construction appears in Becher, Bugeaud, and Slaman (2016), for different purpose.

We will construct $\vec{a}$ and $\vec{b}$ so that $\vec{a}$ is non-decreasing with limit a and so that $\vec{b}$ is strictly increasing with limit b. Simultaneously, we will construct $\vec{m}$, $\vec{q}$ and E in $\mathcal{E}(\vec{q}, \vec{m}, \vec{a})$ as in the proof of Theorem 1. In Theorem 1, we began with $E^J(\vec{m}, \vec{a})$ and $\mathcal{E}(\vec{q}, \vec{m}, \vec{a})$. We defined a sequence of integers d_k and an element of $\mathcal{E}(\vec{q}, \vec{m}, \vec{a})$. At step k, we ensured that the set of real numbers with irrationality exponent greater than $a + 1/2^k$ had small measure with respect to μ. It would have been sufficient to ensure the same fact for the set of numbers with irrationality exponent less than $a + \epsilon_k$, provided that ϵ_k was a non-increasing sequence with limit zero, which is how we will proceed now. Thus, we will construct $(\overline{\alpha}_s : s \in \mathbb{N})$ to be a non-increasing sequence with limit a to stand in for $(a + 1/2^s : s \in \mathbb{N})$.

Consider the problem of ensuring that for a non-negative integer k and for every element x of E, the sequence σ consisting of first $\log(m_k)a_k$ digits in the base-2 expansion of x has Kolmogorov complexity less than $\log(m_k)a_k b$. For this, it would be sufficient to exhibit an uniformly computable map taking binary sequences of length less than $\log(m_k)a_k b$ onto the set of intervals in E_k. For large enough m_k, we can use a binary sequence of length $\log(m_k)(b - b_k)a_k$ to describe the first $k - 1$ steps of the definition of E^J and any initial conditions imposed at the beginning of step k. It will then be sufficient to show that this information is enough to compute a map taking binary sequences of length less than $\log(m_k)a_k b_k$ onto the set of intervals in E_k.

We proceed by recursion on s. During s, we specify three integers, ℓ_s, k_s, and d_s. We specify d_s as we did in Theorem 1. We will specify rational numbers α_s, $\overline{\alpha}_s$ and β_s and use them to specify the values of $\vec{a}$, of up to ℓ_s, specify the levels of E^J up to ℓ_s, which means that we also specify sequence of numbers $\vec{m}$ up to ℓ_s, and we specify the values of $\vec{b}$ up to $\ell_s + 2$. This determines the values of $\vec{q}$ up to ℓ_s. Finally, we specify the first k_s many levels of E.

Initialization of the recursion. Let β_0 be a positive rational number less than b. Let ϵ_0 be a rational number such that $\beta_0+\epsilon_0<b$. Let α_0 and $\overline{\alpha}_0$ be positive rational numbers such that $\alpha_0<a<\overline{\alpha}_0$ and $\overline{\alpha}_0-\alpha_0<1$. Let $E_0=E_0^J=[0,1]$; let m_1 be sufficiently large in the sense of Lemma 6 for the constant sequence $\vec{\alpha}$ with value α_0 and the sequence $\vec{\beta}$ with values $\beta_0+(1-1/2^n)\epsilon_0$, respectively; let $E_1^J=G_{m_1}(\alpha_0)$, the collection of intervals centered at rational numbers $p/m_1\in(0,1)$ with diameter $2/m_1^{\alpha_0}$; let $g_1=\dfrac{q_1}{m_1}-\dfrac{2}{m_1^{\alpha}}$, which is the minimum distance between two intervals in E_1^J; and let d_0 be the least integer d such that $2/d^{\alpha_0}$ is less than g_1 and such that

$$\sum_{j\geq d}\frac{j}{(2j)^{\overline{\alpha}_0}}<\frac{1}{2}.$$

Since $\overline{\alpha}_0>2$, d_0 is well-defined. Since we will define E^J so that Lemma 6 applies, this sum is an upper bound on $\mu_J(B(d_0,\infty,\alpha_0))$. So, this choice of d_0 ensures that $\mu_J(B(d_0,\infty,\alpha_0))$ is less than $1/2$.

Let $\ell_0=1$; let the first two values of $\vec{a}$ be equal to those in $\vec{\alpha}$ and the first 3 values of $\vec{b}$ be the same as those in $\vec{\beta}$. Note that these choices determine q_1. Let $k_0=0$ and let $E_1=E_1^J$.

Recursion: stage $s+1$. Now, suppose that our construction is defined through stage s. Following the proof of Theorem 1, we may assume that we have ensured

$$\mu_J(B(d_s,\infty,\alpha_s)\cap E_{k_s})\ <\ \frac{\mu_J(E_{k_s})}{2^s},$$

subject to our satisfying the hypotheses of Lemma 6.

We ensure that stage $s+1$ of the construction of E is uniformly computable from the construction up to step s and parameters set during the initialization of stage $s+1$ by continuing the construction of E^J recursively in these parameters until the definition of $E_{k_{s+1}}$ is evident.

Initialization of stage $s+1$. Let α_{s+1}, $\overline{\alpha}_{s+1}$ be rational numbers such that

$$\alpha_s<\alpha_{s+1}<a<\overline{\alpha}_{s+1}\leq\overline{\alpha}_s$$

and such that

$$|\overline{\alpha}_{s+1}-\alpha_{s+1}|<\frac{1}{s+1}.$$

Let β_{s+1} be a rational number strictly between $\beta_s+\epsilon_s$ and b such that

$$|b-\beta_{s+1}|<\frac{1}{s+1},$$

and let ϵ_{s+1} be a rational number such that

$$\beta_{s+1}+\epsilon_{s+1}<b.$$

Let d_{s+1} be the least integer d such that $2/d^{\overline{\alpha}_{s+1}} < g_{\ell_s}$ such that $2/d^{\overline{\alpha}_{s+1}}$ is less than the minimum gap length $g_{\ell_s} = \dfrac{q_{\ell_s}}{m_{\ell_s}} - \dfrac{2}{m_{\ell_s}^{\overline{\alpha}_s}}$ and

$$\sum_{j \geq d} \frac{j}{(2j)^{\overline{\alpha}_{s+1}}} < \frac{\mu_J(E_{k_s})}{4 \cdot 2^{s+1}}.$$

Since $2/d_{s+1}^{\overline{\alpha}_{s+1}} < g_{\ell_s}$ and we ensure that our construction of E^J satisfies the hypotheses of Lemma 6, this sum is an upper bound on $\mu_J(B_{d_{s+1}}, \infty, \overline{\alpha}_{\ell_{s+1}})$. Thus, we have ensured that

$$\mu_J(B_{d_{s+1}}, \infty, \overline{\alpha}_{\ell_{s+1}}) < \frac{\mu_J(E_{k_s})}{4 \cdot 2^{s+1}},$$

which is analogous to how we chose d_{s+1} in the proof of Theorem 1.

Let m_{ℓ_s+1} be greater than $f(\ell_s, m_{\ell_s})$ and sufficiently large so that a binary sequence of length $\log(m_{\ell_{s+1}})a(b - \beta_{s+1})$ can describe these parameters together with the first s steps of the construction.

Subrecursion: substage ℓ. Proceed by recursion on substages ℓ starting with initial value ℓ_s+1. Suppose that the termination condition for stage $s+1$ was not realized during substage $\ell - 1$. Define a_ℓ to be equal to α_{s+1} and define $b_{\ell+2}$ to be equal $\beta_{s+1} + \epsilon_{s+1}(1 - 1/2^{\ell-\ell_s})$. If $\ell = \ell_s + 1$, then value of m_{ℓ_s+1} was assigned in the previous paragraph. Otherwise, let m_ℓ be larger than $f(\ell, m_{\ell-1})$. Note that f is defined in terms the values of $\vec{q}$ and $\vec{a}$ up to ℓ and that the values of $\vec{q}$ are defined in terms of $\vec{a}$ and $\vec{m}$ up to ℓ and $\vec{b}$ up to $\ell + 2$, all of which have been determined before the evaluation of f.

Termination of the subrecursion. Note that $\mu_J(E_{k_s})$ is determined at the end of step s, since it is equal to the number of level-k_s intervals in E_{k_s} divided by the number of level-(k_s) intervals in E^J. We say that step ℓ satisfies the termination condition for stage $s+1$ when there is a k between ℓ_s and ℓ such that there is an S in the level k of $\mathcal{E}(\vec{q}, \vec{m}, a)$ satisfying the following conditions.

- There is a d^* in $(d_s, d_s + \ell)$ such that the μ_J-measure of the set of intervals in E^J_ℓ which have non-empty intersection with $B(d_s, d^*, \overline{\alpha}_s) \cap S$ is less than

$$4 \cdot \frac{\mu_J(S)}{2^s} - \sum_{j \geq d^*} \frac{j}{(2j)^{\overline{\alpha}_{s+1}}}.$$

- There is a d^* in $(d_{s+1}, d_{s+1} + \ell)$ such that the μ_J-measure of the set of intervals in E^J_ℓ which have non-empty intersection with $B(d_{s+1}, d^*, \overline{\alpha}_{s+1})$ is less than

$$4 \cdot \frac{\mu_J(S)}{2^{s+1}} - \sum_{j \geq d^*} \frac{j}{(2j)^{\overline{\alpha}_{s+1}}}.$$

- The μ_J measure of the union of the set of intervals in S which contain at least one endpoint of an interval in $B(d_s, d_{s+1}, \overline{\alpha}_{s+1})$ is less than

$$\frac{\mu_J(S)}{2^{s+1}}.$$

If the termination condition is realized for ℓ, then we set ℓ_{s+1} equal to ℓ, and set k_{s+1} and $E_{k_{s+1}}$ to have values equal to those in the least pair k and S which satisfy the termination condition.

Verification. We first note that by inspection of our construction, it satisfies the hypotheses of Lemma 6. Next, we observe that for every stage the subrecursion for that stage eventually realizes its termination condition. For the sake of a contradiction, suppose that there is a stage in the main recursion of the construction whose subrecursion never terminates. Then E^J is defined using an eventually constant sequence $\vec{a}$ and an increasing sequence $\vec{b}$. By the same reasoning as in the proof of Theorem 1: first, there must be a k and an S such that the three sets in question have sufficiently small relative measure in S; and second, for any such k and S, for any $\delta > 0$, there is an ℓ such that the measures of the intersections of those three sets with S is approximated to within δ by the measure of their smallest covers using intervals in E^J_ℓ. But, then the termination condition would apply, a contradiction. Thus, both E and E^J are well-defined, as are their uniform measures μ and μ_J.

By the same argument as in the proof of Theorem 1, μ-almost every real number x has exponent of irrationality equal to a. By Lemma 3, the Mass Distribution Principle applied to μ, if B is a subset of the real numbers and the Hausdorff dimension of B is less than b, then $\mu(B) = 0$. So, μ-almost every x has effective Hausdorff dimension greater than or equal to b.

In order to conclude that every element of E has effective Hausdorff dimension less than or equal to b, it remains to show that for every $x \in E$ and for every $n \in \mathbb{N}$, there is an $m > n$ such that the Kolmogorov complexity of the first m digits in the binary expansion of x is less than or equal to $b \cdot m$. Let such x and n be given, and consider a stage $s+1$ such that m_{s+1} is greater than n. Then, the first s many steps of the construction together with the values of $a_{\ell_{s+1}}$, $b_{\ell_{s+1}}$ and m_{s+1} can be effectively described by a sequence of length m_{s+1}. The result of stage $s+1$ of the construction is effectively determined from these parameters; in particular, $E_{\ell_{s+1}}$ is effectively defined from these parameters. By definition, there are at most $m_{\ell_{s+1}}/q_{\ell_{s+1}}$ many intervals in $E_{\ell_{s+1}}$. By Lemma 6,

$$\frac{1}{q_{\ell_{s+1}}} < m_{\ell_{s+1}}^{a_{\ell_{s+1}}(b_{\ell_{s+1}}+(1-2^{\ell_{s+1}-\ell_s})\epsilon_{s+1})-1} < m_{\ell_{s+1}}^{a_{\ell_{s+1}}(b_{\ell_{s+1}}+\epsilon_{s+1})-1},$$

and so

$$\frac{m_{\ell_{s+1}}}{q_{\ell_{s+1}}} < m_{\ell_{s+1}}^{a_{\ell_{s+1}}(b_{\ell_{s+1}}+\epsilon_{s+1})} = 2^{\log(m_{\ell_{s+1}})a_{\ell_{s+1}}(b_{\ell_{s+1}}+\epsilon_{s+1})}.$$

Thus, we can read off a surjection from the set of binary sequences of length

$$\log(m_{\ell_{s+1}})a_{\ell_{s+1}}(b_{\ell_{s+1}}+\epsilon_{s+1})$$

to the set of intervals in $E_{\ell_{s+1}}$. Each interval I in $E_{\ell_{s+1}}$ has length $2/m_{\ell_{s+1}}^{a_{\ell_{s+1}}}$. Computably, each such interval I restricts the first $\log(m_{\ell_{s+1}})a_{\ell_{s+1}}$ digits in the base-2 expansions of its elements to at most two possibilities. Thus, for each x in E, the sequence of the first $\log(m_{\ell_{s+1}})a_{\ell_{s+1}}$ digits in its base-2 expansion can be uniformly computably described using the information encoded by three sequences, one of length $\log(m_{\ell_{s+1}})a_{\ell_{s+1}}(b-(b_{\ell_{s+1}}+\epsilon_{s+1}))$ to describe the construction up to stage s, one of length $\log(m_{\ell_{s+1}})a_{\ell_{s+1}}(b_{\ell_{s+1}}+\epsilon_{s+1})$ to describe the interval within $E_{\ell_{s+1}}$ that contains x, and one of length 1 to describe which of the two possibilities within that interval apply to x. By the choice of $m_{\ell_{s+1}}$, this sum is less than or equal to $\log(m_{\ell_{s+1}})a_{\ell_{s+1}}b$, as required. □

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Verónica Becher
Departamento de Computación, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires. Pabellón I, Ciudad Universitaria, (1428) Buenos Aires, Argentina. Also, CONICET Argentina.
Email: `vbecher@dc.uba.ar`

Jan Reimann
Department of Mathematics Penn State University. 318B McAllister, University Park, PA 16802
Email: `jan.reimann@psu.edu`

Theodore A. Slaman
The University of California, Berkeley, Department of Mathematics. 719 Evans Hall #3840, Berkeley, CA 94720-3840 USA
Email: `slaman@math.berkeley.edu`